3D Image and Contour Curvature

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1 Surface curvature

In 3D a surface is parameterised by u, v :

$$\boldsymbol{C}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \tag{1}$$

The tangent plane at (u, v) is spanned by two independent vectors given by

$$\boldsymbol{T}_1(u,v) = \boldsymbol{C}_u(u,v) = \begin{pmatrix} x_u(u,v) \\ y_u(u,v) \\ z_u(u,v) \end{pmatrix} \tag{2}$$

$$\boldsymbol{T}_2(u,v) = \boldsymbol{C}_v(u,v) = \begin{pmatrix} x_v(u,v) \\ y_v(u,v) \\ z_v(u,v) \end{pmatrix} \tag{3}$$

and the normal is given by

$$\hat{\boldsymbol{N}}(u,v) = \frac{\boldsymbol{C}_u \times \boldsymbol{C}_v}{|\boldsymbol{C}_u \times \boldsymbol{C}_v|} \tag{4}$$

with $\boldsymbol{T}_1 \cdot \hat{\boldsymbol{N}} = \boldsymbol{T}_2 \cdot \hat{\boldsymbol{N}} = 0$ by definition.

By taking *any* vector $\boldsymbol{a} = a_1\boldsymbol{T}_1 + a_2\boldsymbol{T}_2$ in the tangent plane, the plane spanned by $\hat{\boldsymbol{N}}$ and $\boldsymbol{a}$ cuts the surface $\boldsymbol{C}(u, v)$ to give a parametric line $\boldsymbol{C}_{\boldsymbol{a}}(s)$. The curvature of this line can be found from the two dimensional expression

$$\kappa_{\boldsymbol{a}}(u,v) = \frac{\hat{\boldsymbol{N}}(u,v) \cdot \ddot{\boldsymbol{C}}_{\boldsymbol{a}}(s)}{|\dot{\boldsymbol{C}}_{\boldsymbol{a}}(s)|^2} \tag{5}$$

Since direction $\boldsymbol{a}$ is determined by only two basis vectors, it suggests that the curvature in any direction can be determined by two fundamental curvatures. This is indeed the case, and the mathematical tools required are given by the *Weingarten*

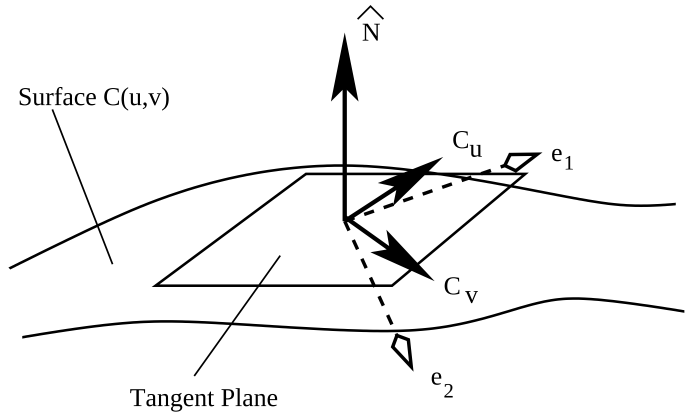

Figure 1: Relationship of tangent plane and normal vectors in determining curvature. $\boldsymbol{C}_u$ and $\boldsymbol{C}_v$ are derivatives of surface with respect to parameters u, v. Normal $\hat{N} = \frac{\boldsymbol{C}_u \times \boldsymbol{C}_v}{|\boldsymbol{C}_u \times \boldsymbol{C}_v|}$. $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are eigenvectors of Weingarten mapping matrix H.

Mapping. We define the *First Fundamental Form*

$$\mathsf{F1} = \begin{pmatrix} \boldsymbol{C}_u \cdot \boldsymbol{C}_u & \boldsymbol{C}_v \cdot \boldsymbol{C}_u \\ \boldsymbol{C}_u \cdot \boldsymbol{C}_v & \boldsymbol{C}_v \cdot \boldsymbol{C}_v \end{pmatrix} \tag{6}$$

which maps movement in parameter space to movement on the surface, and the *Second Fundamental Form*

$$\mathsf{F2} = \begin{pmatrix} \hat{\boldsymbol{N}} \cdot \boldsymbol{C}_{uu} & \hat{\boldsymbol{N}} \cdot \boldsymbol{C}_{uv} \\ \hat{\boldsymbol{N}} \cdot \boldsymbol{C}_{vu} & \hat{\boldsymbol{N}} \cdot \boldsymbol{C}_{vv} \end{pmatrix} \tag{7}$$

which maps change in the tangent direction into the normal direction. We can now define the Weingarten mapping matrix

$$\mathsf{H} = \mathsf{F1}^{-1}\mathsf{F2}\,. \tag{8}$$

This matrix is defined parametrically everywhere on the surface. Since it is symmetric it has an eigendecomposition with real eigenvalues

$$\mathsf{H} = \mathsf{E} \begin{pmatrix} \kappa_{\text{max}} & 0 \\ 0 & \kappa_{\text{min}} \end{pmatrix} \mathsf{E}^{\mathrm{T}}\,. \tag{9}$$

The maximum curvature κ_{max} is in a direction $\boldsymbol{e}_1$ on the tangent plane, and the minimum curvature κ_{min} is in a perpendicular direction $\boldsymbol{e}_2$. More commonly used are the *Gaussian* and *Mean* curvatures defined by

$$\kappa_{\text{Gauss}} = \text{Det}\mathsf{H} = \kappa_{\text{max}}\kappa_{\text{min}} \tag{10}$$

$$\kappa_{\text{Mean}} = \frac{1}{2}\text{Trace}\mathsf{H} = \frac{1}{2}\left(\kappa_{\text{max}} + \kappa_{\text{min}}\right) \tag{11}$$

2 Voxel curvature

To find curvatures from a 3D voxel array $f(x, y, z)$ we need to find equivalent forms for the tangent directions, and the Weingarten Mapping.

The normal is in the direction of the gradient

$$\boldsymbol{n} = \nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \tag{12}$$

which defines direction cosines of a local spherical polar coordinate system

$$\hat{\boldsymbol{n}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = \frac{1}{\sqrt{f_x^2 + f_y^2 + f_z^2}} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \tag{13}$$

We may choose any directions in the tangent plane. Let us arbitrarily choose a direction $\hat{\boldsymbol{p}}$ by taking the vector product of $\hat{\boldsymbol{n}}$ with the Cartesian basis direction $\hat{\boldsymbol{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\hat{\boldsymbol{p}} = \frac{\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}}}{|\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}}|} = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix} = \frac{1}{\sqrt{f_x^2 + f_y^2}} \begin{pmatrix} f_y \\ -f_x \\ 0 \end{pmatrix} \tag{14}$$

Note that this direction lies in the xy plane by choice. We now find a second tangent direction

$$\hat{\boldsymbol{q}} = \frac{\hat{\boldsymbol{n}} \times \hat{\boldsymbol{p}}}{|\hat{\boldsymbol{n}} \times \hat{\boldsymbol{p}}|} = \begin{pmatrix} -\cos\theta\cos\phi \\ -\sin\theta\cos\phi \\ \sin\theta \end{pmatrix} = \frac{1}{\sqrt{(f_x^2 + f_y^2)(f_x^2 + f_y^2 + f_z^2)}} \begin{pmatrix} -f_x f_z \\ -f_y f_z \\ f_x^2 + f_y^2 \end{pmatrix} \tag{15}$$

We choose direction $\hat{\boldsymbol{p}}$ to correspond to $\boldsymbol{C}_u$ and direction $\hat{\boldsymbol{q}}$ to correspond to $\boldsymbol{C}_v$. Since $\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{q}}$ are orthonormal by construction we find that the first fundamental form is the identity matrix. We thus only need to find the second fundamental form which will be identical to the Weingarten mapping matrix H. We can define the directional derivatives

$$\frac{\partial}{\partial p} = \hat{\boldsymbol{p}} \cdot \nabla = \sin\phi \frac{\partial}{\partial x} - \cos\phi \frac{\partial}{\partial y} \tag{16}$$

$$\frac{\partial}{\partial q} = \hat{\boldsymbol{q}} \cdot \nabla = -\cos\theta\cos\phi \frac{\partial}{\partial x} - \sin\theta\cos\phi \frac{\partial}{\partial y} + \sin\phi \frac{\partial}{\partial z} \tag{17}$$

and we need to evaluate

$$
\begin{aligned}
f_{pp} &= \hat{\boldsymbol{n}} \cdot \frac{\partial \hat{\boldsymbol{p}}}{\partial p} = \frac{f_x \frac{\partial f_y}{\partial p} - f_y \frac{\partial f_x}{\partial p}}{\sqrt{(f_x^2 + f_y^2)(f_x^2 + f_y^2 + f_z^2)}} && (18) \\
f_{pq} &= \hat{\boldsymbol{n}} \cdot \frac{\partial \hat{\boldsymbol{q}}}{\partial p} = \frac{-f_x \frac{\partial (f_x f_z)}{\partial p} - f_y \frac{\partial (f_y f_z)}{\partial p} + f_z \frac{\partial (f_x^2 + f_y^2)}{\partial p}}{\sqrt{(f_x^2 + f_y^2)}(f_x^2 + f_y^2 + f_z^2)} && (19) \\
f_{qp} &= \hat{\boldsymbol{n}} \cdot \frac{\partial \hat{\boldsymbol{p}}}{\partial q} = \frac{f_x \frac{\partial f_y}{\partial q} - f_y \frac{\partial f_x}{\partial q}}{\sqrt{(f_x^2 + f_y^2)(f_x^2 + f_y^2 + f_z^2)}} && (20) \\
f_{qq} &= \hat{\boldsymbol{n}} \cdot \frac{\partial \hat{\boldsymbol{q}}}{\partial q} = \frac{-f_x \frac{\partial (f_x f_z)}{\partial q} - f_y \frac{\partial (f_y f_z)}{\partial q} + f_z \frac{\partial (f_x^2 + f_y^2)}{\partial q}}{\sqrt{(f_x^2 + f_y^2)}(f_x^2 + f_y^2 + f_z^2)} && (21)
\end{aligned}
$$

Each of these terms is individually a curvature and we should have $f_{pq} = f_{qp}$. We thus can write

$$
\mathsf{H} = \mathsf{F2} = \begin{pmatrix} f_{pp} & f_{pq} \\ f_{pq} & f_{qq} \end{pmatrix} = \begin{pmatrix} f_{pp} & f_{qp} \\ f_{qp} & f_{qq} \end{pmatrix} \tag{22}
$$