A Note on Image and Contour Curvature

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January 17, 2019

1 Contour curvature

Curvature is an intuitive component of shapes. But how to define it mathematically ? It has something to do with the rate of change of velocity, i.e. *acceleration*. If we imagine travelling at fixed speed along a flat railway track then we feel no acceleration. If the track bends, even if the speed stays constant, we feel acceleration *perpendicular to the velocity direction*. This is sometimes called **centrifugal** (more correctly, **centripetal**) force. The tighter the bend, the stronger the force. The direction of the force tells us which way the track is bending. These ideas allow us to define the concept of curvature mathematically.

Assume a curve is parameterised by s :

$$\boldsymbol{C}(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \tag{1}$$

Then the tangent at s is given by

$$\hat{\boldsymbol{T}}(s) = \frac{1}{L}\begin{pmatrix} \dot{x}(s) \\ \dot{y}(s) \end{pmatrix} \tag{2}$$

where $L = (\dot{x}(s)^2 + \dot{y}(s)^2)^{1/2}$, and the normal is given by

$$\hat{\boldsymbol{N}}(s) = \frac{1}{L}\begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix} \tag{3}$$

with $\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{N}} = 0$ by definition. Now consider that a circle is constructed to be tangential to $\boldsymbol{C}(s)$ at s_1 and s_2 (see figure 1). This is known as an *osculating circle*.The radius of curvature is the distance to the point of intersection of the two lines

$$\begin{pmatrix} x(s_1) \\ y(s_1) \end{pmatrix} + \lambda\hat{\boldsymbol{N}}_1 \quad \text{and} \quad \begin{pmatrix} x(s_2) \\ y(s_2) \end{pmatrix} + \mu\hat{\boldsymbol{N}}_2$$

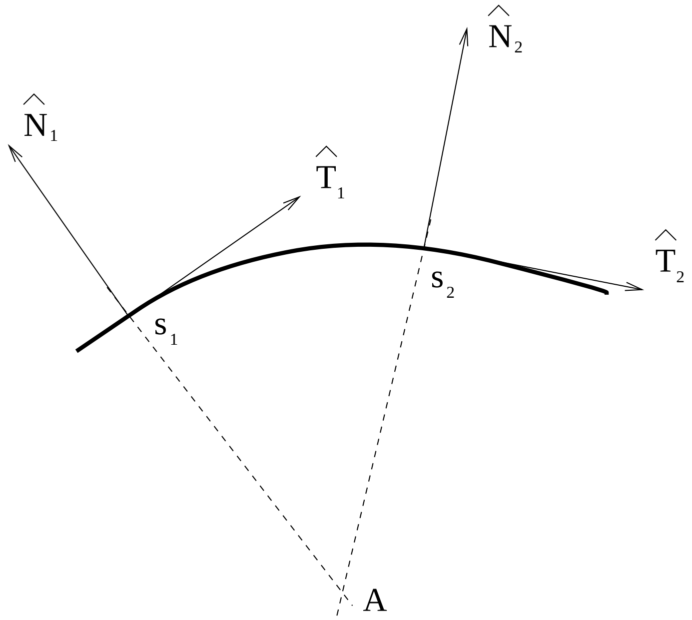

Figure 1: Relationship of tangent and normal vectors in determining curvature. If a circle is tangent at s_1 and s_2 then the radius of curvature is the distance $s_1 A = s_2 A$

Putting $\boldsymbol{\Delta s} = \begin{pmatrix} x(s_2) - x(s_1) \\ y(s_2) - y(s_1) \end{pmatrix}$ and using the standard result that

$$\begin{aligned} \boldsymbol{a} + \lambda \boldsymbol{b} &= \boldsymbol{c} + \mu \boldsymbol{d} \\ \Rightarrow \quad \lambda &= \frac{\boldsymbol{d}_\perp \cdot (\boldsymbol{c} - \boldsymbol{a})}{\boldsymbol{d}_\perp \cdot \boldsymbol{b}} \\ \mu &= \frac{-\boldsymbol{b}_\perp \cdot (\boldsymbol{c} - \boldsymbol{a})}{\boldsymbol{b}_\perp \cdot \boldsymbol{d}} \end{aligned}$$

we get

$$\mu = -\frac{\hat{\boldsymbol{T}}_1 \cdot \boldsymbol{\Delta s}}{\hat{\boldsymbol{T}}_1 \cdot \hat{\boldsymbol{N}}_2}$$

Since $\hat{\boldsymbol{N}}_1$ and $\hat{\boldsymbol{N}}_2$ are of unit length, this value of μ is exactly the distance to the centre of the circle, and thus its radius. But curvature is defined as the reciprocal of the radius of this circle so we get

$$\kappa = -\frac{\hat{\boldsymbol{T}}_1 \cdot \hat{\boldsymbol{N}}_2}{\hat{\boldsymbol{T}}_1 \cdot \boldsymbol{\Delta s}} \tag{4}$$

Now consider the limit as $\Delta s \to 0$. We have

$$\begin{aligned} \boldsymbol{\Delta s} &\to \Delta s \boldsymbol{T}_1 \\ \hat{\boldsymbol{N}}_2 &\to \hat{\boldsymbol{N}}_1 + \Delta s \dot{\hat{\boldsymbol{N}}}(s_1) \end{aligned}$$

so that

$$\kappa \to -\frac{\hat{\boldsymbol{T}}_1 \cdot (\hat{\boldsymbol{N}}_1 + \Delta s \dot{\hat{\boldsymbol{N}}}(s_1))}{\hat{\boldsymbol{T}}_1 \cdot \boldsymbol{T}_1 \Delta s} = -\frac{\hat{\boldsymbol{T}}_1 \cdot (\hat{\boldsymbol{N}}_1 + \Delta s \dot{\hat{\boldsymbol{N}}}(s_1))}{L \Delta s}$$

with the limiting value :

$$\kappa(s) = -\frac{\hat{\boldsymbol{T}}(s) \cdot \dot{\hat{\boldsymbol{N}}}(s)}{L} = -\frac{\boldsymbol{T}(s) \cdot \dot{\hat{\boldsymbol{N}}}(s)}{L^2} \tag{5}$$

Now we can use the parametric definitions of Eqs. 2 and 3 to give

$$\dot{\hat{\boldsymbol{N}}}(s) = \frac{1}{L}\begin{pmatrix} -\ddot{y}(s) \\ \ddot{x}(s) \end{pmatrix} - \frac{\dot{L}}{L^2}\begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix} = \frac{1}{L}\dot{\boldsymbol{N}}(s) - \frac{\dot{L}}{L^2}\boldsymbol{N}(s) \tag{6}$$

leading to

$$\kappa(s) = \frac{\dot{\boldsymbol{T}}(s) \cdot \boldsymbol{N}(s)}{L^3} = -\frac{\boldsymbol{T}(s) \cdot \dot{\boldsymbol{N}}(s)}{L^3} = \frac{\dot{x}(s)\ddot{y}(s) - \dot{y}(s)\ddot{x}(s)}{(\dot{x}(s)^2 + \dot{y}(s)^2)^{3/2}} \tag{7}$$

[Note we used $\boldsymbol{N} \cdot \boldsymbol{T} = 0 \Rightarrow \dot{\boldsymbol{T}} \cdot \boldsymbol{N} + \boldsymbol{T} \cdot \dot{\boldsymbol{N}}$ in the last expression.]

Remark It is worth noting that the expression eq.7 for curvature is *independent of the choice of parameterisation.* This means that curvature is an *invariant* of parameterisation. Suppose that we choose a new parameterisation

$$\tilde{s} = \phi(s)$$

where $\phi(s)$ is a twice differentiable function. Now consider

$$\boldsymbol{C}(\tilde{s}) = \begin{pmatrix} x(\tilde{s}) \\ y(\tilde{s}) \end{pmatrix} = \begin{pmatrix} x(\phi(s)) \\ y(\phi(s)) \end{pmatrix} \tag{8}$$

We have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{\mathrm{d}x}{\mathrm{d}\tilde{s}}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = \dot{x}\dot{\phi} & \qquad \tfrac{\mathrm{d}^2x}{\mathrm{d}s^2} &= \ddot{x}\dot{\phi}^2 + \dot{x}\ddot{\phi} \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \frac{\mathrm{d}y}{\mathrm{d}\tilde{s}}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}s} = \dot{y}\dot{\phi} & \qquad \tfrac{\mathrm{d}^2y}{\mathrm{d}s^2} &= \ddot{y}\dot{\phi}^2 + \dot{y}\ddot{\phi} \end{aligned}$$

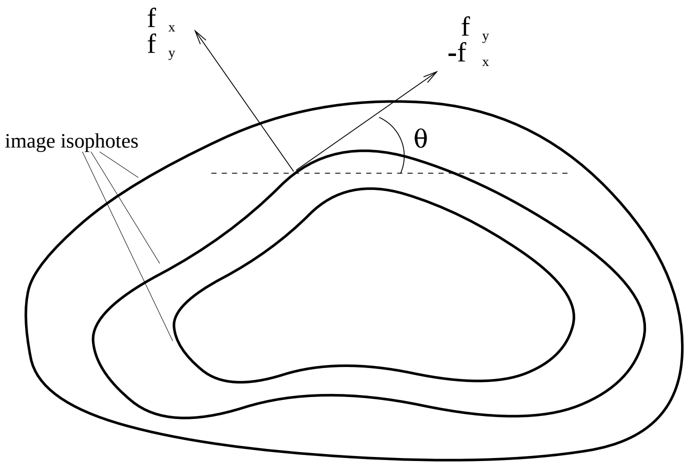

Figure 2: Image gradient is the normal to the lines of equal intensity (isophotes). Gauge coordidates $\hat{\boldsymbol{n}}$ aligned with normal, and $\hat{\boldsymbol{t}}$ aligned with the tangent

Then

$$
\begin{aligned}
\tilde{\kappa} \quad &:= \quad \frac{\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}^2x}{\mathrm{d}s^2}}{\left(\frac{\mathrm{d}x}{\mathrm{d}s}^2 + \frac{\mathrm{d}y}{\mathrm{d}s}^2\right)^{3/2}} \\
&= \quad \frac{\dot{x}\dot{\phi}\left(\ddot{y}\dot{\phi}^2 + \dot{y}\ddot{\phi}\right) - \dot{y}\dot{\phi}\left(\ddot{x}\dot{\phi}^2 + \dot{x}\ddot{\phi}\right)}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}\dot{\phi}^3} \\
&= \quad \frac{\dot{x}\ddot{y}\dot{\phi}^3 + \dot{x}\dot{y}\dot{\phi}^2\ddot{\phi} - \dot{y}\ddot{x}\dot{\phi}^3 - \dot{x}\dot{y}\dot{\phi}^2\ddot{\phi}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}\dot{\phi}^3} \\
&= \quad \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}} =: \kappa
\end{aligned}
\tag{9}
$$

The curve given by eq.1 and that given by eq.8 describe *exactly the same set of points*, and have the same geometrical features. All that changes is the fictious "speed" with which we draw around the contour.

2 Image curvature

The key to defining curvature of *images* is to apply the concepts of parametric curves to the *level sets* of the image. There is no explicit parameterisation of these level sets, but we can define an *implicit* parameterisation. We do this by equating the contour normal to the image gradient (see figure 2) :

$$
\hat{\boldsymbol{N}}(x, y) = \frac{1}{(f_x^2 + f_y^2)^{1/2}}\begin{pmatrix} f_x \\ f_y \end{pmatrix} \tag{10}
$$

which by comparison with Eq. 3 means

$$\dot{x}(s) = f_y \quad \dot{y}(s) = -f_x \tag{11}$$

(the equality can be taken by imposing a suitable parameterisation). We need to define

$$\begin{aligned} \frac{d}{ds} &= \frac{dx}{ds}\frac{d}{dx} + \frac{dy}{ds}\frac{d}{dy} \\ &= \cos\theta \frac{d}{dx} + \sin\theta \frac{d}{dy} \end{aligned}$$

where

$$\begin{aligned} \sin\theta &= \frac{\dot{x}(s)}{(\dot{x}(s)^2 + \dot{y}(s)^2)^{1/2}} = \frac{f_y}{(f_x^2 + f_y^2)^{1/2}} \\ \cos\theta &= \frac{\dot{y}(s)}{(\dot{x}(s)^2 + \dot{y}(s)^2)^{1/2}} = \frac{-f_x}{(f_x^2 + f_y^2)^{1/2}} \end{aligned}$$

which leads to

$$\begin{aligned} \ddot{x}(s) &= \cos\theta f_{yx} + \sin\theta f_{yy} \\ &= \frac{f_y f_{yx}}{(f_x^2 + f_y^2)^{1/2}} + \frac{-f_x f_{yy}}{(f_x^2 + f_y^2)^{1/2}} \\ \ddot{y}(s) &= -\cos\theta f_{xx} - \sin\theta f_{xy} \\ &= -\frac{f_y f_{xx}}{(f_x^2 + f_y^2)^{1/2}} - \frac{-f_x f_{xy}}{(f_x^2 + f_y^2)^{1/2}} \end{aligned}$$

finally giving, from Eq 7

$$\kappa(x, y) = -\frac{f_{xx} f_y^2 + f_{yy} f_x^2 - 2 f_{xy} f_x f_y}{(f_x^2 + f_y^2)^{3/2}} \tag{12}$$

3 Divergence of Normal

There is another expression for the curvature

$$\kappa(x, y) = -\nabla \cdot \hat{\boldsymbol{N}} \tag{13}$$

This can be verified by simple algebra:

$$\begin{aligned}
\nabla \cdot \hat{\boldsymbol{N}} &= \frac{\partial}{\partial x}\frac{f_x}{\left(f_x^2+f_y^2\right)^{1/2}} + \frac{\partial}{\partial y}\frac{f_y}{\left(f_x^2+f_y^2\right)^{1/2}} \\
&= \frac{f_{xx}+f_{yy}}{\left(f_x^2+f_y^2\right)^{1/2}} - \frac{f_x^2 f_{xx} + f_x f_y f_{xy}}{\left(f_x^2+f_y^2\right)^{3/2}} - \frac{f_x f_y f_{xy} + f_y^2 f_{yy}}{\left(f_x^2+f_y^2\right)^{3/2}} \\
&= \frac{(f_{xx}+f_{yy})(f_x^2+f_y^2) - f_x^2 f_{xx} - f_y^2 f_{yy} - 2 f_x f_y f_{xy}}{\left(f_x^2+f_y^2\right)^{3/2}} \\
&= \frac{f_{xx} f_y^2 + f_{yy} f_x^2 - 2 f_{xy} f_x f_y}{(f_x^2+f_y^2)^{3/2}}
\end{aligned}$$

This has an important interpretation in terms of image denoising because it allows for diffusion tangential to an edge but not across the edge.