6 Evolution of Images using Partial Differential Equations

6.1 Variational Methods in Imaging

We here introduce a functional used frequently in image reconstruction and image processing.

$$\Psi(f) = \int_\Omega \psi\left(|\nabla f|\right) \mathrm{d}\mathbf{x} \tag{38}$$

We want to derive the Gateaux derivative for Eq. 38 in direction h, defined to be

$$\Psi'(f)h := \lim_{\epsilon \to 0} \left[\frac{\Psi(f + \epsilon h) - \Psi(f)}{\epsilon} \right] \tag{39}$$

for arbitrary function h satisfying the same boundary conditions as f. $\Psi'(f)$ is a linear operator mapping from the space of functions to a scalar value. If we can show that it does not depend on h then $\Psi'(f)$ is the Fréchet derivative of Ψ and constitutes a generalised notion of a gradient.

Expanding Eq. 38 we have

$$\Psi(f + \epsilon h) = \int_\Omega \psi\left(\sqrt{\nabla f \cdot \nabla f + 2\epsilon \nabla f \cdot \nabla h + \epsilon^2 \nabla h \cdot \nabla h} \right) \mathrm{d}\mathbf{x}$$

Taking the Taylor series we have

$$\Psi(f + \epsilon h) = \Psi(f) + \int_\Omega 2\epsilon \psi'\left(|\nabla f|\right) \frac{1}{2} |\nabla f|^{-1} \nabla f \cdot \nabla h \mathrm{d}\mathbf{x} + O(\epsilon^2)$$

Making the definition

$$\gamma := \frac{\psi'\left(|\nabla f|\right)}{|\nabla f|}$$

we have

$$\Psi'(f)h = \int_\Omega \gamma \nabla f \cdot \nabla h \mathrm{d}\mathbf{x} \quad = \quad -\int_\Omega h \nabla \cdot \gamma \nabla f \mathrm{d}\mathbf{x}$$

where the last expression results from using the divergence theorem. We now define the differential operator

$$\mathcal{L}(f) = -\nabla \cdot \gamma \nabla$$

so that we can state

$$\Psi'(f)h = \int_\Omega h \mathcal{L}(f) f \mathrm{d}x \quad = \langle h, \mathcal{L}(f) f \rangle \quad = \langle h, \delta\Psi \rangle$$

Note that

- $\mathcal{L}(f)$ is a linear operator, but it is parameterised on f because of the function γ.
- $\Psi'(f)$ does not depend on h. It is a linear integral operator with kernel function $\delta\Psi$.
- $\delta\Psi := \mathcal{L}(f) f$ is an image.

We use the kernel of the Fréchet derivative of Ψ to define a *time-domain Partial Differential Equation* (PDE)

$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] f(\mathbf{x}, t) = 0 \tag{40}$$

We can solve Eq. 40 by using the Green's operator

$$(\mathcal{G}f)(\mathbf{x}, t) := \int_\Omega G(\mathbf{x}, \mathbf{x}', t, t') f(\mathbf{x}', t') \mathrm{d}^d x' \mathrm{d}t' \tag{41}$$

where $G(\mathbf{x}, \mathbf{x}', t, t')$ is the *Green's function* of $\mathcal{L}$ (with appropriate boundary conditions), which has the property

$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] G(\mathbf{x}, \mathbf{x}', t, t') = \delta(\mathbf{x}, \mathbf{x}', t, t') \tag{42}$$

It follows that

$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] \mathcal{G} f = \mathcal{G} \left[\frac{\partial}{\partial t} + \mathcal{L}\right] f = f \tag{43}$$

and the solution to any inhomogoeous PDE

$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] f(\mathbf{x}, t) = q(\mathbf{x}, t) \tag{44}$$

is given by

$$f = \mathcal{G} q \,. \tag{45}$$

6.2 Gaussian Convolution as a Diffusion Process

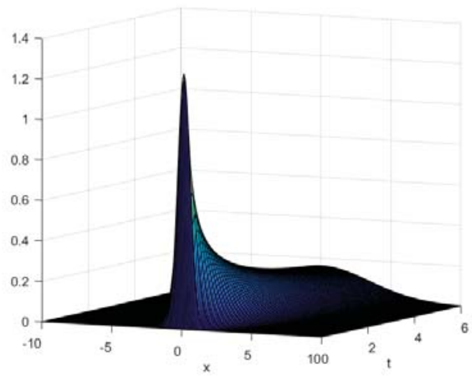

Figure 7: 3D plot of Green's function for the homogeneous diffusion equation (Eq. 46)

We first consider the case where $\psi(s) = \frac{1}{2}s^2 \Leftrightarrow \psi'(s) = s$. It follows that $\gamma = 1$ and

$$\mathcal{L}(f) = -\nabla^2$$

which is the stationary Laplacian (does not depend on f). The resultant diffusion is *isotropic* and the resultant Green's function is also stationary

$$G(\mathbf{x}, \mathbf{x}', t, t') = G^{\mathrm{H}}(\mathbf{x} - \mathbf{x}', t - t') := \frac{1}{\sqrt{2\pi(t-t')}^d} \exp\left[-\frac{|\mathbf{x} - \mathbf{x}'|^2}{4(t-t')}\right] \quad t > t' \tag{46}$$

so that the solution to the equation

$$\left[\frac{\partial}{\partial t} - \nabla^2\right] f(\mathbf{x}, t) = q(\mathbf{x}, t) \tag{47}$$

is given by the stationary convolution

$$f = \int_{\mathbb{R}^d} G^{\mathrm{H}}(\mathbf{x} - \mathbf{x}', t - t') q(\mathbf{x}', t') \mathrm{d}^n x' \mathrm{d}t' = G^{\mathsf{H}} * q\,. \tag{48}$$

Importantly, the converse is also true : *Convolution of an image f_0 with a Gaussian G_σ of standard deviation σ is the same as evolving the isotropic equation Eq. 47 with initial condition $q = f_0|_{t=0}$ for a time $t = \frac{\sigma^2}{2}$.* This gives us a third mechanism (in addition to spatial and Fourier domain convolution) for performing Gaussian smoothing. The advantage comes when we consider efficient methods for solving Eq. 47 in a discrete setting in section 6.4. Figure 7 shows a 3D plot of the function Eq. 46.

6.3 Anisotropic Diffusion

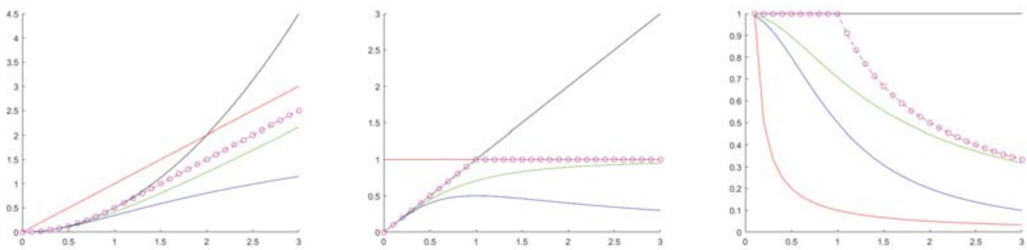

Figure 8: Left to right : i) Different varational forms for $\psi = \frac{1}{2}s^2$ (black), Total Variation (red), Smoothed Total Variation (green), Perona-Malik (blue), Huber (magenta circles); ii) the derivative $\psi'(s)$; iii) diffusivity $\gamma = \frac{\psi'(s)}{s}$

When the function ψ in Eq. 38 is not the quadratic function, the resultant diffusion scheme is *anisotropic*. We'll consider some common examples

1. *Total Variation.* If $\psi(s) = s \Leftrightarrow \psi'(s) = 1$, then $\gamma = \frac{1}{|\nabla f|}$

$$\mathcal{L}(f) f = -\nabla \cdot \frac{1}{|\nabla f|} \nabla f = \kappa$$

where κ is the image curvature.

2. *Smoothed Total Variation.* Because of the division by zero of $\gamma = \frac{1}{|\nabla f|}$ when f is constant, total variation is often implemented by some approximation such as

$$\psi(s) = T\sqrt{s^2+T^2} - T^2 \Leftrightarrow \psi'(s) = \frac{sT}{\sqrt{s^2+T^2}} \Leftrightarrow \gamma = \frac{T}{\sqrt{|\nabla f|^2+T^2}} = \frac{1}{\sqrt{1+\left(\frac{|\nabla f|}{T}\right)^2}}$$

where T is a threshold.

3. *Perona-Malik.* If $\psi(s) = \frac{T^2}{2}\log\left(1+\left(\frac{s}{T}\right)^2\right) \Leftrightarrow \psi'(s) = \frac{sT^2}{T^2+s^2}$ then $\gamma = \frac{T^2}{T^2+|\nabla f|^2}$

$$\mathcal{L}(f)f = -\nabla \cdot \frac{1}{1+\left(\frac{|\nabla f|)}{T}\right)^2}\nabla f$$

where T is a threshold

4. *Huber.* This functional is designed to give a smooth transition from a quadratic functional when $|\nabla f| \to 0$ and a linear functional (as in Total Variation) when $|\nabla f| \to \infty$. It is achieved using

$$\psi(s) = \begin{cases} Ts - \frac{T^2}{2} & s > T \\ \frac{s^2}{2} & s \le T \end{cases} \Leftrightarrow \psi'(s) = \begin{cases} T & s > T \\ s & s \le T \end{cases} \Leftrightarrow \gamma = \begin{cases} \frac{T}{|\nabla f|} & |\nabla f| > T \\ 1 & |\nabla f| \le T \end{cases}$$

These schemes do not correspond to stationary convolution, instead they are a way of implementing a blurring process that depends on the local image structure.

6.4 Implementation of Diffusion using Finite Differencing

We make use of the discrete differentiation methods we introduced in section 4.1. We will assume that an image is sampled on a regular grid of spacing $\Delta x, \Delta y$ centred at an origin (x_0, y_0) , and that the time series is at regular intervals $\{t_n = t_0 + n\Delta t\}$ leading to $\{f^{(n)}_{i,j} = f(x_0 + ih\Delta x, y_0 + j\Delta y, t_0 + n\Delta t)\}$.

6.4.1 Explicit Methods

In the explicit scheme, all derivatives are approximated at $t = t_n$ and the subsequent point in time is evaluated as

$$\begin{aligned} f^{(n+1)}_{ij} &= f^{(n)}_{ij} + \Delta t \Big[\gamma_{i+\frac{1}{2},j}(f^{(n)}_{i+1,j} - f^{(n)}_{i,j}) - \gamma_{i-\frac{1}{2},j}(f^{(n)}_{i,j} - f^{(n)}_{i-1,j}) \\ &\qquad + \gamma_{i,j+\frac{1}{2}}(f^{(n)}_{i,j+1} - f^{(n)}_{i,j}) - \gamma_{i,j-\frac{1}{2}}(f^{(n)}_{i,j} - f^{(n)}_{i,j-1})\Big] \qquad (49) \\ &= \left(1 - \Delta t\left(\gamma_{i+\frac{1}{2},j} + \gamma_{i-\frac{1}{2},j} + \gamma_{i,j+\frac{1}{2}} + \gamma_{i,j-\frac{1}{2}}\right)\right) f^{(n)}_{i,j}. \\ &\qquad + \Delta t\left(\gamma_{i+\frac{1}{2},j} f^{(n)}_{i+1,j} + \gamma_{i-\frac{1}{2},j} f^{(n)}_{i-1,j} + \gamma_{i,j+\frac{1}{2}} f^{(n)}_{i,j+1} + \gamma_{i,j-\frac{1}{2}} f^{(n)}_{i,j-1}\right) \qquad (50) \end{aligned}$$

where the inter-pixel diffusivity is the average of its neighbours $\gamma_{i\pm\frac{1}{2},j} = \frac{1}{2}(\gamma_{i,j} + \gamma_{i\pm1,j})$, $\gamma_{i,j\pm\frac{1}{2}} = \frac{1}{2}(\gamma_{i,j} + \gamma_{i,j\pm1})$. We see that Eq. 50 can be written in matrix-vector form as

$$\mathbf{f}^{(n+1)} = [\mathsf{I} + \Delta t \mathsf{A}]\, \mathbf{f}^{(n)} \tag{51}$$

where the matrix A has the structure

$$A_{i,j} = \begin{cases} \gamma_{i,j-\frac{1}{2}} & j = i - N \\ \gamma_{i-\frac{1}{2},j} & j = i - 1 \\ -\left(\gamma_{i+\frac{1}{2},j} + \gamma_{i-\frac{1}{2},j} + \gamma_{i,j+\frac{1}{2}} + \gamma_{i,j-\frac{1}{2}}\right) & i = j \\ \gamma_{i+\frac{1}{2},j} & j = i + 1 \\ \gamma_{i,j+\frac{1}{2}} & j = i + N \end{cases} \tag{52}$$

which results from reading out the two-dimensional image f in row-order, i.e. pixels neighbouring in the x-direciton have an index differeing by one, whereas those in the y-direction have an index differing by N. This structure is referred to as *tridiagonal with fringes*. The explicit scheme Eq. 50 is efficient because it does not require the explicit construction of matrix A, only the smal number of summations and multiplications shown. However, due to the Courant-Friedrichs-Levy (CFL) condition for such schemes it is only *conditionally stable*. In fact it is required that $\Delta t \leq \frac{\min(\Delta x, \Delta y)}{2}$ for the scheme to converge. This step may be too small for some requirements.

6.4.2 Implicit Methods

The implicit scheme evaluates derivatives by backward differencing in time, leading to

$$[\mathsf{I} - \Delta t \mathsf{A}]\, \mathbf{f}^{(n+1)} = \mathbf{f}^{(n)} \tag{53}$$

This requires the inversion of a large $N^2 \times N^2$ matrix. However, since the structure is of the special form given in Eq. 52 it can be efficiently decomposed and inverted. The explicit scheme is *unconditionally stable* : it can be evaluated at arbitrarily large values of the time-step Δt.

6.4.3 Alternating Direction Implicit and Semi-implicit methods

A method that is more optimal is Alternating Direction Implicit (ADI). Here it is noted that the matrix in Eq. 52 can be split as $\mathsf{A} = \mathsf{A}^{(\mathsf{x})} + \mathsf{A}^{(\mathsf{y})}$ where

$$A^{(\mathsf{x})}_{i,j} = \begin{cases} \gamma_{i-\frac{1}{2},j} & j = i - 1 \\ -\left(\gamma_{i+\frac{1}{2},j} + \gamma_{i-\frac{1}{2},j}\right) & i = j \\ \gamma_{i+\frac{1}{2},j} & j = i + 1 \end{cases} \qquad A^{(\mathsf{y})}_{i,j} = \begin{cases} \gamma_{i,j-\frac{1}{2}} & j = i - N \\ -\left(\gamma_{i,j+\frac{1}{2}} + \gamma_{i,j-\frac{1}{2}}\right) & i = j \\ \gamma_{i,j+\frac{1}{2}} & j = i + N \end{cases} \tag{54}$$

and the ADI schemes is written in two steps

$$\left[\mathsf{I} - \frac{\Delta t}{2}\mathsf{A}^{(\mathsf{x})}\right]\mathbf{f}^{(n+\frac{1}{2})} = \left[\mathsf{I} + \frac{\Delta t}{2}\mathsf{A}^{(\mathsf{y})}\right]\mathbf{f}^{(n)} \tag{55}$$

$$\left[\mathsf{I} - \frac{\Delta t}{2}\mathsf{A}^{(\mathsf{y})}\right]\mathbf{f}^{(n+1)} = \left[\mathsf{I} + \frac{\Delta t}{2}\mathsf{A}^{(\mathsf{x})}\right]\mathbf{f}^{(n+\frac{1}{2})} \tag{56}$$

In Eq. 55 the matrix on the left is precisely tridiagonal (no fringes) and may be inverted in $\mathcal{O}(n)$ time by LU-decomposition and forward/backward substitution. In Eq. 56 the same method can be employed by first reading out the image $f^{(n+\frac{1}{2})}$ in *column order* : this means that y-neighbours have index differing by 1 and x-neighbours have index differing by N.

The ADI scheme is both computationally fast and unconditionally stable.

References

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