Relation algebra reducts of cylindric algebras and an application to proof theory

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Abstract

We confirm a conjecture about neat embeddings of cylindric algebras made in 1969 by J. D. Monk, confirm a later conjecture by Maddux about relation algebras obtained from cylindric algebras, and solve a problem raised by Tarski and Givant. These results in algebraic logic have the following consequence for predicate logic: for every finite cardinal $\alpha \geq 3$ there is a logically valid sentence φ, in a first-order language with equality and exactly one nonlogical binary relation symbol, such that φ contains only 3 variables (each of which can occur arbitrarily many times), φ has a proof containing exactly $\alpha+1$ variables, but no proof containing only α variables.

1 Introduction and summary

Cylindric algebras are an algebraisation of the theory of α-ary relations. For the special case of binary relations, there is an alternative approach to algebraisation, using relation algebras. This paper is concerned with the connection between the two approaches and the consequences for α-variable proofs in first-order logic.

We use the notation of [5, 6]. In particular, CA_α is the class of α-dimensional cylindric algebras, and, for any class K of algebras, SK is the class of all algebras that embed into some algebra in K. For any ordinal $\alpha \geq 3$, there is a well-known method of obtaining a relation algebra $\mathfrak{Ra}\mathfrak{C}$ from an α-dimensional cylindric algebra $\mathfrak{C}$. $\mathfrak{Ra}\mathfrak{C}$ is constructed by taking the two-dimensional elements of $\mathfrak{C}$ and using the third dimension to define conversion and composition [6, 5.3.7]. $\mathfrak{Ra}\mathfrak{C}$ is called the **relation-algebraic reduct** of $\mathfrak{C}$, defined formally in definition 3. $\mathsf{Ra}\,\mathsf{CA}_\alpha$ is the class $\{\mathfrak{Ra}\mathfrak{C} : \mathfrak{C} \in \mathsf{CA}_\alpha\}$. If $\alpha \geq 4$ then $\mathfrak{Ra}\mathfrak{C}$ is a relation algebra [6, 5.3.8]. If $\alpha = 3$, $\mathfrak{Ra}\mathfrak{C}$ may not be a relation algebra [4, 6]. Thus $\mathsf{Ra}\,\mathsf{CA}_4$ is a class of relation algebras, but $\mathsf{Ra}\,\mathsf{CA}_3$ contains algebras that are not relation algebras.

The question arises as to whether it is possible to work backwards. Which relation algebras can be obtained as subalgebras of $\mathfrak{Ra}\mathfrak{C}$ for some α-dimensional cylindric algebra $\mathfrak{C}$? Put another way, if a relation algebra $\mathfrak{A}$ is not obtainable as a subalgebra of $\mathfrak{Ra}\mathfrak{C}$ for some α-dimensional cylindric algebra $\mathfrak{C}$, does this leave any traces in the relation algebra structure of $\mathfrak{A}$?

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By 1960 Tarski [4] had already constructed a 3-dimensional cylindric algebra in which composition is not associative in the relation-algebraic reduct $\mathfrak{A}$, so $\mathfrak{A}$ could not be the relation-algebraic reduct of any four-dimensional cylindric algebra. He started with an algebra found by J. C. C. McKinsey that satisfies all the equational axioms for relation algebras except the associative law, and built a cylindric algebra $\mathfrak{C}$ such that $\mathfrak{Ra}\mathfrak{C}$ is McKinsey's algebra. In his dissertation [13], J. D. Monk showed that every relation algebra is the relation-algebraic reduct of some 3-dimensional cylindric algebra and in Maddux's dissertation [9], it was shown that every relation algebra is embeddable in the relation-algebraic reduct of some 4-dimensional cylindric algebra; see [6, Theorem 5.3.17]. To obtain his result, Maddux defined the notion of an *α-dimensional cylindrical basis* for a relation algebra $\mathfrak{A}$, and he showed that if $\mathfrak{A}$ has such a basis B then an α-dimensional cylindric algebra $\mathfrak{Ca}B$ can be constructed such that $\mathfrak{A}$ embeds into $\mathfrak{Ra}\mathfrak{Ca}B$ [9, theorem 10(13)], [10, theorem 10]. We will use this fact later to show that the algebras we construct can be embedded into relation algebra reducts of cylindric algebras. However, the converse is not valid. Not having a basis is not enough to show that an algebra cannot be embedded. Indeed, every representable relation algebra $\mathfrak{A}$ must belong to $\mathsf{SRa\,CA}_\alpha$ for every $\alpha \geq 3$, and $\mathfrak{A}$ can even be embedded in a relation algebra that has an α-dimensional cylindric basis. On the other hand, $\mathfrak{A}$ itself may not even have a five-dimensional cylindrical basis. Examples can be found among the relation algebras constructed from projective geometries by Lyndon [8].

Taking the relation algebra reduct of a cylindric algebra has an analogue within the realm of cylindric algebras alone: one takes the *neat reduct* to some lower dimension. Given $\beta < \alpha$ and an α-dimensional cylindric algebra $\mathfrak{C}$, one extracts its β-dimensional elements and then takes the reduct to the signature of β-dimensional cylindric algebras. The result is indeed a β-dimensional cylindric algebra, denoted $\mathfrak{Nr}_\beta\mathfrak{C}$. It is evident from the definitions that if we preface a relation algebra reduct with such a neat reduct for $\beta \geq 3$, the outcome is unchanged: $\mathfrak{Ra}\mathfrak{Nr}_\beta\mathfrak{C} = \mathfrak{Ra}\mathfrak{C}$. It follows that any relation algebra that is the relation algebra reduct of an α-dimensional cylindric algebra is also the relation algebra reduct of a β-dimensional cylindric algebra, for all $3 \leq \beta < \alpha$. So $\mathsf{Ra\,CA}_\alpha \supseteq \mathsf{Ra\,CA}_{\alpha+1}$ whenever $\alpha \geq 3$, and of course the same holds for the closure of these classes under subalgebras: $\mathsf{SRa\,CA}_\alpha \supseteq \mathsf{SRa\,CA}_{\alpha+1}$.

In this paper, we prove

THEOREM 1 *Each of the inclusions* $\mathsf{SRa\,CA}_3 \supseteq \mathsf{SRa\,CA}_4 \supseteq \cdots$ *is strict.*

Theorem 1 confirms a conjecture of Maddux [1, Problem 17, p. 734]. That some of these are strict is already known. The first inclusion is certainly strict, because if $\mathfrak{C}$ is a 3-dimensional cylindric algebra then as we saw, $\mathfrak{Ra}\mathfrak{C}$ is not necessarily associative, whereas if $\mathfrak{C}$ is a 4-dimensional cylindric algebra then $\mathfrak{Ra}\mathfrak{C}$ is a relation algebra and so is definitely associative (see [11, pp377–378]; in fact, $\mathsf{SRa\,CA}_4$ is the class of all relation algebras). In [12] it was proved that $\mathsf{SRa\,CA}_\alpha \neq \mathsf{SRa\,CA}_{3\alpha-7}$ for every finite $\alpha > 3$, so that infinitely many inclusions (and in particular, the second one) are strict. It is also noteworthy that for all finite $\alpha \geq 5$ there is a representable atomic relation algebra $\mathfrak{A}$ with an α-dimensional cylindrical basis (so that $\mathfrak{A} \in \mathsf{Ra\,CA}_\alpha$) but with no $(\alpha+1)$-dimensional cylindrical basis (see [10, p. 954] and [9]). However, the latter does not imply that $\mathfrak{A} \notin \mathsf{SRa\,CA}_{\alpha+1}$. Here we prove that all the inclusions are strict by constructing certain algebras $\mathfrak{N}_\alpha^\beta$ (for $4 \leq \alpha \leq \beta < \omega$) that belong to $\mathsf{SRa\,CA}_\alpha$ but do not embed into any algebra in $\mathsf{Ra\,CA}_{\alpha+1}$ whenever β is sufficiently large compared to α. See theorems 7 and 9 below. The first of these is simple to prove, using results of [10]; the second involves some combinatorial argument (cf. [3]). The algebras $\mathfrak{N}_\alpha^\beta$ themselves are not too complicated. The case $\alpha = 3$ is covered by known results (see above).

Theorem 1 has consequences for neat embeddings of cylindric algebras. J. D. Monk formulated the following conjecture in 1969 [14]. Let R be the set of triples of finite cardinals $\langle\alpha,\kappa,\gamma\rangle$ having the property that there exists a cylindric algebra $\mathfrak{C}$ of dimension α that can be neatly embedded in a cylindric algebra of dimension $\alpha+\kappa$ but can not be embedded in a cylindric algebra of dimension $\alpha+\gamma$, *i.e.*, $\mathfrak{C}\in\mathsf{CA}_\alpha$, $\mathfrak{C}\in\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa}$, but $\mathfrak{C}\notin\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\gamma}$. Various results about cylindric algebras (see [5, Prob. 2.12, p. 464]) imply that this situation is interesting only if $3\leq\alpha$ and possible only if $\kappa<\gamma$. Monk conjectured that $\langle\alpha,\kappa,\kappa+1\rangle\in R$ for all $\kappa<\omega$ and $\alpha\geq 3$; this implies $\langle\alpha,\kappa,\gamma\rangle\in R$ for all possible and interesting $\langle\alpha,\kappa,\gamma\rangle$. The proof of this conjecture provides a negative solution to the *neat embedding problem* [5, Problem 2.12, p. 464]: For $2<\alpha<\omega$, is there a $\kappa<\omega$ such that $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa}=\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$? As Monk [14, page 342, line 28] also conjectured, the answer is "no".

Corollary 2 *For $2<\alpha<\omega$, there is no $\kappa<\omega$ such that* $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa}=\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$.

Proof:

Let $3\leq\beta\leq\alpha<\omega$. First, observe that $\mathsf{SRa}\,\mathsf{Nr}_\beta\mathsf{CA}_\alpha=\mathsf{SRa}\,\mathsf{SNr}_\beta\mathsf{CA}_\alpha$, and the same for $\alpha+1$. The inclusion '$\subseteq$' is trivial; for '$\supseteq$', note that if $\mathfrak{A}\subseteq\mathfrak{Ra}\mathfrak{B}$ for $\mathfrak{B}\subseteq\mathfrak{C}\in\mathsf{Nr}_\beta\mathsf{CA}_\alpha$, then $\mathfrak{A}\subseteq\mathfrak{Ra}\mathfrak{C}$, so that $\mathfrak{A}\in\mathsf{SRa}\,\mathsf{Nr}_\beta\mathsf{CA}_\alpha$. Here, '$\subseteq$' denotes 'subalgebra'. As mentioned above, it is clear that $\mathsf{SRa}\,\mathsf{CA}_\alpha=\mathsf{SRa}\,\mathsf{Nr}_\beta\mathsf{CA}_\alpha$.

Now assume for contradiction that $\mathsf{SNr}_\beta\mathsf{CA}_\alpha=\mathsf{SNr}_\beta\mathsf{CA}_{\alpha+1}$. Then

$$\begin{aligned}\mathsf{SRa}\,\mathsf{CA}_\alpha &= \mathsf{SRa}\,\mathsf{Nr}_\beta\mathsf{CA}_\alpha\\ &= \mathsf{SRa}\,\mathsf{SNr}_\beta\mathsf{CA}_\alpha\\ &= \mathsf{SRa}\,\mathsf{SNr}_\beta\mathsf{CA}_{\alpha+1}\\ &= \mathsf{SRa}\,\mathsf{Nr}_\beta\mathsf{CA}_{\alpha+1}\\ &= \mathsf{SRa}\,\mathsf{CA}_{\alpha+1},\end{aligned}$$

contradicting theorem 1. So $\mathsf{SNr}_\beta\mathsf{CA}_\alpha\neq\mathsf{SNr}_\beta\mathsf{CA}_{\alpha+1}$. This proves the corollary. ∎

Theorem 1 also has interesting consequences for proof theory. We will construct, for each finite $\alpha\geq 4$, a certain relation algebra $\mathfrak{N}^\beta_\alpha$ that belongs to $\mathsf{SRa}\,\mathsf{CA}_\alpha$ but not $\mathsf{SRa}\,\mathsf{CA}_{\alpha+1}$. This algebra is generated by a single element, e. [16] gives a first-order language $\mathcal{L}$ with a single binary predicate symbol $\mathbf{E}$ and the equality symbol, together with axioms and rules for this language, and a second language, $\mathcal{L}^+$, obtained by extending $\mathcal{L}$ with operators corresponding to the relation algebra operators. The two formalisms are shown to be equipollent in means of expression and equipollent in means of proof [16, section 2.3, parts (vii) and (ix)]. Properties of the algebra $\mathfrak{N}^\beta_\alpha$ translate to properties of $\mathcal{L}$. A consequence of $\mathfrak{N}^\beta_\alpha\in\mathsf{SRa}\,\mathsf{CA}_\alpha\setminus\mathsf{SRa}\,\mathsf{CA}_{\alpha+1}$ is that there is a valid three-variable sentence ϕ_α in $\mathcal{L}$ with a proof (in either formalism) using $\alpha+1$ variables but with no proof containing only α variables. See theorem 14. The underlying signature of ϕ_α consists of a single binary relation symbol, so is independent of α.

Notation We generally use the same notation for an algebra as for its domain. So, for example, $X\subseteq\mathfrak{A}$ will indicate that X is a set of elements of the algebra $\mathfrak{A}$ (but note that we also write $\mathfrak{B}\subseteq\mathfrak{A}$ to denote that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$). As usual, an ordinal is the set of all smaller ordinals. If $\mathfrak{A}$ is a boolean algebra with operators, $At\mathfrak{A}$ denotes the set of atoms, or minimal non-zero elements, of the boolean part of $\mathfrak{A}$.

2 Relation algebra reducts of cylindric algebras

We assume a basic knowledge of relation algebras and cylindric algebras. See [11] for an introduction, and [5, 6] for a comprehensive study. The *j-for-i substitutor,* s^i_j, is defined by

$$\mathsf{s}^i_j x = \begin{cases} x & \text{if } i = j, \\ \mathsf{c}_i(x \cdot \mathsf{d}_{ij}) & \text{otherwise.} \end{cases}$$

Given a cylindric algebra $\mathfrak{C}$, it is possible to construct from it an algebra $\mathfrak{Ra}\mathfrak{C}$, called the *neat relation algebra reduct* or simply the *relation algebra reduct* of $\mathfrak{C}$.

Definition 3 [6, 5.3.7] *Let $3 \leq \alpha < \omega$, and let $\mathfrak{C}$ be any α-dimensional cylindric algebra. Define $\mathfrak{Ra}\mathfrak{C}$ to be the algebra*

$$\langle \{\mathsf{c}_2\mathsf{c}_3 \cdots \mathsf{c}_{\alpha-1}c : c \in \mathfrak{C}\}, +, \cdot, \bar{\ }, 0, 1, ;, \breve{\ }, \mathsf{d}_{01} \rangle$$

where:

- *converse is defined by $\breve{a} = \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2 a$, for $a \in \mathfrak{Ra}\mathfrak{C}$*
- *composition is defined by $a_1 \,;\, a_2 = \mathsf{c}_2(\mathsf{s}^1_2 a_1 \cdot \mathsf{s}^0_2 a_2)$, for $a_1, a_2 \in \mathfrak{Ra}\mathfrak{C}$.*

In [6, 5.3.8], it is shown that when $\alpha \geq 4$ and $\mathfrak{C} \in \mathsf{CA}_\alpha$, the algebra $\mathfrak{Ra}\mathfrak{C}$ is a relation algebra.

3 A strict hierarchy

In this section we define a class of finite, symmetric integral (hence, simple) non-representable relation algebras called $\mathfrak{N}^\beta_\alpha$, where $4 \leq \alpha \leq \beta < \omega$, each of which is generated by a single atom. We will prove that $\mathfrak{N}^\beta_\alpha \in \mathsf{Ra}\,\mathsf{CA}_\alpha$ in theorem 7.

Definition 4 Define $f : \omega \to \omega$ by:

$$\begin{aligned} f(0) &= 0, \\ f(n+1) &= 1 + n \cdot f(n), \quad \text{for } n < \omega. \end{aligned}$$

In theorem 9, we prove that $\mathfrak{N}^\beta_\alpha \notin \mathsf{SRa}\,\mathsf{CA}_{1+\alpha}$ whenever $\beta > 2f(\alpha - 1) + 1$, by an encoding of the standard proof of a special case of Ramsey's Theorem [3].[1]

3.1 The algebras $\mathfrak{N}^\beta_\alpha$

Let $4 \leq \alpha \leq \beta < \omega$, and write $\mathfrak{B} = \langle B, +, \cdot, \bar{\ } \rangle$ for a finite Boolean algebra having exactly $\alpha + \beta$ atoms, say

$$1', e, p_1, \ldots, p_{\alpha-2}, q_0, \ldots, q_{\beta-1}.$$

We will define a relation algebra $\mathfrak{N}^\beta_\alpha = \langle \mathfrak{B}, ;, \breve{\ }, 1' \rangle$, with Boolean part $\mathfrak{B}$. The converse $\breve{x}$ of an element $x \in B$ is defined in the trivial way, namely, $\breve{x} = x$, and the atom 1' is a distinguished

[1] Indeed, [3, § 5] shows that any value of β greater than the integer part of $(\alpha - 2)!\,2e$ would suffice for our result.

element. We write 0' ('diversity') for the boolean complement $\overline{1'}$ of 1'. For any atoms a, b, c of $\mathfrak{B}$, let $[a, b, c]$ be the closure of $\langle a, b, c\rangle$ under permutations:

$$[a,b,c] = \{\, \langle a,b,c\rangle\, , \langle a,c,b\rangle\, , \langle b,c,a\rangle\, , \langle b,a,c\rangle\, , \langle c,a,b\rangle\, , \langle c,b,a\rangle\, \}\, .$$

Such a set is called a **cycle**. The following cycles are *forbidden:*

$$\begin{array}{rll} [1', x, y] & \text{if } x \neq y & (1)\\ [q_i, q_j, q_k] & \text{for all } i, j, k < \beta & (2)\\ [r, s, t] & \text{if } r, s, t \in \{p_i, q_i\} \text{ for some } i \text{ with } 1 \leq i < \alpha - 1 & (3)\\ [r, s, t] & \text{for all } r, s, t \in \{e, q_0\} & (4)\\ [e, q_i, q_j] & \text{if } i, j < \beta \text{ and } |i - j| > 1. & (5) \end{array}$$

Let C be the set of all triples of atoms *except* for the forbidden cycles, listed above. C determines the binary operation ; as follows. For any elements $x, y \in B$, set

$$x\,;y = \sum\{c : [a, b, c] \subseteq C, a \leq x, b \leq y\}.$$

The algebras are symmetric because conversion is trivial. The definition of $[a, b, c]$ is restricted to this case.

The conditions on α and β are sufficient for our purposes, but not entirely necessary. The definitions apply to any α and β, but we assume $\alpha \leq \beta$ for simplicity and to avoid exploring fringe cases. The algebras constructed here were obtained from algebras in [10] by the addition of all the cycles of the form $[e, q_\kappa, q_\kappa]$.

We could extend these results to cover the case $\beta = \omega$, though the algebras are no longer finite in this case. In some ways the argument is simplified when $\beta = \omega$ in that we do not have to worry about the size of 'large' sets in the construction — they are simply infinite. However here we use some finite value for β greater than $2f(\alpha - 1) + 1$ (definition 4), and this makes $\mathfrak{N}_\alpha^\beta$ into a finite relation algebra.

THEOREM 5 *For* $4 \leq \alpha \leq \beta < \omega$, $\mathfrak{N}_\alpha^\beta$ *is a finite symmetric integral relation algebra with cardinality* $2^{\alpha+\beta}$.

PROOF:

The Boolean part of $\mathfrak{N}_\alpha^\beta$ is determined by the cardinality of its set of atoms, since there is, up to isomorphism, exactly one atomic Boolean algebra with $\alpha+\beta$ atoms. The cardinality of the Boolean part of $\mathfrak{N}_\alpha^\beta$ (and hence $\mathfrak{N}_\alpha^\beta$ as well) is $2^{\alpha+\beta}$.

The algebra $\mathfrak{N}_\alpha^\beta$ satisfies the **identity law** $1'\,;x = x = x\,;1'$ because of the inclusion of the identity cycles $[1', a, a]$ in C and the exclusion from C of all cycles of the form $[1', a, b]$ where a and b are *distinct* atoms.

$\mathfrak{N}_\alpha^\beta$ satisfies the **cycle law** for symmetric algebras, namely, for all $x, y, z \in \mathfrak{N}_\alpha^\beta$,

$$(x\,;y \cdot z = 0) \iff (x\,;z \cdot y = 0) \iff (z\,;y \cdot x = 0),$$

since *any* union of cycles, such as C, produces an algebra that satisfies the cycle law. What remains is the associative law; but this follows from the fact, proved below, that $\mathfrak{N}_\alpha^\beta$ has a 4-dimensional cylindrical basis. Still, it is instructive to try a direct proof. By the distributivity of ; over + and the fact that each variable in the associative law appears exactly once on each side, the associative law for ; need only be verified for atoms. If one of those atoms is 1', then associativity follows from the identity law. So one need only show associativity for three diversity atoms $a, b, c \leq 0$'. This follows readily from the observation that $a;(b;c) \geq 0$' and $(a;b);c \geq 0$'. To show, for example, that $a;(b;c) \geq 0$' it suffices to find, for every diversity atom $d \leq 0$', an atom f such that $[f, b, c]$ and $[a, f, d]$ are not forbidden cycles. One of e, p_1, or p_2 will work. Here we use $4 \leq \alpha \leq \beta$. ∎

3.2 $\mathfrak{N}^\beta_\alpha$ is an α-dimensional reduct

We refer to [10] for the notions in this section. In particular, $B_\alpha\mathfrak{N}^\beta_\alpha$ is the set of all α-dimensional basic matrices over $\mathfrak{N}^\beta_\alpha$.

LEMMA 6 *For $4 \leq \alpha \leq \beta < \omega$, $B_\alpha\mathfrak{N}^\beta_\alpha$ is an α-dimensional cylindrical basis for $\mathfrak{N}^\beta_\alpha$.*

PROOF:

By [10, theorem 7], $B_\alpha\mathfrak{A}$ is an α-dimensional cylindrical basis for $\mathfrak{A}$ if we have $0 \neq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$ whenever

$$x, y \in {}^{\alpha-2}\{e, p_1, p_2, \ldots, p_{\alpha-2}, q_0, q_1, q_2, \ldots, q_{\beta-1}\}.$$

We may assume $x \neq y$, since otherwise $1' \leq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$.

There are $\alpha - 2$ pairs of the form $\{x_\kappa, y_\kappa\}$, where $\kappa < \alpha - 2$. (Some of them may be singletons, but at least one of them is a pair, since $x \neq y$.) There are also $\alpha - 2$ pairs of the form $\{p_\gamma, q_\gamma\}$, with $1 \leq \gamma < \alpha - 1$. We consider two cases that are exhaustive but *not* mutually exclusive.

Suppose that for every $\kappa < \alpha - 2$ there is some γ such that $\{x_\kappa, y_\kappa\} \subseteq \{p_\gamma, q_\gamma\}$ and $1 \leq \gamma < \alpha - 1$. Then $[e, x_\kappa, y_\kappa]$ is not a forbidden cycle so in this case we have $e \leq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$.

Suppose that there is some γ such that $1 \leq \gamma < \alpha - 1$ and for every $\kappa < \alpha - 2$, $\{x_\kappa, y_\kappa\} \nsubseteq \{p_\gamma, q_\gamma\}$. Then $[p_\gamma, x_\kappa, y_\kappa]$ is not a forbidden cycle, hence $p_\gamma \leq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$. ∎

THEOREM 7 $\mathfrak{N}^\beta_\alpha \in \mathsf{Ra}\,\mathsf{CA}_\alpha$.

PROOF:

$B_\alpha\mathfrak{N}^\beta_\alpha$ is a cylindrical basis, so, by [10, theorem 10], we may apply the complex-algebra operator $\mathfrak{Ca}$ and obtain a cylindric algebra $\mathfrak{C} = \mathfrak{Ca}B_\alpha\mathfrak{N}^\beta_\alpha$ with $\mathfrak{C} \in \mathsf{CA}_\alpha$ and $\mathfrak{N}^\beta_\alpha \cong \mathfrak{Ra}\mathfrak{C}$. ∎

3.3 $\mathfrak{N}^\beta_\alpha$ is one-generated

THEOREM 8 *$\mathfrak{N}^\beta_\alpha$ is generated by e without using 1'.*

PROOF:

First, note that $\overline{e;e} = e + q_0$, so $q_0 = \overline{e;e} \cdot \overline{e}$. Next, $e;q_0 = q_1 + p_1 + \cdots + p_{\alpha-2}$ and $q_0;q_0 = 1' + p_1 + \cdots + p_{\alpha-2}$, so $1' = q_0;q_0 \cdot \overline{e;q_0}$ and $q_1 = e;q_0 \cdot \overline{q_0;q_0}$. This last equation is the first case of $q_{\kappa+1} = e;q_\kappa \cdot \overline{q_0;q_0} \cdot \overline{q}_\kappa \cdot \overline{q}_{\kappa-1}$, which holds for every $0 < \kappa < \beta$. Thus we get all the q_κ's. To get the p_κ's, notice that for every $\kappa \in \{1, \ldots, \alpha-2\}$ we have $q_\kappa;q_\kappa = \overline{p_\kappa} \cdot (1' + e + p_1 + \cdots + p_{\alpha-2}) = \overline{p_\kappa} \cdot (e + q_0;q_0)$, so $p_\kappa = \overline{q_\kappa;q_\kappa} \cdot (e + q_0;q_0)$. Thus all the atoms are generated by e. Every finite algebra is generated by its atoms (using only $+$), so $\mathfrak{N}^\beta_\alpha$ is generated by e. ■

3.4 $\mathfrak{N}^\beta_\alpha$ is not an $(\alpha+1)$-dimensional reduct

THEOREM 9 *If* $4 \leq \alpha < \omega$ *and* $2(f(\alpha-1)+1) \leq \beta < \omega$ *then* $\mathfrak{N}^\beta_\alpha \notin \mathsf{SRa\,CA}_{\alpha+1}$.

As can be checked, the condition implies that $\beta \geq \alpha$, as we prefer. A more succinct sufficient condition is $\beta \geq f(\alpha)$. This result will complete the section. Because the proof is rather technical, it may be worth sketching some of the ideas behind the argument. We saw that theorem 7 was easily proved using the following property: with only $\alpha - 2$ pairs of diversity atoms $x_\kappa, y_\kappa \in \mathfrak{N}^\beta_\alpha$, $(\kappa < \alpha - 2)$, it was impossible to force $\prod_{\kappa<\alpha-2} x_\kappa;y_\kappa = 0$. However, moving up one dimension, it is quite easy to find, for each $\kappa < (\alpha+1)-2$, a pair of diversity atoms $x_\kappa, y_\kappa \in \mathfrak{N}^\beta_\alpha$ with $\prod_{\kappa<\alpha-1} x_\kappa;y_\kappa = 0$. (For example, take $x_0 = q_0$, $y_0 = q_2$, and $x_\kappa = y_\kappa = p_\kappa$ for $1 \leq \kappa < \alpha - 1$, and consider the forbidden cycles.) Of course, this does not prove that $\mathfrak{N}^\beta_\alpha \notin \mathsf{S}\mathfrak{Ra}\mathsf{CA}_{\alpha+1}$; it merely shows that the proof of theorem 7 does not go through at this higher dimension.

In order to prove our result we will assume, for contradiction, that $\mathfrak{N}^\beta_\alpha \subseteq \mathfrak{Ra}\mathfrak{C}$ for some cylindric algebra $\mathfrak{C} \in \mathsf{CA}_{\alpha+1}$. We derive a contradiction by proving the existence of certain 'large' subsets S_t of $\mathfrak{C}$ $(1 \leq t \leq \alpha - 1)$. (The notion of 'large' depends on t and weakens as t rises, so that $S_{\alpha-1}$ merely has to have two distinct elements. This is formalized by requiring that $|S_t| > f(\alpha - t)$ — see definition 4.) Each $x \in S_t$ will determine a 'network' of 'colors' — atoms $x(i,t)$ of $\mathfrak{N}^\beta_\alpha$ such that $x \leq \mathsf{s}^0_i \mathsf{s}^1_t x(i,t)$. The chief properties that S_t will have is that (i) for each $x \in S_t$, $x(0,t)$ will be a q_γ for γ even and unique to x, and (ii) for each i with $1 \leq i < t$ there is a single color $p_{\kappa(i)}$ for some $1 \leq \kappa(i) < \alpha - 1$, such that the (i,t)th color $x(i,t)$ of any $x \in S_t$ is $p_{\kappa(i)}$. The color $p_{\kappa(i)}$ is independent of $x \in S_t$ (and even independent of t).

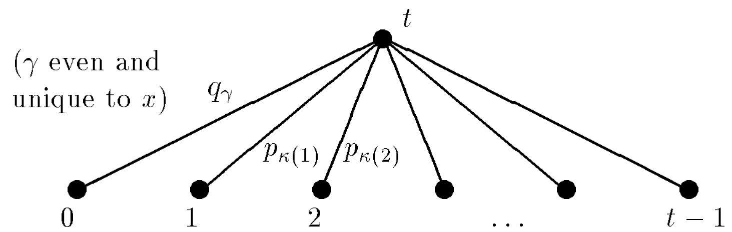

Figure 1: Colors of an element $x \in S_t$

To start $(t = 1)$, as there are no values of i to consider, S_1 is built in a straightforward way, as $\{q_\gamma : \gamma < \beta \text{ even}\}$. It is large, as there is a very large number $(f(\alpha-1))$ of atoms of

$\mathfrak{N}_\alpha^\beta$ of the form q_γ for $\gamma < \beta$, γ even. To build S_{t+1} from S_t, we begin by 'glueing together' any two elements x, z of S_t, after first 'moving' the tth coordinate of x to $t+1$ by taking $\mathsf{s}^t_{t+1}x$. See figure 2. By fixing z and varying x, a large number of different elements can be obtained

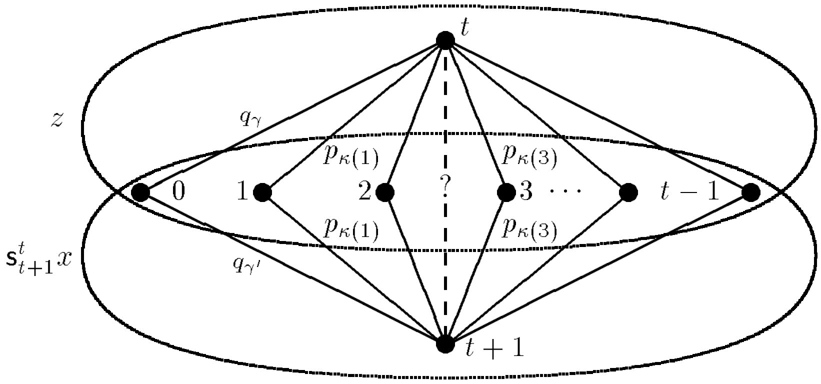

Figure 2: Amalgamation

in this way. Because they lie beneath elements obtained from S_t, their $(i, t+1)$th color is the same as the (i, t)th color for S_t, if $i < t$; but their $(t, t+1)$th colors are unpredictable and may not all be the same. So we use the pigeon-hole principle to pick out a large subset S_{t+1} of them in which the $(t, t+1)$th color is fixed; the notion of 'large' is chosen so that this is possible.

It follows from the definition of composition in $\mathfrak{N}_\alpha^\beta$, that for each t, the colors $p_{\kappa(i)}$ $(1 \leq i < t)$ are all distinct. This gives a contradiction when $t = \alpha - 1$, because if $x, z \in S_{\alpha-1}$ are distinct, we can 'glue them together' to form a non-zero element whose (i, α)th color is $p_{\kappa(i)}$, for all $1 \leq i < \alpha$. Again, the colors $p_{\kappa(i)}$ must all be distinct — but now there are not enough colors to permit this, as there are $\alpha - 1$ values of i, but only $\alpha - 2$ available colors.

Now let us prove our result formally. Here, recall that

$$\mathsf{s}^i_j x = \begin{cases} x & \text{if } i = j, \\ \mathsf{c}_i(x \cdot \mathsf{d}_{ij}) & \text{otherwise.} \end{cases}$$

We will freely use the facts about substitutions given below. If one thinks of the elements of an $(\alpha+1)$-dimensional cylindric algebra $\mathfrak{C}$ as $(\alpha+1)$-ary relations R on some set X, then for $x_0, \ldots, x_\alpha \in X$, $(x_0, \ldots, x_\alpha) \in \mathsf{s}^i_j R$ iff $(x_0, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_\alpha) \in R$. In this light, the facts seem intuitively natural. However, the reader should beware: $\mathfrak{C}$ may not be representable as $(\alpha+1)$-ary relations, and not all 'natural facts' about substitutions are valid in cylindric algebras. In fact, the study of the strength of various substitution identities is very extensive; see [5, 6].

Fact 10 The following equations are valid in $\mathsf{CA}_{\alpha+1}$, for all $i, j, k, l \leq \alpha$.

1. $\mathsf{s}^i_j(x \cdot y) = \mathsf{s}^i_j x \cdot \mathsf{s}^i_j y$, $\mathsf{s}^i_j(x + y) = \mathsf{s}^i_j x + \mathsf{s}^i_j y$, $\mathsf{s}^i_j 0 = 0$, and $\mathsf{s}^i_j 1 = 1$ [5, 1.5.3].
2. If $x = \mathsf{c}_i x$ then $\mathsf{s}^i_j x = x$ [5, 1.5.8(i)].

3. $\mathsf{c}_i\mathsf{s}^j_i x = \mathsf{c}_j\mathsf{s}^i_j x$ [5, 1.5.9(i)].

4. $\mathsf{s}^i_j\mathsf{c}_k x = \mathsf{c}_k\mathsf{s}^i_j x$ if $k \neq i, j$ [5, 1.5.8(ii)].

5. $\mathsf{s}^i_j\mathsf{s}^k_l x = \mathsf{s}^k_l\mathsf{s}^i_j x$ if $\{i,j\}\cap\{k,l\} = \emptyset$ [5, 1.5.10(iii)].

6. $\mathsf{s}^i_j x = \mathsf{s}^i_k\mathsf{s}^i_j x$ if $i \neq j$ [5, 1.5.10(i)].

7. $\mathsf{s}^i_k\mathsf{s}^j_i x = \mathsf{s}^j_k\mathsf{s}^i_j x$ [5, 1.5.10(vi)].

8. $\mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y$ and $\mathsf{c}_i(x\cdot\mathsf{c}_i y) = \mathsf{c}_i x\cdot\mathsf{c}_i y$. [5, theorem 1.2.6 and definition 1.1.1 C3].

Recall that we are assuming for contradiction that $\mathfrak{N}^\beta_\alpha \subseteq \mathfrak{Ra}\mathfrak{C}$ for some $\mathfrak{C}\in\mathsf{CA}_{\alpha+1}$.The domain of $\mathfrak{N}^\beta_\alpha$ is therefore a subset of the domain of $\mathfrak{C}$, and in fact, each $a\in\mathfrak{N}^\beta_\alpha$ is a two-dimensional element of $\mathfrak{C}$ ($a = \mathsf{c}_i a$ whenever $2\le i\le\alpha$). Of course, $\breve{a} = a$ in $\mathfrak{N}^\beta_\alpha$.

We need two observations on the way elements of $\mathfrak{N}^\beta_\alpha$ behave in $\mathfrak{C}$. First, $a = \mathsf{s}^i_j a$ whenever $a\in\mathfrak{N}^\beta_\alpha$, $i,j<\alpha+1$, and $i\ge 2$. Second, composition uses only the first three dimensions of $\mathfrak{C}$: $a\,;b = \mathsf{c}_2(\mathsf{s}^1_2 a\cdot\mathsf{s}^0_2 b)$ for $a,b\in\mathfrak{Ra}\mathfrak{C}$. These dimensions can be 'moved', using substitutions. In [6, 5.3.8(*)] it is proved that $a;b = \mathsf{c}_j(\mathsf{s}^1_j a\cdot\mathsf{s}^0_j b)$ for $j=3$, and the proof works for any $3\le j<\alpha+1$. So

$$a;b = \mathsf{c}_j(\mathsf{s}^1_j a\cdot\mathsf{s}^0_j b) \text{ for } 2\le j<\alpha+1. \tag{6}$$

In terms of the discussion at the start of section 3.4, this allows us to relate the (i,j)th color of an element of $\mathfrak{C}$, for varying i,j, to composition in $\mathfrak{N}^\beta_\alpha$, using the following lemma.

Lemma 11 *Let $a,b\in\mathfrak{N}^\beta_\alpha$ and $i<j<k<\alpha+1$. Then $\mathsf{s}^0_i\mathsf{s}^1_j a\cdot\mathsf{s}^0_j\mathsf{s}^1_k b\le\mathsf{s}^0_i\mathsf{s}^1_k(a\,;b)$.*

Proof:

We show that $\mathsf{s}^0_i\mathsf{s}^1_k(a;b) = \mathsf{c}_j(\mathsf{s}^0_i\mathsf{s}^1_j a\cdot\mathsf{s}^0_j\mathsf{s}^1_k b)$, from which the lemma follows immediately. It follows from $i<j$ that $1\le j$, so there are two cases, $1=j$ and $1<j$. First consider the case $1<j$.

$$\begin{aligned}
\mathsf{s}^0_i\mathsf{s}^1_k(a;b) &= \mathsf{s}^0_i\mathsf{s}^1_k\mathsf{c}_j(\mathsf{s}^1_j a\cdot\mathsf{s}^0_j b) && (6)\\
&= \mathsf{c}_j\mathsf{s}^0_i\mathsf{s}^1_k(\mathsf{s}^1_j a\cdot\mathsf{s}^0_j b) && \text{fact 10.4}\\
&= \mathsf{c}_j(\mathsf{s}^0_i\mathsf{s}^1_k\mathsf{s}^1_j a\cdot\mathsf{s}^0_i\mathsf{s}^1_k\mathsf{s}^0_j b) && \text{fact 10.1}\\
&= \mathsf{c}_j(\mathsf{s}^0_i\mathsf{s}^1_j a\cdot\mathsf{s}^0_j\mathsf{s}^1_k b) && \text{facts 10.5, 10.6}
\end{aligned}$$

In case $j=1$, we have $0=i$ and $k\ge 2$. Choose m so that $2\le m<\alpha+1$ and

$k \neq m$.

$$\begin{aligned}
\mathsf{s}^0_i\mathsf{s}^1_k(a\,;b) &= \mathsf{s}^1_k\mathsf{c}_m(\mathsf{s}^1_m a\cdot \mathsf{s}^0_m b) && i=0, (6)\\
&= \mathsf{c}_m\mathsf{s}^1_k(\mathsf{s}^1_m a\cdot \mathsf{s}^0_m b) && \text{fact 10.4}\\
&= \mathsf{c}_m(\mathsf{s}^1_k\mathsf{s}^1_m a\cdot \mathsf{s}^1_k\mathsf{s}^0_m b) && \text{fact 10.1}\\
&= \mathsf{c}_m(\mathsf{s}^1_m a\cdot \mathsf{s}^1_m\mathsf{s}^1_k\mathsf{s}^0_m b) && \text{fact 10.6}\\
&= \mathsf{c}_m\mathsf{s}^1_m(a\cdot \mathsf{s}^1_k\mathsf{s}^0_m b) && \text{fact 10.1}\\
&= \mathsf{c}_1\mathsf{s}^m_1(a\cdot \mathsf{s}^0_m\mathsf{s}^1_k b) && \text{facts 10.3, 10.5}\\
&= \mathsf{c}_1(a\cdot \mathsf{s}^m_1\mathsf{s}^0_m\mathsf{s}^1_k b) && \text{facts 10.1, 10.2}\\
&= \mathsf{c}_1(a\cdot \mathsf{s}^0_1\mathsf{s}^m_0\mathsf{s}^1_k b) && \text{fact 10.7}\\
&= \mathsf{c}_1(a\cdot \mathsf{s}^0_1\mathsf{s}^1_k b) && \text{facts 10.5, 10.2}\\
&= \mathsf{c}_j(\mathsf{s}^0_i\mathsf{s}^1_j a\cdot \mathsf{s}^0_j\mathsf{s}^1_k b) && i=0, j=1
\end{aligned}$$

■

We construct by induction on t, with $1 \leq t < \alpha$, a set $S_t \subseteq \mathfrak{Nr}_{t+1}\mathfrak{C} \setminus \{0\}$ with the following properties:

1. $|S_t| > f(\alpha - t)$.
2. If $x, y \in S_t$ then $\mathsf{c}_t x = \mathsf{c}_t y$.
3. If $x \in S_t$, there is an even number $k < \beta$ with $x \leq \mathsf{s}^1_t q_k$.
4. If $x, y \in S_t$, $k < \beta$, and $x, y \leq \mathsf{s}^1_t q_k$, then $x = y$.
5. There is a one-one map $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ such that for each i with $1 \leq i < t$ and each $x \in S_t$, we have $x \leq \mathsf{s}^0_i\mathsf{s}^1_t p_{\kappa(i)}$.

We let $S_1 = \{q_k : k < \beta,\ k \text{ is even}\}$. Note that $S_1 \subseteq \mathfrak{Nr}_2\mathfrak{C} \setminus \{0\}$. Let us check that S_1 has the required properties. We use fact 10 freely. Clearly, $q_k\,; 1 = 1$ for any k. That is, $1 = \mathsf{c}_2(\mathsf{s}^1_2 q_k \cdot \mathsf{s}^0_2 1) = \mathsf{c}_2\mathsf{s}^1_2 q_k = \mathsf{c}_1\mathsf{s}^2_1 q_k = \mathsf{c}_1 q_k$. It follows that property 2 above holds for S_1. We have $q_k = \mathsf{s}^1_1 q_k$ for each k, so property 3 holds. Property 4 holds because the q_k (for $k < \beta$) are pairwise disjoint. This gives property 1 because $|S_1| = |\{k < \beta : k \text{ is even}\}| = \lfloor \frac{\beta}{2} \rfloor \geq f(\alpha - 1) + 1 > f(\alpha - 1)$. Since $t = 1$, the last condition is vacuously true with $\kappa = \emptyset$.

Assume now that S_t is constructed, for some t with $1 \leq t < \alpha - 1$. Fix some $z \in S_t$. We show how to obtain S_{t+1}.

LEMMA 12 *Suppose that $x \in S_t$, $i < t$, and $a \in \mathfrak{N}^\beta_\alpha$ satisfy $x \leq \mathsf{s}^0_i\mathsf{s}^1_t a$. Then $\mathsf{s}^t_{t+1} x \leq \mathsf{s}^0_i\mathsf{s}^1_{t+1} a$.*

PROOF:

By fact 10, $\mathsf{s}^t_{t+1} x \leq \mathsf{s}^t_{t+1}\mathsf{s}^0_i\mathsf{s}^1_t a = \mathsf{s}^0_i\mathsf{s}^t_{t+1}\mathsf{s}^1_t a = \mathsf{s}^0_i\mathsf{s}^1_{t+1}\mathsf{s}^t_1 a = \mathsf{s}^0_i\mathsf{s}^1_{t+1} a$, the last equality holding trivially if $t = 1$, and because $a \in \mathfrak{Nr}_2\mathfrak{C}$ if $t \geq 2$. ■

For each atom $p \in \{p_1, \ldots, p_{\alpha-2}\}$ and $x \in S_t \setminus \{z\}$, define

$$\tau_x^p = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p).$$

Let $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha-2\}$ be the one-one function given by property 5 for S_t, and let $D = \{p_k : 1 \leq k < \alpha - 1, k \notin rng(\kappa)\}$. Note that $|D| = \alpha - 2 - |rng(\kappa)| = \alpha - 1 - t$.

Lemma 13 *If $x \in S_t \setminus \{z\}$, then $z \leq \sum\{\tau_x^p : p \in D\}$.*

Proof:

Fix $x \in S_t \setminus \{z\}$.

Claim. $z \cdot \mathsf{s}_{t+1}^t x \leq \sum_{p \in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p$.

Proof of Claim. Using fact 10.1, we have

$$z \cdot \mathsf{s}_{t+1}^t x \leq 1 = \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 1 = \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 \Big(\sum_{a \in At \mathfrak{N}_\alpha^\beta} a \Big) = \sum_{a \in At \mathfrak{N}_\alpha^\beta} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a.$$

So it suffices to show that if $a \in At\mathfrak{N}_\alpha^\beta$ and $(z \cdot \mathsf{s}_{t+1}^t x) \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$, then $a \notin \{1\text{'}, e, q_k : k < \beta\} \cup \{p_l : l \in rng(\kappa)\}$.

Therefore let $w = z \cdot \mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a$, for some atom a of $\mathfrak{N}_\alpha^\beta$, and assume that $w \neq 0$. Properties 3 and 4 for S_t give us distinct, even numbers $l, m < \beta$ such that $z \leq \mathsf{s}_t^1 q_l$ and $x \leq \mathsf{s}_t^1 q_m$. Observe that by lemma 12, $\mathsf{s}_{t+1}^t x \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m$, so that $w \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m$. So w satisfies:

$$w \leq \mathsf{s}_0^0 \mathsf{s}_t^1 q_l, \quad w \leq \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a, \quad w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m.$$

Hence, $w \leq \mathsf{s}_0^0 \mathsf{s}_t^1 q_l \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a$, and by lemma 11, $w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 (q_l \,; a)$. We now see that $w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 ((q_l \,; a) \cdot q_m)$, whence $(q_l \,; a) \cdot q_m \neq 0$ — i.e., $[a, q_l, q_m]$ is not a forbidden cycle.

Now, from the definition of composition in $\mathfrak{N}_\alpha^\beta$, $a = 1\text{'}$ is impossible, since as $l \neq m$, this would give a forbidden cycle of type (1), $a = q_k$ is impossible, for $k < \beta$, since this gives a forbidden cycle of type (2), and $a = e$ is impossible, since this gives a forbidden cycle of type (5) (using the fact that $|l - m| > 1$).

To rule out p_l for $l \in rng(\kappa)$ when $t > 1$, suppose that $i \in dom(\kappa)$ is such that $\kappa(i) = l$, so that $x, z \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_l$ by property 5. So $w \leq z \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_l$, and by lemma 12, $w \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_l$. As $w \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$, we have $\mathsf{s}_i^0 \mathsf{s}_t^1 p_l \cdot \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_l \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$. By lemma 11, this implies that $\mathsf{s}_i^0 \mathsf{s}_{t+1}^1 (p_l \,; a \cdot p_l) \neq 0$. Thus, $a \neq p_l$, since $[p_l, p_l, p_l]$ is a forbidden cycle of type (3). This proves the claim.

Now, to prove the lemma, observe that

$$\begin{array}{rcll}
z & = & z \cdot \mathsf{c}_t z \;=\; z \cdot \mathsf{c}_t x & \text{by property 2 of } S_t, \\
 & = & z \cdot \mathsf{c}_t \mathsf{s}_t^{t+1} x \;=\; z \cdot \mathsf{c}_{t+1} \mathsf{s}_{t+1}^t x & \text{by fact 10.3,} \\
 & = & \mathsf{c}_{t+1}(z \cdot \mathsf{s}_{t+1}^t x) & \text{since } z \in \mathfrak{Nr}_{t+1}\mathfrak{C}, \\
 & = & \mathsf{c}_{t+1}(z \cdot \mathsf{s}_{t+1}^t x \cdot \sum_{p \in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by the claim,} \\
 & \leq & \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \sum_{p \in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by fact 10.8,} \\
 & = & \mathsf{c}_{t+1} \sum_{p \in D}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by distributing,} \\
 & = & \sum_{p \in D} \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by fact 10.8} \\
 & = & \sum_{p \in D} \tau_x^p & \text{by the definition.}
\end{array}$$

■

Our path is now clear. Since the lemma holds for all x, by boolean manipulation we get

$$z \leq \prod_{x \in S_t \setminus \{z\}} \sum_{p \in D} \tau_x^p = \sum_{g: S_t \setminus \{z\} \to D} \prod_{x \in S_t \setminus \{z\}} \tau_x^{g(x)}.$$

So as $z \neq 0$, there is $g : S_t \setminus \{z\} \to D$ with $z \cdot \prod_{x \in S_t \setminus \{z\}} \tau_x^{g(x)} \neq 0$. Choose $X \subseteq S_t \setminus \{z\}$ with $|X| > f(\alpha - (t+1))$ on which g is constant. This is possible, for if $|g^{-1}(p)| \leq f(\alpha - t - 1)$ for all $p \in D$, then $|S_t \setminus \{z\}| \leq |D| \cdot f(\alpha - t - 1) = (\alpha - t - 1) \cdot f(\alpha - t - 1) = f(\alpha - t) - 1$, so that $|S_t| \leq f(\alpha - t)$, contradicting property 1 for S_t. Let p be the constant value of g on X, and define:

$$\begin{array}{rcll} \zeta & = & z \cdot \prod_{x \in X} \tau_x^p \neq 0, & \\ x' & = & \mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p \cdot \zeta, & \text{for each } x \in X, \\ S_{t+1} & = & \{x' : x \in X\}. & \end{array}$$

We check the required properties for S_{t+1}. Recall that $\tau_x^p = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p)$. To begin, note that $\mathsf{c}_i y = y$ for every $y \in S_{t+1}$ and $i \geq t+2$, so that $S_{t+1} \subseteq \mathfrak{Nr}_{t+2}\mathfrak{C}$. For property 2, observe that $\mathsf{c}_{t+1}\zeta = \zeta$, so for all $x \in X$,

$$\mathsf{c}_{t+1} x' = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p \cdot \zeta) = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) \cdot \zeta = \tau_x^p \cdot \zeta = \zeta.$$

This also shows that the $x' \in S_{t+1}$ are non-zero, since $\zeta \neq 0$.

Consider properties 1, 3, and 4. Let $x \in X$. By property 3 for S_t, $x \leq \mathsf{s}_t^1 q_k$ for some even k. By definition of x' and lemma 12, $x' \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_{t+1}^1 q_k$. This proves property 3. For property 4, note that if also $x' \leq \mathsf{s}_{t+1}^1 q_l$, then $x' \leq \mathsf{s}_{t+1}^1 q_k \cdot \mathsf{s}_{t+1}^1 q_l = \mathsf{s}_{t+1}^1(q_k \cdot q_l)$, and as $x' \neq 0$, we have $k = l$. It follows that if $x, y \in X$ and $x', y' \leq \mathsf{s}_{t+1}^1 q_l$ for some l, then $x, y \leq q_l$, so that by property 4 for S_t, $x = y$. Hence, $x' = y'$, giving property 4 for S_{t+1}. This also shows that $x' = y' \Rightarrow x = y$, so property 1 holds by assumption on $|X|$.

Finally, we arrange property 5. Recall that $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ is the existing (possibly empty) one-one function for S_t. Let $1 \leq i < t$ and $x \in X$. Then $x \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_{\kappa(i)}$, so by lemma 12, $x' \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_{\kappa(i)}$. Finally, for the case $i = t$, we have $x' \leq \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p$ by definition of x'. So we may define $\kappa' : \{1, \ldots, t\} \to \{1, \ldots, \alpha - 2\}$ suitable for S_{t+1} by: $\kappa'|_{\{1,\ldots,t-1\}} = \kappa$, and $p_{\kappa'(t)} = p$. Then κ' is one-one, because $p \in D$. This finishes the inductive construction.

In the case $t = \alpha - 1$, the construction yields a set S_t with $|S_t| > f(\alpha - t) = 1$. So there are distinct $z, x \in S_t$. In this case, the one-one map $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ is surjective, so that $D = \emptyset$. Applying lemma 13 to x now shows that $x \leq \sum\{\tau_x^p : p \in D\} = 0$. This is a contradiction, and proves theorem 9. ∎

4 Application to logic

We apply the main result to the logical systems of [16] and solve a problem raised by Tarski and Givant[2] by showing that the first displayed inequality on page 93 of [16] does hold for every $n > 4$. (It was known to fail for 3, hold for 4, and hold for infinitely many n.) We review here the concepts we need from [16]. $\mathcal{L}$ is a language of first order logic that has the

[2] [16, p. 93], [2, p. 208, problem 15], and [1, p. 735, problem 23]

equality symbol $\overset{\circ}{\mathbf{1}}$ and one additional binary predicate $\mathbf{E}$ (for "elementhood"). The set of variables of $\mathcal{L}$ is

$$\mathbf{\Upsilon} = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots\}.$$

The only connectives used in [16] are the implication and negation symbols. The only quantifier is the universal quantifier. The other connectives and the existential quantifier are introduced as abbreviations. Atomic formulas are formed as usual; $\mathbf{v}_0\mathbf{E}\mathbf{v}_1$ and $\mathbf{v}_2\overset{\circ}{\mathbf{1}}\mathbf{v}_3$ are examples of atomic formulas. Formulas are formed in the standard way from atomic formulas using propositional connectives and quantifiers. A sentence is a formula with no free variables. The set of formulas of $\mathcal{L}$ is $\mathbf{\Phi}$, and the set of sentences of $\mathcal{L}$ is $\mathbf{\Sigma}$. For any formula $X \in \mathbf{\Phi}$, the universal closure $[X]$ of X is formed by universally quantifying X with respect to all its free variables, where the order of quantification is determined by the indices of the variables. There are nine logical axioms for $\mathcal{L}$, namely, (AI)–(AIX), adopted from [15]. All these axioms are sentences, and *modus ponens* is the only rule of inference. (The rule of "generalization" is not needed.) If $\Psi \subseteq \mathbf{\Sigma}$ is any set of sentences, then $\Theta\eta\,\Psi$ is the $\mathcal{L}$-theory generated by Ψ, that is, the set of sentences deducible from Ψ using the axioms and rules of $\mathcal{L}$. By the completeness theorem, a sentence X is in $\Theta\eta\,\emptyset$ iff X is logically valid (true in all models).

Tarski's extension of $\mathcal{L}$, called $\mathcal{L}^+$, is obtained by adding operators on predicates and an another equality symbol, namely $=$ (which is why $\overset{\circ}{\mathbf{1}}$ is used earlier). There are two binary and two unary predicate operators, corresponding to the basic operations of relation algebras. For example, if P and Q are binary predicates of $\mathcal{L}^+$, then so are $P{+}Q$, $P{\odot}Q$, $\overline{P}$, and $\breve{P}$. The set of predicates so obtained, starting from $\mathbf{E}$ and $\overset{\circ}{\mathbf{1}}$, is $\mathbf{\Pi}$. The set of variables of $\mathcal{L}^+$ is $\mathbf{\Upsilon}$ (as it is for $\mathcal{L}$). The atomic formulas of $\mathcal{L}^+$ are formed using $\mathbf{\Pi}$ instead of $\{\mathbf{E}, \overset{\circ}{\mathbf{1}}\}$. For example, $\mathbf{v}_0\mathbf{E} \odot \mathbf{E}\mathbf{v}_1$ is an atomic formula of $\mathcal{L}^+$. Additionally, $\mathcal{L}^+$ contains another kind of atomic formula, namely the atomic sentence $P = Q$ for any two predicates $P, Q \in \mathbf{\Pi}$. The formulas and sentences of $\mathcal{L}^+$ are obtained from this larger set of atomic formulas in the same way as in $\mathcal{L}$. $\mathbf{\Phi}^+$ and $\mathbf{\Sigma}^+$ are the sets of formulas and sentences of $\mathcal{L}^+$, respectively. The logical axioms (AI)–(AIX) of $\mathcal{L}$ are included among the logical axioms of $\mathcal{L}^+$. However, some axioms are added that serve as definitions of the predicate operators, namely (DI)–(DV). For example, (DIII) and (DV) consist respectively of all sentences of the form

$$\forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0P{\odot}Q\,\mathbf{v}_1 \iff \exists_{\mathbf{v}_2}(\mathbf{v}_0P\mathbf{v}_2 \wedge \mathbf{v}_2Q\mathbf{v}_1)) \quad \text{(DIII)}$$
$$P = Q \iff \forall_{\mathbf{v}_0}\forall_{\mathbf{v}_1}(\mathbf{v}_0P\mathbf{v}_1 \iff \mathbf{v}_0Q\mathbf{v}_1) \quad \text{(DV)}$$

where $P, Q \in \mathbf{\Pi}$. $\Theta\eta\,\emptyset$ and $\Theta\eta^+\emptyset$ are the sets of logical consequences, in $\mathcal{L}$ and $\mathcal{L}^+$, respectively, of the empty set of sentences. $\mathcal{L}^+$ is an extension of $\mathcal{L}$. Indeed, an immediate consequence of the definitions is that $\mathbf{\Phi} \subseteq \mathbf{\Phi}^+$ and $\mathbf{\Sigma} \subseteq \mathbf{\Sigma}^+$. In [16] it is shown that $\mathcal{L}^+$ is actually equipollent with $\mathcal{L}$ in means of both expression and proof; there is a recursive function $\mathbf{G}$ that eliminates occurences of the predicate operators in accordance with (DI)–(DV), preserving provability and meaning; see Chapter 2 for details. For $3 \leq \alpha < \omega$, the formalisms $\mathcal{L}_\alpha$ and $\mathcal{L}^+_\alpha$ are obtained from $\mathcal{L}$ and $\mathcal{L}^+$ by restricting the set of variables to $\mathbf{\Upsilon}_\alpha = \{\mathbf{v}_0, \ldots, \mathbf{v}_{\alpha-1}\}$ (see [16, p. 91].) The sets of formulas and sentences of $\mathcal{L}_\alpha$ and $\mathcal{L}^+_\alpha$ are $\mathbf{\Phi}_\alpha$, $\mathbf{\Phi}^+_\alpha$, $\mathbf{\Sigma}_\alpha$, and $\mathbf{\Sigma}^+_\alpha$. The logical axioms of $\mathcal{L}_\alpha$ are (AI)-(AVIII) and (AIX′), restricted to $\mathbf{\Upsilon}_\alpha$. Axiom (AIX′) replaces (AIX) for various technical reasons discussed in [16, Section 3.7]; it makes it possible to respell bound variables by allowing a proof that "alphabetic variants" are equivalent. $\Theta\eta_\alpha\emptyset$ and $\Theta\eta^+_\alpha\emptyset$ are the sets of logical consequences of $\emptyset$ in $\mathcal{L}_\alpha$ and $\mathcal{L}^+_\alpha$.

Theorem 14 *If* $4 \leq \alpha < \omega$ *then*

$$\Theta\eta_{\,\alpha}\emptyset \cap \mathbf{\Sigma}_3 \neq \Theta\eta_{\,\alpha+1}\emptyset \cap \mathbf{\Sigma}_3$$

and

$$\Theta\eta^{\,+}_{\,\alpha}\emptyset \cap \mathbf{\Sigma}^+_3 \neq \Theta\eta^{\,+}_{\,\alpha+1}\emptyset \cap \mathbf{\Sigma}^+_3.$$

Proof:

The proof is based on the proof of [10, Theorem 25]. Suppose $\beta > \alpha$. Later we will choose $\beta \geq 2(f(\alpha-1)+1)$. We use the algebra $\mathfrak{N}^\beta_\alpha$ to form an algebraic realization of $\mathcal{L}^+_\alpha$. The set $B_\alpha\mathfrak{N}^\beta_\alpha$ is an α-dimensional cylindrical basis for $\mathfrak{N}^\beta_\alpha$, and it is easy to see that $B_\alpha\mathfrak{N}^\beta_\alpha$ is also "permutation closed" (meaning that $B_\alpha\mathfrak{N}^\beta_\alpha$ satisfies condition (C_3) in [10, Definition 16]), so $\mathfrak{M} = \langle \mathfrak{N}^\beta_\alpha, e, B_\alpha\mathfrak{N}^\beta_\alpha\rangle$ is an α-dimensional algebraic realization [10, Definition 16].

The set $\mathbf{\Pi}$ of predicates forms an algebra under the predicate operations, namely

$$\mathfrak{P} = \left\langle \mathbf{\Pi}, +, ^-, \odot, ^\smile, \mathring{\mathbf{1}} \right\rangle.$$

The algebra $\mathfrak{P}$ is an absolutely free algebra of similarity type $\langle 2,1,2,1,0\rangle$, freely generated by $\mathbf{E}$ (see [16, p. 238]). Treating the connectives and quantifiers as operations on formulas, we construct another absolutely free algebra, this time having the same similarity type as $(\alpha+1)$-dimensional cylindric algebras, namely,

$$\mathfrak{C} = \left\langle \mathbf{\Phi}^+_{\alpha+1}, \wedge, \vee, \neg, \exists_{\mathbf{v}_\kappa}, \mathbf{v}_\kappa \mathring{\mathbf{1}} \mathbf{v}_\lambda \right\rangle_{\kappa,\lambda<\alpha+1}.$$

Define a binary relation $\simeq^+_{\alpha+1}$ on $\mathbf{\Phi}^+_{\alpha+1}$ as follows. If $X, Y \in \mathbf{\Phi}^+_{\alpha+1}$, then

$$X \simeq^+_{\alpha+1} Y \quad \text{iff} \quad [\,X \Leftrightarrow Y\,] \in \Theta\eta^{\,+}_{\,\alpha+1}\emptyset.$$

It is easy (but somewhat tedious) to show, using axioms (AI)–(AVIII) and (AIX′), that $\simeq^+_{\alpha+1}$ is a congruence relation on $\mathfrak{C}$ and that $\mathfrak{C}/\simeq^+_{\alpha+1} \in \mathsf{CA}_{\alpha+1}$. Note that $\mathfrak{Ra}\left(\mathfrak{C}/\simeq^+_{\alpha+1}\right)$ is a relation algebra since $\alpha+1 > 4$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{Ra}\left(\mathfrak{C}/\simeq^+_{\alpha+1}\right)$ generated by $\mathbf{v}_0\mathbf{E}\mathbf{v}_1/\simeq^+_{\alpha+1}$. Let h be the homomorphism from $\mathfrak{P}$ onto $\mathfrak{A}$ determined by the condition that $h(\mathbf{E}) = \mathbf{v}_0\mathbf{E}\mathbf{v}_1/\simeq^+_{\alpha+1}$. With the help of (DI)–(DV), it is easy to prove that $h(P) = \mathbf{v}_0 P \mathbf{v}_1/\simeq^+_{\alpha+1}$ for every $P \in \mathbf{\Pi}$. (Alternatively, one can define the mapping h by this formula, and then use (DI)–(DV) to prove that h is a homomorphism.) The denotation function $D^{\mathfrak{M}}$ [10, Definition 17] maps predicates in $\mathbf{\Pi}$ to elements of $\mathfrak{N}^\beta_\alpha$ and formulas in $\mathbf{\Phi}^+_\alpha$ to subsets of $B_\alpha\mathfrak{N}^\beta_\alpha$. Restricted to $\mathbf{\Pi}$, $D^{\mathfrak{M}}$ is a homomorphism from $\mathfrak{P}$ onto $\mathfrak{N}^\beta_\alpha$. Since $\mathfrak{P}$ is absolutely freely generated by $\mathbf{E}$, this homomorphism is completely determined by the condition that $D^{\mathfrak{M}}(\mathbf{E}) = e$. It follows from [10, Theorem 22(iii)] that

$$D^{\mathfrak{M}}(X) = B_\alpha\mathfrak{N}^\beta_\alpha \text{ for every } X \in \Theta\eta^{\,+}_{\,\alpha}\emptyset. \tag{7}$$

$$\begin{tikzcd}
\mathfrak{P} \arrow[r, two heads, "h"] \arrow[dr, "D^{\mathfrak{M}}"'] & \mathfrak{A} \arrow[d, dashed, "g"] \\
& \mathfrak{N}^{\beta}_{\alpha}
\end{tikzcd}$$

Figure 3: The homomorphism g

Next we will derive a contradiction by assuming that $D^{\mathfrak{M}}(P) = D^{\mathfrak{M}}(Q)$ whenever $h(P) = h(Q)$ and $P, Q \in \mathbf{\Pi}$. This assumption implies that $D^{\mathfrak{M}}$ can be factored through h, yielding a homomorphism g from $\mathfrak{A}$ onto $\mathfrak{N}^{\beta}_{\alpha}$ where $g(h(P)) = D^{\mathfrak{M}}(P)$ for every $P \in \mathbf{\Pi}$. See figure 3. Since $\mathsf{SRa}\,\mathsf{CA}_{\alpha+1}$ is a variety, it contains its subalgebras and homomorphic images, including $\mathfrak{A}$ and $\mathfrak{N}^{\beta}_{\alpha}$. But $\mathfrak{N}^{\beta}_{\alpha}$ is not in $\mathsf{SRa}\,\mathsf{CA}_{\alpha+1}$ if we choose $\beta \geq 2(f(\alpha-1)+1)$. From this contradiction it follows that there must be $P, Q \in \mathbf{\Pi}$ with $h(P) = h(Q)$ and $D^{\mathfrak{M}}(P) \neq D^{\mathfrak{M}}(Q)$. From $h(P) = h(Q)$ we get $[\,\mathbf{v}_0 P \mathbf{v}_1 \iff \mathbf{v}_0 Q \mathbf{v}_1] \in \Theta\eta^{\,+}_{\,\alpha+1}\emptyset$, which implies $P = Q \in \Theta\eta^{\,+}_{\,\alpha+1}\emptyset$. On the other hand, $D^{\mathfrak{M}}(P) \neq D^{\mathfrak{M}}(Q)$ implies $D^{\mathfrak{M}}(P = Q) \neq B_{\alpha}\mathfrak{N}^{\beta}_{\alpha}$ by [10, lemma 21(x)], hence by (7), $P = Q \notin \Theta\eta^{\,+}_{\,\alpha}\emptyset$. Note that $P = Q$ is in $\mathbf{\Sigma}^{+}_{3}$ since it is a sentence with no variables at all. Thus the second inequality holds. To get a sentence witnessing the first inequality, just apply the elimination mapping $\mathbf{G}$ [16, 2.3(iii)] to $P = Q$, obtaining a sentence $\mathbf{G}(P = Q) \in \mathbf{\Sigma}_3$. By the equipollence results [16, §3.8] this gives $\mathbf{G}(P = Q) \notin \Theta\eta_{\,\alpha}\emptyset \cap \mathbf{\Sigma}_3$ and $\mathbf{G}(P = Q) \in \Theta\eta_{\,\alpha+1}\emptyset \cap \mathbf{\Sigma}_3$. ∎

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