GI12/4C59 – Homework #4 (Due 12am, November 18, 2004)

Aim: To get familiarity with channel coding and capacity. Presentation, clarity, and synthesis of exposition will be taken into account in the assessment of these exercises.

This document is available at `http://www.cs.ucl.ac.uk/staff/M.Pontil/courses/IT-homework4.pdf`

1. [**30pts**] Consider a 26-key typewriter

   (a) If pushing a key results in printing the associated letter, what is the capacity C in bits?

   (b) Now suppose that pushing a key results in printing that letter or the next (with equal probability)? Thus $A \to A$ or $B, \ldots, Z \to Z$ or A. What is the capacity?

   (c) What is the highest rate code with block length one that you can find that achieves *zero* probability of error for the channel in part (b) .

2. [**40 pts**] Let X be a random variable with values in the set $\mathcal{X} = \{0, 1, \ldots, n-1\}$, where n is a positive integer. Consider the discrete memoryless channel $Y = (X + Z)$ (modulo n), where $Z \in \{-1, 0, 1\}$.

   (a) Assume that Z is independent of X. Compute the channel capacity as a function of n and the entropy of Z. What is the maximizing $p(x)$?

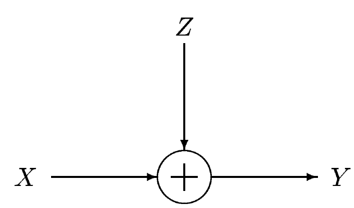
   (b) Under the same assumption in (a), derive an optimal code in the case that $n = 8$ and $p(Z = 0) = 0$, $p(Z = 1) = p(Z = -1) = \frac{1}{2}$.

   (c) Is the assumption that Z and X are independent necessary for the result derived in question 1 to hold? Explain your observation.

   (d) Compute the channel capacity when $n = 3$ and Z *depends* on X according to the following transition matrix

$$p(z|x) = \begin{bmatrix} & X=0 & X=1 & X=2 \\ Z=-1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ Z=0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ Z=1 & 0 & 0 & 0 \end{bmatrix}$$

3. [**30 pts**] Find the channel capacity of the following discrete memoryless channel:



   where $\Pr\{Z = 0\} = \Pr\{Z = a\} = \frac{1}{2}$. The alphabet for x is $\mathbf{X} = \{0, 1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

Solutions to Homework #4

1. (a) If the typewriter prints out whatever key is struck, then the output, Y, is the same as the input, X, and

$$C = \max I(X;Y) = \max H(X) = \log 26, \tag{1}$$

attained by a uniform distribution over the letters.

(b) In this case, the output is either equal to the input (with probability $\frac{1}{2}$) or equal to the next letter (with probability $\frac{1}{2}$). Hence $H(Y|X) = \log 2$ independent of the distribution of X, and hence

$$C = \max I(X;Y) = \max H(Y) - \log 2 = \log 26 - \log 2 = \log 13, \tag{2}$$

attained for a uniform distribution over the output, which in turn is attained by a uniform distribution on the input.

(c) A simple zero error block length one code is the one that uses every alternate letter, say A,C,E,...,W,Y. In this case, none of the codewords will be confused, since A will produce either A or B, C will produce C or D, etc. The rate of this code,

$$R = \frac{\log(\#\text{ codewords})}{\text{Block length}} = \frac{\log 13}{1} = \log 13. \tag{3}$$

In this case, we can achieve capacity with a simple code with zero error.

2. (a) $C = \log n - H(Z)$. The maximizing $p(x)$ is the uniform distribution.

(b) An optimal code of length 1 is $\{1, 2, 5, 6\}$.

(c) In general, it is not possible. In the special case that $n = 3$, it is possible.

(d) This channel is symmetric and therefore we have $C = \log 3 - H((\frac{1}{3}, \frac{2}{3}, 0)) = \frac{2}{3}$.

3.

$$Y = X + Z \qquad X \in \{0,1\}, \quad Z \in \{0, a\} \tag{4}$$

We have to distinguish various cases depending on the values of a.

$a = 0$ In this case, $Y = X$, and $\max I(X;Y) = \max H(X) = 1$. Hence the capacity is 1 bit per transmission.

$a \neq 0, \pm 1$ In this case, Y has four possible values $0, 1, a$ and $1 + a$. Knowing Y, we know the X which was sent, and hence $H(X|Y) = 0$. Hence $\max I(X;Y) = \max H(X) = 1$, achieved for an uniform distribution on the input X.

$a = 1$ In this case Y has three possible output values, $0, 1$ and 2, and the channel is identical to the binary erasure channel discussed in class, with $a = 1/2$. As derived in class, the capacity of this channel is $1 - a = 1/2$ bit per transmission.

$a = -1$ This is similar to the case when $a = 1$ and the capacity here is also $1/2$ bit per transmission.