GI12/4C59 – Homework #1 (Due 12am, October 14, 2004)

Aim: to get familiarity with basic probability and practising mathematical reasoning. Presentation, clarity, and synthesis of exposition will be taken into account in the assessment of these exercises.

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1. [**30 pts**] A fair die is thrown twice. What is the probability that

   (a) a four turns up exactly once?

   (b) both numbers are even?

   (c) the sum of the scores is 4.

   (d) the sum of the scores is divisible by 3.

2. [**30 pts**] Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $P(\omega_1) = P(\omega_2) = P(\omega_3) = \frac{1}{3}$. Let $X, Y, Z : \Omega \to \mathbb{R}$ be defined by

$$X(\omega_1) = 1, \;\; X(\omega_2) = 2, \;\; X(\omega_3) = 3,$$

$$Y(\omega_1) = 2, \;\; Y(\omega_2) = 3, \;\; Y(\omega_3) = 1,$$

$$Z(\omega_1) = 2, \;\; Z(\omega_2) = 2, \;\; Z(\omega_3) = 1.$$

   (a) Show that X and Y have the same probability mass functions.

   (b) Find the probability mass functions of $X + Y$, XY and X/Y.

   (c) Find the conditional mass functions $p_{Y|X}$ and $p_{Z|Y}$.

3. [**20 pts**] Let X and Y be independent discrete random variables, and let $g, h : \mathbb{R} \to \mathbb{R}$. Show that $g(X)$ and $h(Y)$ are also independent.

4. [**20 pts**] Sam wants to buy a car which costs N pounds. He starts with n pounds, where $0 < n < N$, and tries to win the remainder by the following gamble with his boss. He tosses a fair coin repeatedly; if it comes up head then the boss pays him one pound otherwise he pays one pound to the boss. He plays the game repeatedly until one of two events occurs: either he runs out of money or he wins enough to buy the car. What is the probability than he runs out of money?

GI12/4C59 – Solutions to Homework #1

1. (a) $P(4\ on\ first\ die) \times P(no\ 4\ on\ second\ die) \times 2 = \frac{1}{6} \times \frac{5}{6} \times 2 = \frac{5}{18}$
   (b) $P(first\ number\ is\ even) \times P(second\ number\ is\ even) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
   (c) $P(1,3) + P(3,1) + P(2,2) = \frac{3}{36} = \frac{1}{12}$
   (d) $P(sum\ is\ 3) + P(sum\ is\ 6) + P(sum\ is\ 9) + P(sum\ is\ 12) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{1}{3}$

2. (a) $p_X(i) = p_Y(i) = \frac{1}{3}$, $i = 1, 2, 3$.
   (b) $X + Y$ takes the values $3, 4, 5$ with equal probabilities $(\frac{1}{3})$. In fact, $X + Y(\omega_1) = 2$, $X + Y(\omega_2) = 5$, $X + Y(\omega_3) = 4$ and by hypothesis each of these have the same probability.
   Likewise, XY takes the equally probable values $2, 3, 6$ and X/Y takes the equally probable values $\frac{1}{2}, \frac{2}{3}, 3$.
   (c) $p(y|x) = \frac{p(x,y)}{p(x)}$ from which we get: $p(y|x) = 1$ if $(y, x) = (2, 1), (3, 2), (1, 3)$ and zero in all other cases. Similarly, $p(z|y) = 1$ if $(z, y) = (2, 2), (2, 3), (1, 1)$ and zero otherwise.

3. By hypothesis $p(x, y) = P(X = x, Y = y) = P(X = x)P(Y = y) = p(x)p(y)$. Let $\{c_1, \ldots, c_\ell\}$ and $\{d_1, \ldots, d_q\}$ be the value sets for the r.v. $g(X)$ and $h(Y)$, where $\ell \le |\mathcal{X}|$ and $q \le |\mathcal{Y}|$. Thus

$$p(g(X) = c) = \sum_{x:g(x)=c} p(x), \qquad p(h(Y) = d) = \sum_{y:h(y)=d} p(y)$$

   and we conclude that

$$P(g(X) = c, h(Y) = d) = \sum_{x:g(x)=c} \sum_{y:h(y)=d} p(x, y) = \sum_{x:g(x)=c} \sum_{y:h(y)=d} p(x)p(y) = \sum_{x:g(x)=c} p(x) \sum_{y:h(y)=d} p(y)$$

4. Let A denotes the event "Sam runs out of money", B the event "the first toss of the coin shows head" and $P_k(A)$ the probability of A when the initial sum is k pounds. We have

$$P_k(A) = P_k(A|B)P(B) + P_k(A|\bar{B})P(\bar{B}).$$

   Clearly if the toss is a head the capital becomes $k + 1$ pounds, so $P_k(A|B) = P_{k+1}(A)$. Likewise, $P_k(A|\bar{B}) = P_{k-1}(A)$. Writing $P_k(A) = p_k$, last equation becomes

$$p_k = \frac{1}{2}(p_{k+1} + p_{k-1}), \quad 0 < k < N$$

   This is a linear difference equation subject to the boundary conditions $p_0 = 1$, $p_N = 0$. We define $b_k = p_k - p_{k-1}$ to obtain $b_k = b_{k-1}$ and hence $b_k = b_1$ for all k. Thus

$$p_k = b_1 + p_{k-1} = 2b_1 + p_{k-2} = \cdots = kb_1 + p_0.$$

   The boundary conditions give $p_0 = 1$ and $b_1 = -1/N$, so we have

$$p_k = 1 - \frac{k}{N}.$$