Interlude: Solving systems of Equations

- Solving Ax = b
- What happens to x under Ax?
- The singular value decomposition
  - Rotation matrices
  - Singular matrices
  - Condition number
  - Null space
- Solving Ax = 0 under constraint $x^T x = 1$

Least Squares Solution **Ax=b**

How do we solve the system of equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

CASE I: The matrix **A** is square and invertible. In this case, we simply use the matrix inverse.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

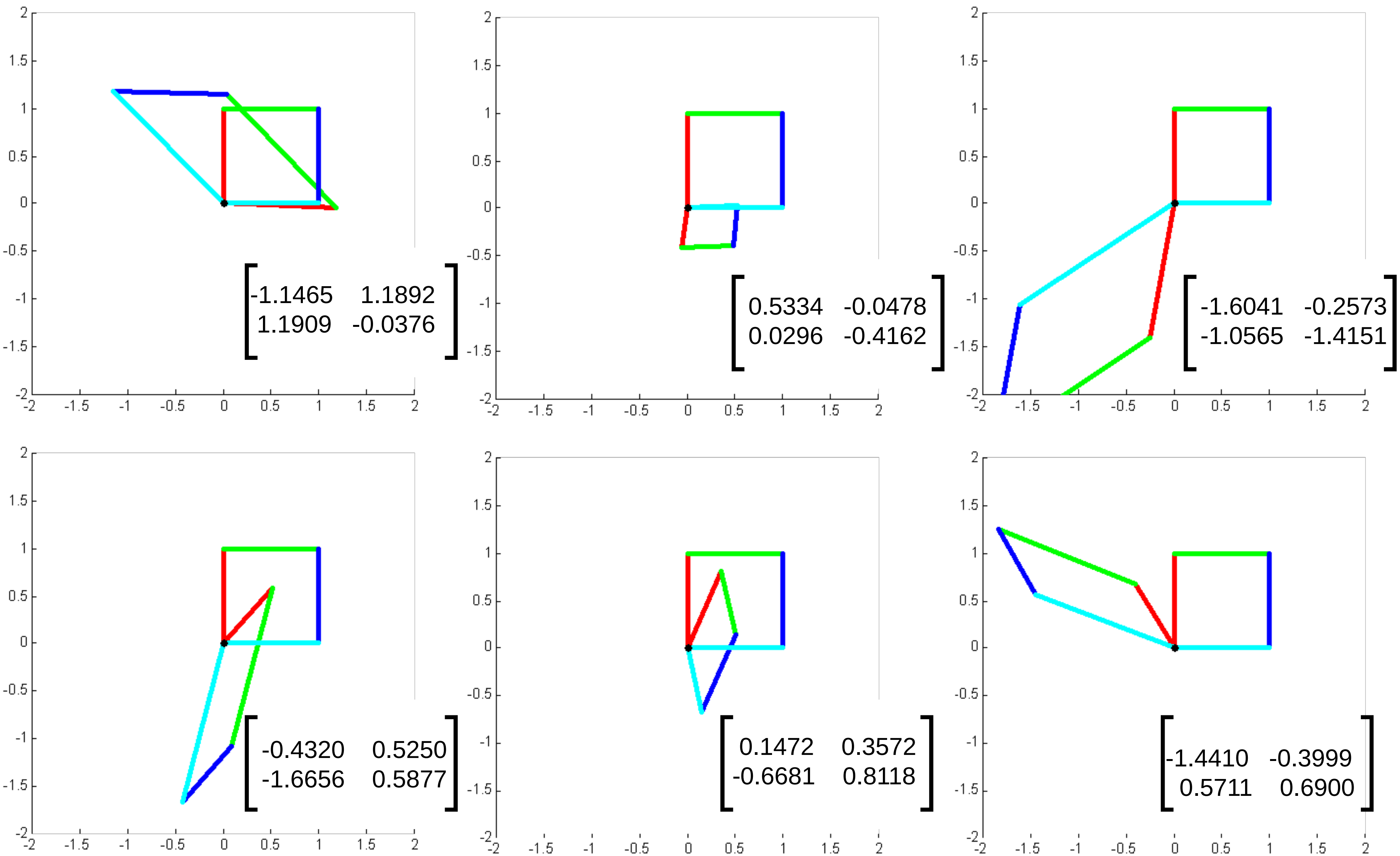
CASE II: The system of equations is *over-determined:* there are more constraints on the solution than there are unknowns. In general there is no unique solution. Define the objective function to minimize:

$$E = (\mathbf{A}\mathbf{x} - \mathbf{b})^T(\mathbf{A}\mathbf{x} - \mathbf{b})$$

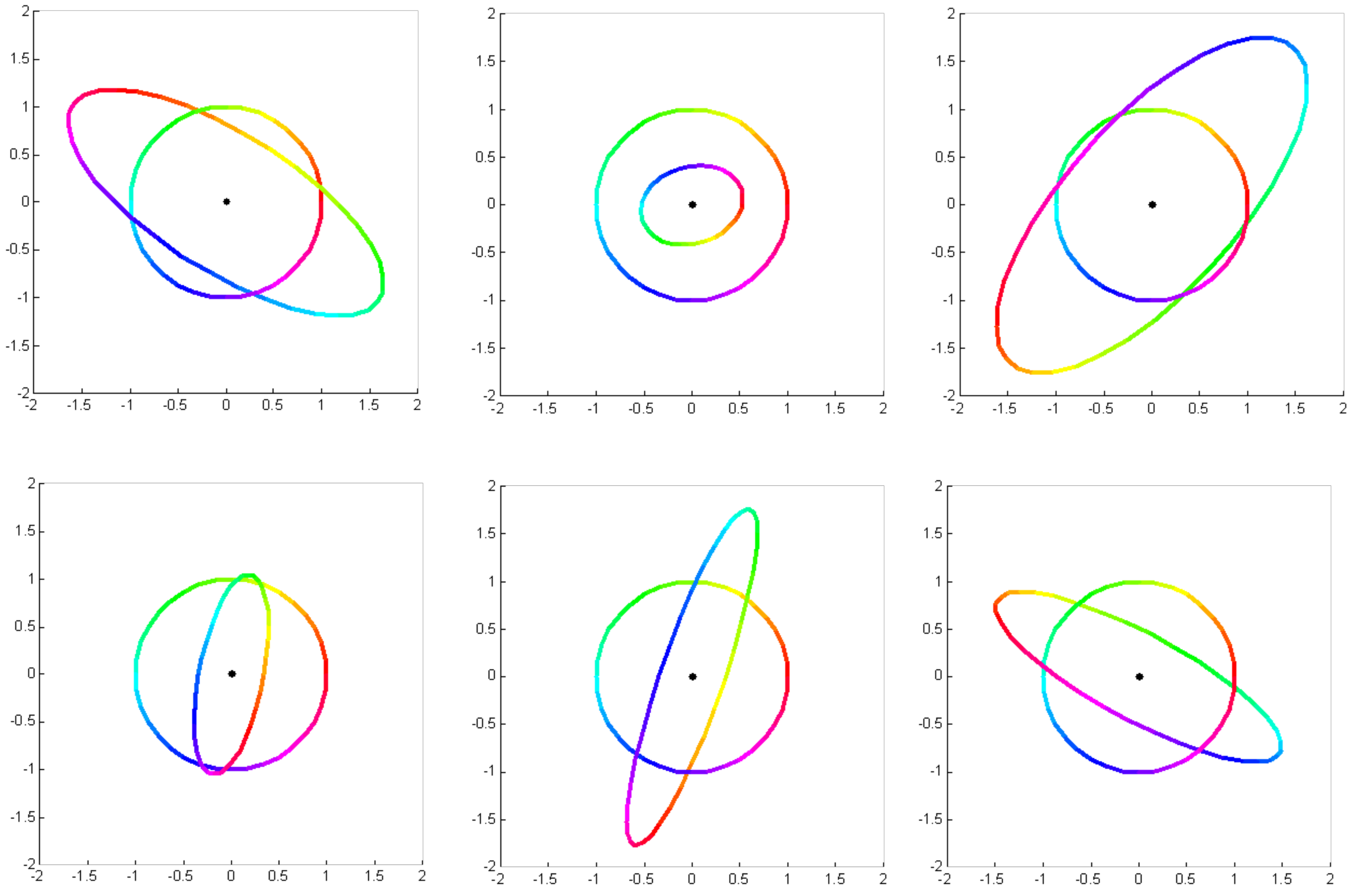
Now take the derivative w.r.t. ***x*** *and re-arrange to get the solution:*

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

What happens to x under Ax?
Unit Square



What happens to x under Ax?
Transformation of unit circle



Important Points to note:

- The origin is always mapped to itself
- Points on constant radial angle from origin all mapped to a different constant radial angle from origin
- Parallelism maintained
- There is a certain direction where magnitude increases the most and a certain direction where it increases least.
- What happens when the matrix is singular?

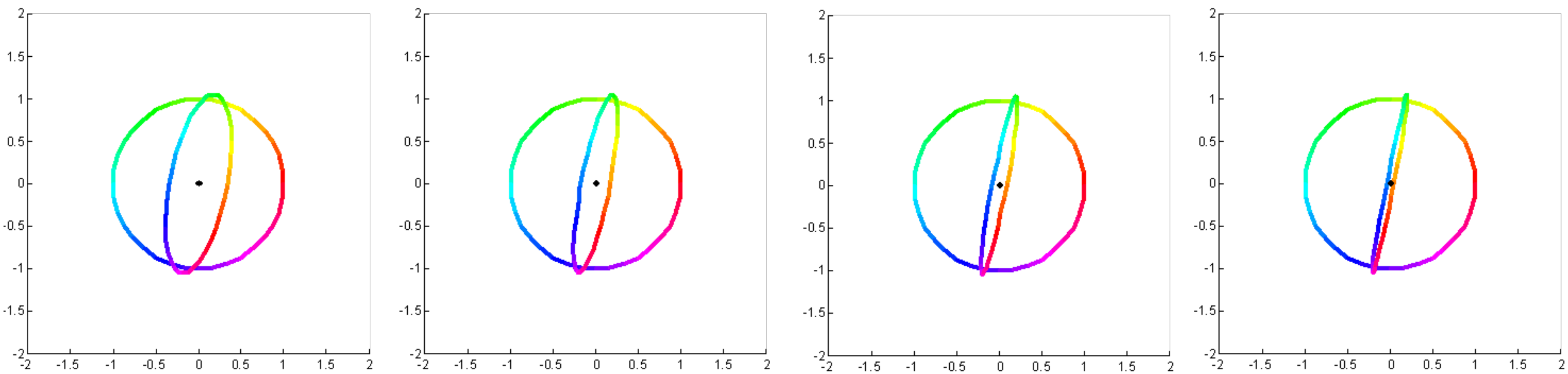
Condition Number & Determinant

cond =
0.2070
det =
0.3581

cond =
0.1114
det =
0.1791

cond =
0.0579
det =
0.0895

cond =
0.0296
det =
0.0448

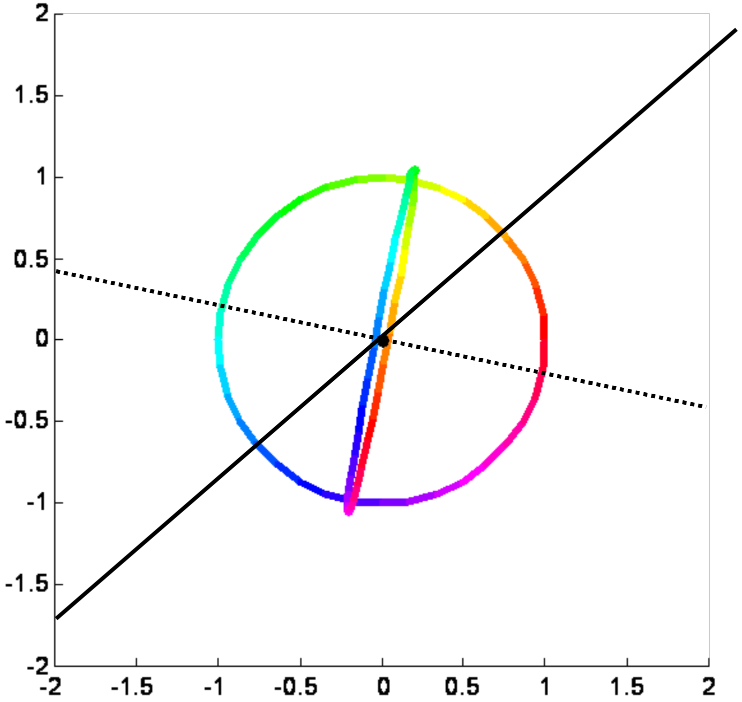


Condition number: ratio of largest:smallest expansion

Determinant: area of unit square under transformation

Both are measures of how “invertible” the matrix is.

Singular Matrices & the Null Space



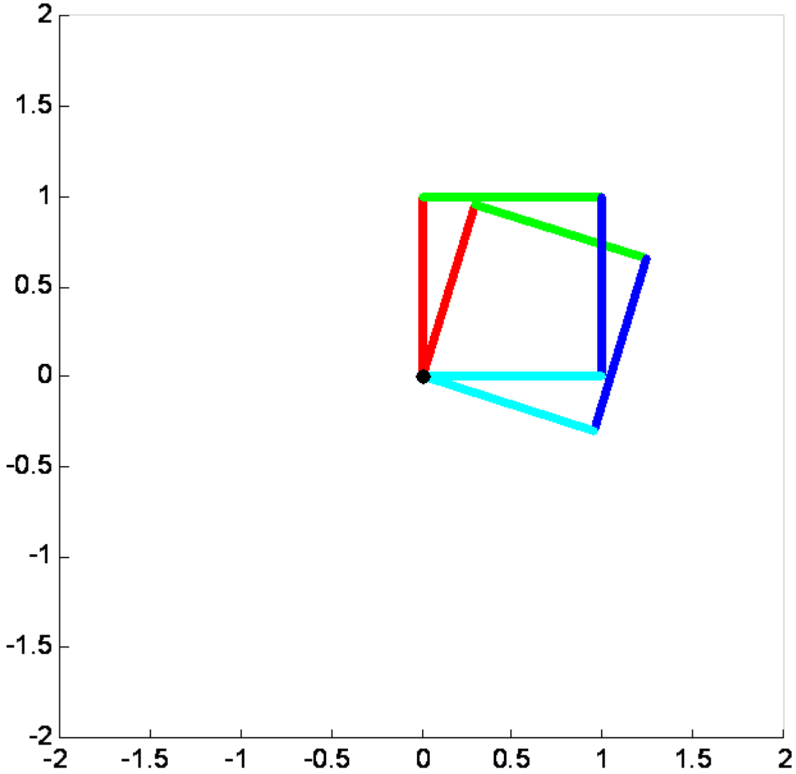
If we continued this process (not shown) then eventually the circle will be mapped to a line.

When this happens, multiple points in the plane are mapped to the same position: in other words the transformation can no longer be inverted (the matrix is singular)

There is a certain direction in which all of the points will be mapped to zero (black solid line). This is the null space of the matrix.

Rotation Matrices

- Rotation matrices do just what they say: they rotate the plane about the origin.



$$\begin{bmatrix} 0.9553 & 0.2955 \\ -0.2955 & 0.9553 \end{bmatrix}$$

Everything stays a constant distance from origin. No distortion.

Properties of a Rotation Matrix

$$R \doteq [r_1, r_2, r_3] \in \mathbb{R}^{3\times 3}$$

What are the properties of a 3D rotation matrix?
Which of the below are 3D rotation matrices?

$$\begin{bmatrix} 0.9003 & -0.0280 & -0.0871 \\ -0.5377 & 0.7826 & -0.9630 \\ 0.2137 & 0.5242 & 0.6428 \end{bmatrix}$$

$$\begin{bmatrix} -0.5841 & 0.4108 & -0.1381 \\ 0.4100 & -0.4376 & 0.8003 \\ -0.7005 & -0.7999 & 0.5835 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

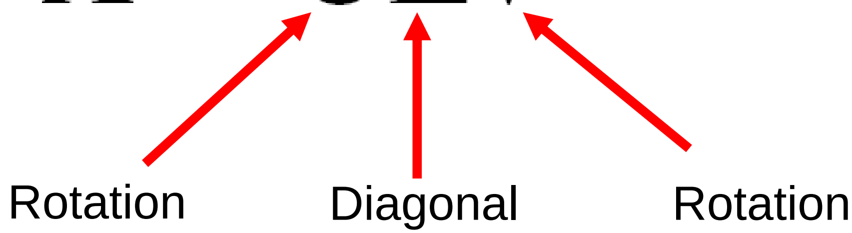
$$\begin{bmatrix} 0.4263 & 0.5673 & 0.7046 \\ -0.3538 & 0.8214 & -0.4474 \\ -0.8325 & -0.0586 & 0.5509 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Singular Value Decomposition

Every matrix can be decomposed into the product of three other matrices.

$$\mathbf{A} = \mathbf{U}\mathbf{L}\mathbf{V}^T$$



Works for matrices of every size, doesn’t have to be square, doesn’t have to be invertible.

SVDs in Matlab

- Example SVD 1

$$\mathbf{A} = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix}$$

```
[u l v] = svd(A)
```

$$\mathbf{A} = \mathbf{ULV}^T = \begin{bmatrix} -0.9193 & -0.3936 \\ 0.3936 & 0.9193 \end{bmatrix} \begin{bmatrix} 1.2212 & 0 \\ 0 & 0.2633 \end{bmatrix} \begin{bmatrix} -0.7897 & -0.6135 \\ -0.6135 & 0.7897 \end{bmatrix}$$

- Example SVD 2

$$\mathbf{A} = \begin{bmatrix} 0.8913 & 0.0185 \\ 0.7621 & 0.8214 \\ 0.4565 & 0.4447 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{ULV}^T = \begin{bmatrix} -0.5138 & 0.8568 & -0.0441 \\ -0.7437 & -0.4705 & -0.4749 \\ -0.4276 & -0.2112 & 0.8790 \end{bmatrix} \begin{bmatrix} 1.4647 & 0 \\ 0 & 0.5577 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8329 & -0.5534 \\ 0.5534 & -0.8329 \end{bmatrix}$$

- Note that there is an ordering of the elements of L by size.
- The diagonal elements of L are called the singular values

Using SVD to understand action of matrix

- Instead of thinking about what happens to x under:

$$\mathbf{A}\mathbf{x}$$

- Think instead about what happens under:

$$\mathbf{U}\mathbf{L}\mathbf{V}^T\mathbf{x}$$

- Example

$$\mathbf{A} = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix}$$

```
[u l v] = svd(A)
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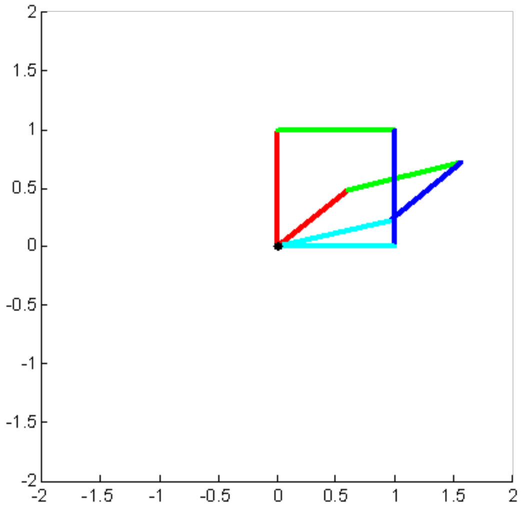
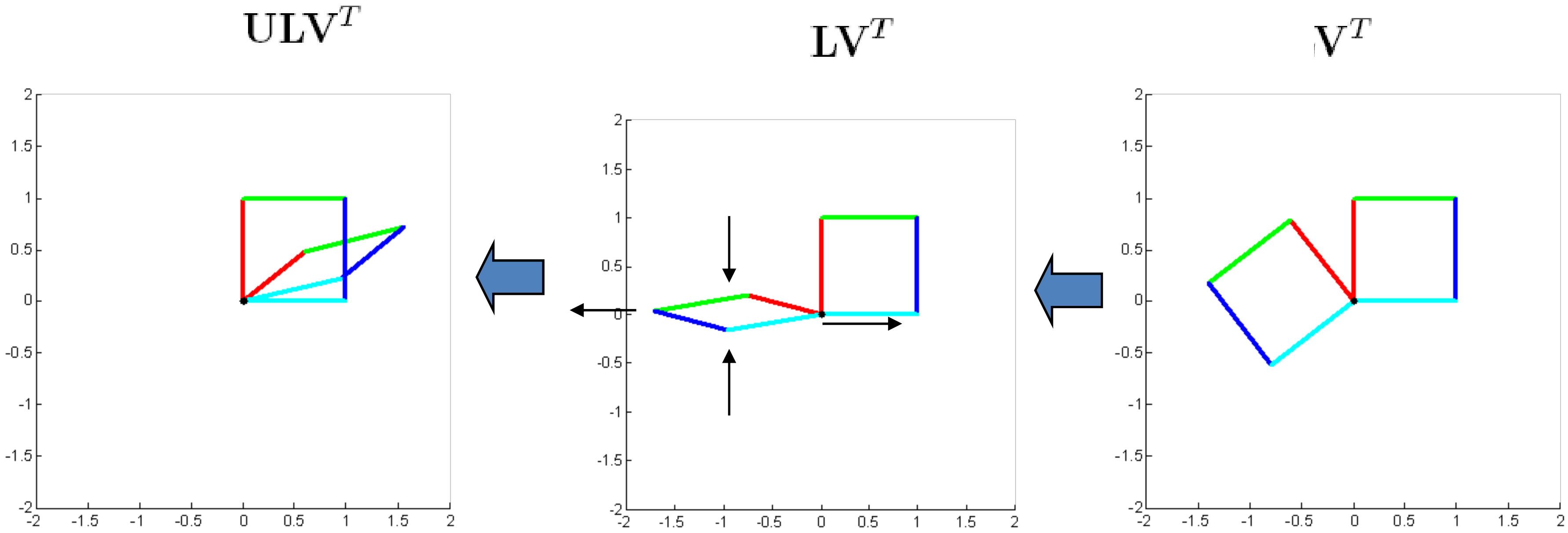


$$\mathbf{U}\mathbf{L}\mathbf{V}^T\mathbf{x}$$

- First rotate by $\mathbf{V}^T$
- Then scale along axes by **L**
- Then rotate again by **U**

$$\mathbf{A} = \mathbf{ULV}^T = \begin{bmatrix} -0.9193 & -0.3936 \\ 0.3936 & 0.9193 \end{bmatrix} \begin{bmatrix} 1.2212 & 0 \\ 0 & 0.2633 \end{bmatrix} \begin{bmatrix} -0.7897 & -0.6135 \\ -0.6135 & 0.7897 \end{bmatrix}$$



Then rotate again by **U** | Then scale along axes by **L** | First rotate by **VT**

Questions:

- What direction in the original image gets stretched the most?
- What direction gets stretched the least?

What can you learn from SVD?

1. Rotation Matrix – singular values all 1

$$\begin{bmatrix} 0.8437 & 0.5368 \\ -0.5368 & 0.8437 \end{bmatrix} = \begin{bmatrix} 0.8437 & 0.5368 \\ -0.5368 & 0.8437 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Ill-conditioned matrix – ratio of highest to lowest singular value is high

$$\begin{bmatrix} -0.4408 & 0.0799 \\ -1.6634 & 0.2997 \end{bmatrix} = \begin{bmatrix} -0.2562 & -0.9666 \\ -0.9666 & 0.2562 \end{bmatrix} \begin{bmatrix} 1.7845 & 0 \\ 0 & 0.0005 \end{bmatrix} \begin{bmatrix} 0.9841 & -0.1774 \\ -0.1774 & -0.9841 \end{bmatrix}$$

3. Singular (non-invertible) Matrix – at least one zero singular value

$$\begin{bmatrix} -0.0203 & -0.1341 \\ 0.1112 & 0.7354 \end{bmatrix} = \begin{bmatrix} -0.1794 & 0.9838 \\ 0.9838 & 0.1794 \end{bmatrix} \begin{bmatrix} 0.7560 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1496 & 0.9888 \\ 0.9888 & -0.1496 \end{bmatrix}$$

Nullspace and SVD

$$\begin{bmatrix} -0.3741 & 0.1702 & 0.1367 \\ 2.2175 & 1.0091 & -0.8103 \\ -0.1619 & -0.0737 & 0.0592 \end{bmatrix} =$$

$$\begin{bmatrix} -0.1659 & 0.9711 & 0.1716 \\ 0.9835 & -0.1503 & 0.1005 \\ -0.0718 & 0.1854 & 0.9800 \end{bmatrix} \begin{bmatrix} 2.6105 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8637 & 0.3930 & -0.3156 \\ 0.5008 & -0.5974 & 0.6264 \\ -0.0576 & 0.6990 & 0.7128 \end{bmatrix}$$

The vectors in V that correspond to zero singular values span the nullspace of the matrix. The nullspace is the space of directions which are projected to the origin by the action of the matrix.

Inverse of Matrix

$$\begin{aligned}\mathbf{A}^{-1} &= \left(\mathbf{U}\mathbf{L}\mathbf{V}^T\right)^{-1} \\ &= \left(\mathbf{V}^T\right)^{-1}(\mathbf{L})^{-1}(\mathbf{U})^{-1} \\ &= \mathbf{V}\mathbf{L}^{-1}\mathbf{U}^T\end{aligned}$$

Note that for the singular matrix we saw on the last slide, you cannot invert L so you cannot compute the inverse in this way either.

$$\begin{bmatrix}-0.0203 & -0.1341 \\ 0.1112 & 0.7354\end{bmatrix} = \begin{bmatrix}-0.1794 & 0.9838 \\ 0.9838 & 0.1794\end{bmatrix}\begin{bmatrix}0.7560 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}0.1496 & 0.9888 \\ 0.9888 & -0.1496\end{bmatrix}$$

Determinants of Matrix

The determinant of the matrix is the product of the singular values

$$\begin{bmatrix} 0.8437 & 0.5368 \\ -0.5368 & 0.8437 \end{bmatrix} = \begin{bmatrix} 0.8437 & 0.5368 \\ -0.5368 & 0.8437 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determinant = 1

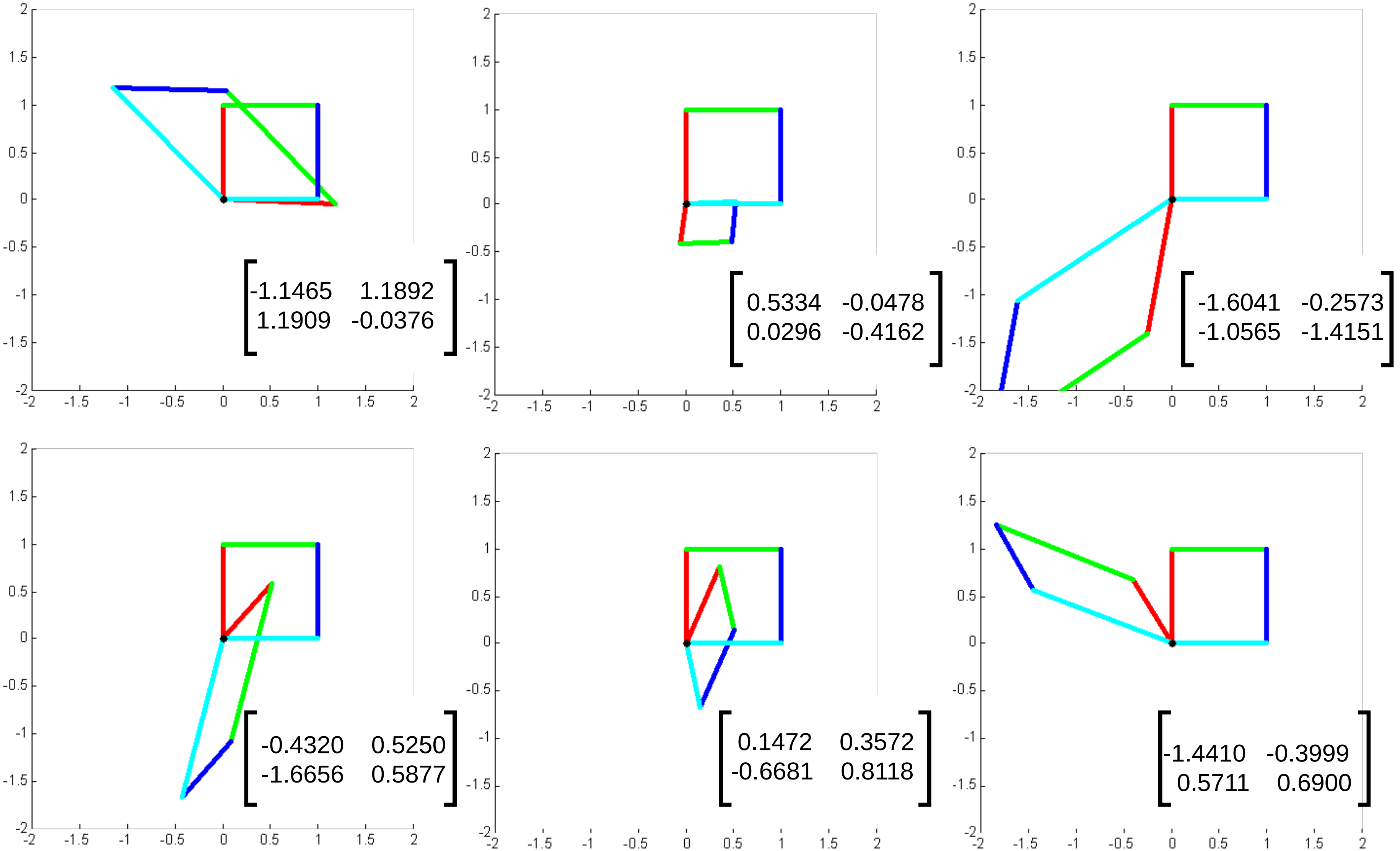
$$\begin{bmatrix} -0.4408 & 0.0799 \\ -1.6634 & 0.2997 \end{bmatrix} = \begin{bmatrix} -0.2562 & -0.9666 \\ -0.9666 & 0.2562 \end{bmatrix} \begin{bmatrix} 1.7845 & 0 \\ 0 & 0.0005 \end{bmatrix} \begin{bmatrix} 0.9841 & -0.1774 \\ -0.1774 & -0.9841 \end{bmatrix}$$

Determinant = 0.00089

$$\begin{bmatrix} -0.0203 & -0.1341 \\ 0.1112 & 0.7354 \end{bmatrix} = \begin{bmatrix} -0.1794 & 0.9838 \\ 0.9838 & 0.1794 \end{bmatrix} \begin{bmatrix} 0.7560 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1496 & 0.9888 \\ 0.9888 & -0.1496 \end{bmatrix}$$

Determinant =0

Determinant is area of unit area after transformation



Least Squares Solution **Ax=0**

How do we solve the system of equations:

$$\mathbf{Ax=0}$$

Don't want trivial solution **x**=**0**. Note that solution is ambiguous to scale. Define a similar objective function to before. E = |Ax|. The solution is found via the *singular value decomposition* (SVD) of A.

$$\mathbf{A} = \mathbf{ULV}^T$$

Rotation Diagonal Rotation

Least Squares Solution **Ax=0**

$$\mathbf{A} = \mathbf{U}\mathbf{L}\mathbf{V}^T$$

$$\mathbf{A} = \begin{bmatrix} 0.9501 & 0.4565 & 0.9218 \\ 0.2311 & 0.0185 & 0.7382 \\ 0.6068 & 0.8214 & 0.1763 \\ 0.4860 & 0.4447 & 0.4057 \\ 0.8913 & 0.6154 & 0.9355 \\ 0.7621 & 0.7919 & 0.9169 \end{bmatrix}$$

Select column of **v** associated with smallest singular value.

$$= \underbrace{\begin{bmatrix} -0.4971 & -0.2276 & -0.6777 & -0.1522 & -0.4570 & 0.0992 \\ -0.2210 & -0.6135 & 0.3985 & 0.0237 & -0.1868 & -0.6168 \\ -0.3162 & 0.7332 & 0.0576 & -0.2086 & -0.1544 & -0.5401 \\ -0.2760 & 0.1435 & -0.0062 & 0.9473 & -0.0750 & 0.0112 \\ -0.5148 & -0.1013 & -0.1445 & -0.0689 & 0.8337 & -0.0633 \\ -0.5127 & 0.0574 & 0.5981 & -0.1749 & -0.1779 & 0.5603 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 2.7754 & 0 & 0 \\ 0 & 0.7516 & 0 \\ 0 & 0 & 0.2481 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} -0.6122 & -0.4815 & -0.6272 \\ 0.1465 & 0.7104 & -0.6883 \\ -0.7770 & 0.5133 & 0.3644 \end{bmatrix}}_{\mathbf{V}^T}$$