Proof-theoretic Semantics for Intuitionistic Multiplicative Linear Logic (Extended Abstract)

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Proof-theoretic semantics (P-tS) [16] is an alternative approach to model-theoretical semantics, in which proof-theoretic validity replaces model-theoretic validity to provide meaning of logic. There are two major branches of P-tS: proof-theoretic validity (P-tV) in the Dummett-Prawitz tradition (see, for example, Schroeder-Heister [15]), and base-extension semantics (B-eS) in the sense of, for example, Sandqvist [13, 11, 12]. As shown by Gheorghiu and Pym [6], this B-eS captures precisely the declarative content of P-tV.

We focus on B-eS, in particular we take Sandqvist's B-eS for intuitionistic propositional logic (IPL) [12] as the point of departure. There, validity of formulas is defined inductively from a *base*, which contains proof rules that give the validity of atoms in terms of derivability.

Definition 1. A *base* $\mathscr{B}$ is a set of second-level inference rules of the form $(Q_1 \triangleright q_1, \ldots, Q_n \triangleright q_n) \Rightarrow q$, where each Q_i is a finite set of atoms, and q_i is an atom. *Derivability in a base* $\mathscr{B}$ is the least relation $\vdash_{\mathscr{B}}$ satisfying the following:

(Ref-IPL) $S, q \vdash_{\mathscr{B}} q$.

(App-IPL) If atomic rule $(Q_1 \triangleright q_1, \ldots, Q_n \triangleright q_n) \Rightarrow q$ is in $\mathscr{B}$, and $S, Q_i \vdash_{\mathscr{B}} q_i$ for all $i = 1, \ldots, n$, then $S \vdash_{\mathscr{B}} q$.

The semantics is in Figure 1, and is shown to be sound and complete. It features in a distinct treatment of disjunction ($\vee$) and absurdity ($\bot$), which takes the form of their elimination rules.

Our work is the first study on B-eS for a substructural logic. We give a sound and complete B-eS for intuitionistic multiplicative linear logic (IMLL), the $(\otimes, I, \multimap)$-fragment of intuitionistic linear logic. Towards this, we start with an alternative definition of B-eS for IPL which is closer to IMLL.

Revisit B-eS for IPL. The defining clauses for $\vee$ and $\bot$ in the B-eS for IPL are distinct from its possible-world semantics (see Beth [2], Kripke [10]): while in the latter $\vee$ and $\bot$ are defined in terms of their introduction rules,

(At-IPL)	$\Vdash_{\mathscr{B}} p$	iff	$\vdash_{\mathscr{B}} p$
$(\to)$	$\Vdash_{\mathscr{B}} \varphi \to \psi$	iff	$\varphi \Vdash_{\mathscr{B}} \psi$
$(\wedge)$	$\Vdash_{\mathscr{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathscr{B}} \varphi$ and $\Vdash_{\mathscr{B}} \psi$
$(\vee)$	$\Vdash_{\mathscr{B}} \varphi \vee \psi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$ and any $p \in \mathbb{A}$, if $\varphi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$, then $\Vdash_{\mathscr{C}} p$
$(\bot)$	$\Vdash_{\mathscr{B}} \bot$	iff	$\Vdash_{\mathscr{B}} p$ for any $p \in \mathbb{A}$
(Inf-IPL)	$\Gamma \Vdash_{\mathscr{B}} \varphi$	iff	for any $\mathscr{C}$ such that $\mathscr{B} \subseteq \mathscr{C}$, if $\Vdash_{\mathscr{C}} \gamma$ for any $\gamma \in \Gamma$, then $\Vdash_{\mathscr{C}} \varphi$

Figure 1: Sandqvist's Support in a Base

in the B-eS they are defined in terms of their elimination rules. This can be explained by the principle of definitional reflection (DR) (see Hallnäs [8, 9] and Schroeder-Heister [14]):

> whatever follows from all the premisses of an assertion also follows from the assertion itself

Taking the perspective that the introduction rules are definitions, DR provides an answer for the way in which the elimination rules follow. In particular, DR gives the following *generalized* elimination rule for $\wedge$:

$$\frac{\varphi \wedge \psi \qquad \begin{array}{c}[\varphi, \psi] \\ \chi\end{array}}{\chi}$$

Accordingly, we may modify the B-eS for IPL by replacing $(\wedge)$ with the following:

$(\wedge')$ $\Vdash_{\mathscr{B}} \varphi \wedge \psi$ iff for any $\mathscr{C} \supseteq \mathscr{B}$ and any $p \in \mathbb{A}$, if $\varphi, \psi \Vdash_{\mathscr{C}} p$, then $\Vdash_{\mathscr{C}} p$

This clause motivates the later definition of the multiplicative conjunction $(\otimes)$ in IMLL. The resulting semantics is shown to be also sound and complete.

B-eS for IMLL. We turn to the B-eS for IMLL. There are two distinguishing features of our approach. First, following the principle of DR, all the connectives are defined in terms of their elimination rules. Second, the strict context management in IMLL requires extra care of multiplicity. The latter means that in IMLL, sequents are of the form $\Gamma \triangleright \varphi$ where the context Γ is a *multiset* of formulas, and the deduction rules are multiplicity sensitive, using multiset union denoted as $\fatcomma$. The yields the celebrated 'resource interpretations' of linear logic — see Girard [7], Allwein and Dunn [1], and Coumans et al. [5]. The leading example of which is, perhaps, the number-of-uses reading in which a proof of a formula $\varphi \multimap \psi$ determines a function that *uses* its arguments exactly ones. This reading is, however, entirely proof-theoretic and is not at all reflected in

(At)	$\Vdash^{P}_{\mathscr{B}} p$	iff	$P \vdash_{\mathscr{B}} p$
$(\otimes)$	$\Vdash^{P}_{\mathscr{B}} \varphi \otimes \psi$	iff	for every $\mathscr{X} \supseteq \mathscr{B}$, atomic multiset U, and atom p, if $\varphi \mathbin{,} \psi \Vdash^{U}_{\mathscr{X}} p$, then $\Vdash^{P \mathbin{,} U}_{\mathscr{X}} p$
(I)	$\Vdash^{P}_{\mathscr{B}} I$	iff	for every base $\mathscr{X} \supseteq \mathscr{B}$, atomic multiset U, and atom p, if $\Vdash^{U}_{\mathscr{X}} p$, then $\Vdash^{P \mathbin{,} U}_{\mathscr{X}} p$
$(\multimap)$	$\Vdash^{P}_{\mathscr{B}} \varphi \multimap \psi$	iff	$\varphi \Vdash^{P}_{\mathscr{B}} \psi$
$(\mathbin{,})$	$\Vdash^{P}_{\mathscr{B}} \Gamma \mathbin{,} \Delta$	iff	there are U and V such that $P = U \mathbin{,} V$ and $\Vdash^{U}_{\mathscr{B}} \Gamma$ and $\Vdash^{V}_{\mathscr{B}} \Delta$
(Inf)	$\Gamma \Vdash^{P}_{\mathscr{B}} \varphi$	iff	for any $\mathscr{X} \supseteq \mathscr{B}$ and any U, if $\Vdash^{U}_{\mathscr{X}} \Gamma$, then $\Vdash^{P \mathbin{,} U}_{\mathscr{X}} \varphi$

Figure 2: Base-extension Semantics for IMLL

the truth-functional semantics of IMLL — see Girard [7] — though it is reflected in the categorical semantics — see Seely [17] and Biermann [4, 3].

The multiplicity is reflected at two points in the B-eS: first in the derivability relation between atoms, second in the supporting relation between formulas. For the former, the rules in the bases for IMLL are multiset counterpart of that for IPL in Definition 1.

Definition 2. A *base* is a (possibly infinite) set of second-level rules of the form $(P_1 \triangleright p_1, \ldots, P_n \triangleright p_n) \Rightarrow p$, where each P_i is a finite multiset of atoms, and p_i is an atom. *Derivability in a base* $\mathscr{B}$ is the least relation $\vdash_{\mathscr{B}}$ satisfying the following:

(Ref) $p \vdash_{\mathscr{B}} p$

(App) If $S_i \mathbin{,} P_i \vdash_{\mathscr{B}} p_i$ for $i = 1, \ldots, n$ and $(P_1 \triangleright p_1, \ldots, P_n \triangleright p_n) \Rightarrow p \in \mathscr{B}$, then $S_1 \mathbin{,} \ldots \mathbin{,} S_n \vdash_{\mathscr{B}} p$.

Note the differences between Definition 1 and Definition 2: first, in (Ref), no redundant atoms are allowed to appear, while in (Ref-IPL) they may; second, in (App), the multisets S_1,...,S_n are collected together as a multiset, while in (App-IPL), there is one set. These differences reflect the fact in the multiplicative setting that 'resources' can neither be discharged nor shared.

Second, according to the aforementioned resource reading of linear logic, we expect that in (Inf-IPL), φ being supported in a base $\mathscr{B}$ relative to some multiset of formulas Γ means that the resources garnered by Γ suffice to produce φ. We may express this by enriching the notion of support with multisets of resources P and U combined with multiset union — denoted $\mathbin{,}$. Then, that the resources garnered by Γ are given to φ is captured by the following behaviour:

$$\Gamma \Vdash^{P}_{\mathscr{B}} \varphi \quad \text{iff} \quad \text{for any } \mathscr{X} \supseteq \mathscr{B} \text{ and any } U, \text{ if } \Vdash^{U}_{\mathscr{X}} \Gamma, \text{ then } \Vdash^{P \mathbin{,} U}_{\mathscr{X}} \varphi$$

where

$$\Vdash^{U}_{\mathscr{B}} \Gamma_1 \mathbin{,} \Gamma_2 \quad \text{iff} \quad \text{there are } U_1 \text{ and } U_2 \text{ s.t. } U = U_1 \mathbin{,} U_2 \text{ and } \Vdash^{U_1}_{\mathscr{X}} \Gamma_1 \text{ and } \Vdash^{U_1}_{\mathscr{X}} \Gamma_2$$

The resulting semantics is defined in Figure 2. In particular, we read (Inf) as saying that, $\Gamma \Vdash^{P}_{\mathscr{B}} \varphi$ means, for arbitrary extension $\mathscr{X}$ of $\mathscr{B}$, if Γ is supported in $\mathscr{X}$ with some resource U (i.e. $\Vdash^{U}_{\mathscr{X}} \Gamma$), then φ is also supported in $\mathscr{B}$ with the combined resource U and S.

The resulting validity relation $\Vdash$ — defined as that $\Vdash^{\varnothing}_{\mathscr{B}}$ holds for every base $\mathscr{B}$ with the empty multiset resource $\varnothing$ — is sound and complete for IMLL.

Theorem 1. $\Gamma \vdash \varphi$ *iff* $\Gamma \Vdash \varphi$.

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