Maths revision for algorithmic analysis

The course will assume familiarity with a small range of standard mathematical functions, in particular **powers** (as in **polynomials**), **exponentials**, and **logarithms.**

Power functions

Suppose x is a real number and that for simplicity k is an integer. $\mathbf{x^k}$ or **'x to the power of k'** is a shorthand notation denoting the product

$$\underbrace{x \times x \times \ \dots \ \times x}_{\text{k times}}$$

k is referred to here as the **exponent**. For example, $2^3=8$, $10^2=100$, $3.1^2=9.61$.

If we multiply x^k by x^l we have (k+l) occurrences of x, so

$$x^k \times x^l = x^{k+l}$$

On **multiplication, exponents are added.**

Negative exponents indicate that divisions by x are taking place:

$$x^{-k} = \frac{1}{x^k} = \frac{1}{x \times x \times \dots \times x}$$

For example $2^{-2} = 1/2^2 = 0.25$, $10^{-3} = 1/10^3 = 0.001$.

If we divide x^k by x^l we have (k-l) occurrences of x, so

$$\frac{x^k}{x^l} = x^{k-l}$$

On **division, exponents are subtracted.**

(The addition/subtraction rules still apply if k is not an integer but in almost all the cases encountered in this course it will be.)

We can therefore also give a meaning to x^0, since

$$x^0 = x^{k-k} = \frac{x \times x \times \ldots \times x}{x \times x \times \ldots \times x} = 1$$

the k occurrences of x in the numerator cancel the k x's in the denominator

Polynomials

Any function which takes the form of a sum of powers (with constant multipliers) is known as a polynomial. For example $3x^2 + 2x + 1$ is a *quadratic polynomial* (fastest-growing part is proportional to x^2), $2.45x^3 - 5.1x^2 + 0.56x + 1.3$ is a *cubic polynomial* (fastest-growing part is proportional to x^3).

Growth of power functions

- If $x > y > 0$, and $k > 0$, then $x^k > y^k$, ie x^k is an increasing function of x.
- If $x > 1$, and $k > l > 0$, then $x^k > x^l$, ie the power for a fixed non-fractional x is an increasing function of the exponent.

Exponential functions

In an exponential function the variable x is the exponent, the power to which some other (fixed) number is raised. For example 2^x, $(½)^x$ are exponential functions.

Exponential functions behave very differently depending on whether the number being raised to the power x is in magnitude smaller or larger than 1.0 -- if it's smaller, like ½, the function tends to zero as x increases but if it's larger, like 2, the function increases *very* rapidly. In the algorithmics context where the functions are time-demands it will be the latter which is the case -- exponential growth of a time-demand is always bad news.

Logarithms

Suppose y is equal to the exponential function of x with **base** (the number being raised to the power) a:

$$y = a^x$$

An equivalent way to express this relationship is to say that **x is the logarithm (log) to base a of y**:

$$x = \log_a y$$

Common bases for logarithms are 2 ($\log_2$ is sometimes written 'lg', 10 ($\log_{10}$ is often just written as 'log') and e=2.7182818... ($\log_e$ is usually written 'ln').

($y = e^x$ has the useful property that $dy/dx = e^x$, ie its derivative is the same as the function itself -- but these derivative properties of exponential functions won't be needed in this course).

For example:
Base 2: $\log_2(8) = 3$ ($2^3 = 8$)
Base 10: $\log_{10}(0.0001) = -4$ ($10^{-4} = 0.0001$)

Logarithms were originally introduced as an aid to calculation before calculators were available. To multiply two numbers you looked up their logarithms (to base 10) in a book of tables, added the logs (since if $c = 10^x$ and $d = 10^y$, $c \times d = 10^{x+y}$) and then looked up the antilogarithm (= $c \times d$) of this sum. To divide two numbers a similar process was followed except in this case the log values were subtracted ($c / d = 10^{x-y}$).

The use of log tables is now a thing of the past but logarithms are still important in computer mathematics because $1 + \lfloor \log_2 n \rfloor$ is **the number of bits needed to represent the value n in binary code** (where $\lfloor x \rfloor$ is the *floor* of x, the largest integer not greater than x).

Some useful properties of logarithms

(NB the proofs are just here for interest, you don't need to memorise them.)

i. $\mathbf{\log_a(1) = 0}$ (as $a^0 = 1$, for any a)

ii. $\mathbf{\log_a a = 1}$ (as $a^1 = a$, for any a)

iii. $\mathbf{\log_a(x^n) = n \log_a x}$

Proof: Let $y = \log_a x$, so $x = a^y$.
$x^n = (a^y)^n = a^{ny}$
Hence $\log_a x^n = ny = n\log_a x$

iv. $x = a^{\log_a x}$

Proof: $\log_a$ of LHS = $\log_a x$
$\log_a$ of RHS = $\log_a(a^{\log_a x})$
$= \log_a x \times \log_a a$ (by iii)
$= \log_a x$ (by ii)

v. $\mathbf{\log_a(xy) = \log_a x + \log_a y}$

Proof: Using iv, $x = a^{\log_a x}$, $y = a^{\log_a y}$
$xy = a^{\log_a x} \times a^{\log_a y} = a^{\log_a x + \log_a y}$
$\rightarrow \log_a(xy) = \log_a x + \log_a y$

vi. $\mathbf{\log_a(x/y) = \log_a x - \log_a y}$

Proof: $x/y = a^{\log_a x} / a^{\log_a y} = a^{\log_a x - \log_a y}$
$\rightarrow \log_a(x/y) = \log_a x - \log_a y$

vii. $\mathbf{(\log_a b)(\log_b a) = 1}$

Proof: Let $x = \log_a b$ (so $b = a^x$), $y = \log_b a$ (so $a = b^y$)
Then $b = (\underbrace{b^y}_{a})^x = b^{xy}$

$\rightarrow xy = (\log_a b)(\log_b a) = 1$

viii. **Change of base** (from b to a):

$\mathbf{\log_a x = (\log_b x)(\log_a b)}$

Proof: $\log_a x = \log_a[b^{\log_b x}]$ (using iv)
$= (\log_b x)(\log_a b)$ (using iii)