Applications of Stochastic Differential Equations (SDE)

- Modelling with SDE: Ito vs. Stratonovich (6.1 & 6.2)
- Parameter Estimation (6.4)
- Optimal Stochastic Control (6.5)
- Filtering (6.6)

Ito SDE or Stratonovich SDE? (6.1 + 6.2)

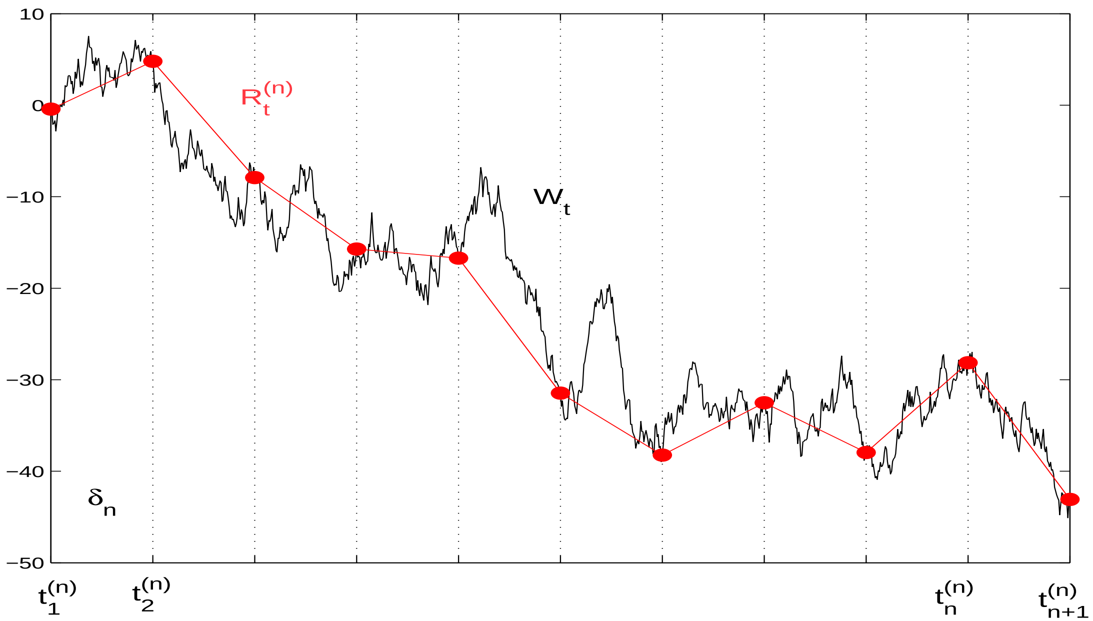
- From a purely mathematical viewpoint both the Ito and Stratonovich calculi are correct;
- Ito or Stratonovich? This question can only be discussed in the context of a particular application;
- Ito SDE is appropriate when the continuous approximation of a discrete system is concerned (many examples in the biological sciences);
- Stratonovich SDE is appropriate when the idealization of a smooth real noise process is concerned (many examples in engineering and the physical sciences).

Idealization of smooth noise processes

Example: a random differential equation

$$dX_t^{(n)} = aX_t^{(n)}dt + bX_t^{(n)}dR_t^{(n)}$$

where $R_t^{(n)}$ is the piecewise differentiable linear interpolation of a Wiener process W_t on a partition $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T$



$\longrightarrow$ The solution by classical calculus

$$X_t^{(n)} = X_{t_0} \exp\left(a(t-t_0) + b(R_t^{(n)} - R_{t_0}^{(n)})\right)$$

When $n \to \infty$ and $\max_{1\leq j\leq n} \left| t_{j+1}^{(n)} - t_j^{(n)} \right| \longrightarrow 0$

- The noise process $\longrightarrow W_t$
- The solution $\longrightarrow X_t = X_{t_0} \exp\left(a(t-t_0) + b(W_t - W_{t_0})\right)$;
- The random DE $\longrightarrow$ a Stratonovich SDE

$$dX_t = aX_t dt + bX_t \circ dW_t$$

Continuous approximation of discrete systems

- Example 1: a random walk

$$X_{k+1}^{(N)} = X_k^{(N)} + \frac{1}{\sqrt{N}}\xi_k$$

at times $t_k^{(N)} = \frac{k}{N}$, $k = 0, 1, ..., N$, where ξ_k take $+1$ or -1 with probability $\frac{1}{2}$ $\overset{N\to\infty}{\longrightarrow}$ a standard Wiener process in $[0,1]$ (Central Limit Theorem);

- Example 2: a random walk

$$X_{k+1}^{(N)} = X_k^{(N)} + a_N\left(X_k^{(N)}\right)\frac{1}{N} + b_N\left(X_k^{(N)}\right)\frac{1}{\sqrt{N}}\xi_k$$

with consistency conditions such as

$$\lim_{n\to\infty} NE\big(X_{k+1}^{(N)} - X_k^{(N)}\big|X_k^{(N)} = x\big) = \lim_{N\to\infty} E(a_N(x)) = a(x)$$

$$\lim_{n\to\infty} NE\big(\big(X_{k+1}^{(N)} - X_k^{(N)}\big)^2\big|X_k^{(N)} = x\big) = \lim_{N\to\infty} E(b_N^2(x)) = b^2(x)$$

$$\lim_{n\to\infty} NE\big(\big|X_{k+1}^{(N)} - X_k^{(N)}\big|^3\big|X_k^{(N)} = x\big) = 0$$

$\overset{N\to\infty}{\longrightarrow}$ an Ito stochastic differential equation

$$dX_t = a(X_t)dt + b(X_t)dW_t$$

- Example 3: a population of $2N$ genes with two alleles a and A.
  - Suppose that there are i genes of type a and $2N - i$ of type A at the k-th generation;
  - The probability of j genes of type a at the $k+1$-th generation is $B(2N, \frac{i}{2N})$;
  - Further all time intervals between two successive generations are of length $\frac{1}{N^2}$;
  - Let's define $X_k^{(N)} = \frac{i}{2N}$. When N goes infinity, we obtain a process which is a solution of the Ito SDE

$$dX_t = X_t(1 - X_t)dW_t.$$

Parameter Estimation (6.4): An example

$$dX_t = \alpha \cdot a(X_t)dt + dW_t$$

To determine a maximum likelihood estimate of α when the trajectory $X(t)$ of a solution process over the time interval $[0, T]$ is given.

- the likelihood ratio

$$L(\alpha, T) = \exp\left(+\frac{1}{2}\alpha^2 \int_0^T \alpha^2(X_t)dt - \alpha \int_0^T \alpha(X_t)dX_t \right)$$

  - the Euler scheme:

$$X_{i+1} = X_i + \alpha a(X_i)\Delta + \Delta W_i$$

    for $i = 0$, 1, ..., N - 1 where $\Delta = \frac{T}{N}$;
  - $\Delta W_0, ..., \Delta W_{N-1}$ are i.i.d. and $\Delta W_i \sim N(0; \Delta) \quad \forall i$
  - $\Delta X_0, ..., \Delta X_{N-1}$ are i.i.d. and $\Delta X_i \sim N(0; \Delta) \quad \forall i$

– the Radon-Nikodyn derivation of the process X_t w.r.t. W_t

$$\frac{P(\Delta X_0, ..., \Delta X_{N-1})}{P(\Delta W_0, ..., \Delta W_{N-1})} \stackrel{N\to\infty}{\longrightarrow} L(\alpha, T)$$

- the maximum likelihood estimator

$$\hat{\alpha}(T) = \frac{\int_0^T a(X_t)dt}{\int_0^T a^2(X_t)dt}$$

If $E\left(\int_0^T a^2(X_t)dt\right) < \infty$, the SDE above has a stationary solution with density $\bar{p} \longrightarrow$

$$\hat{\alpha}(T) - \alpha = \frac{\int_0^T a(X_t)dW_t}{\int_0^T a^2(X_t)dt} \cdot \frac{1/T}{1/T} \stackrel{T\to\infty}{\longrightarrow} \frac{0}{\int a^2(x)\bar{p}(x)dx} = 0$$

the Central Limit Theorem $\stackrel{T\to\infty}{\longrightarrow}$

$$T^{1/2}(\hat{\alpha}(T) - \alpha) \sim N\left(0, \frac{1}{\int a^2(x)\bar{p}(x)dx}\right)$$

Optimal Stochastic Control

- problem formulation
  - state $X \in \Re^d$:

$$dX_t = a(t, X_t, u)dt + b(t, X_t, u)dW_t$$

  - control parameter $u \in \Re^k$:
    * $u = u(t, X_t)$ (Markov feedback control)
    * $u = u(t, \omega)$ (open-loop control)
  - the cost functional for Markov feedback controls

$$J(s, x; u) = E\left(K(\tau, X_\tau) + \int_s^\tau F(t, X_t, u)dt \Big| X_s = x\right)$$

    where K and F are given functions and τ is a specified Markov time

- the Hamilton-Jacobi-Bellman (HJB) equation

The minimum cost functional

$$H(s,x) = \min_{u(\cdot)} J(s,x,u(\cdot))$$

The HJB equation

$$\min_{u \in \Re^k} \{F(s,x,u) + L_u H(s,x)\} = 0$$

with the final time condition $H(T,x) = K(T,x)$
where $\tau = T$ and

$$L_u = \frac{\partial}{\partial s} + \sum_{i=1}^{d} a^i(s,x,u) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} D^{i,j}(s,x,u) \frac{\partial^2}{\partial x_i \partial x_j}$$

with $D = bb^\top$

- the linear-quadratic regulator problem
  - $K(T, X_T) = X_T^\top R X_T$ *($R \in \Re^{d\times d}$, symmetric and positive semi-definite)*
  - $F(t, X_t, u) = X_t^\top C(t) X_t + u^\top G(t) u$ *($C \in \Re^{d\times d}$, symmetric and positive semi-definite, $G \in \Re^{k\times k}$, symmetric and positive definite)*
  - $a(t, X_t, u) = A(t) X_t + M(t) u$ *($A \in \Re^{d\times d}$ $M \in \Re^{d\times k}$)*
  - $b(t, X_t, u) = \sigma(t)$ *($\sigma \in \Re^{d\times m}$)*

  $\rightarrow$ a guess solution

$$H(s, x) = x^\top S(s) x + a(s)$$

  with the final time condition

$$S(T) = R \quad \text{and} \quad a(T) = 0;$$

  $\rightarrow$ the left side of HJB

$$x^\top S'(s)x + a'(s) + x^\top C(s)x + u^\top G(s)u$$

$$+(A(s)x + M(s)u)^\top (S(s)x + S^\top(s)x) + \sum_{ij} (\sigma\sigma^\top)_{ij} S_{ij}$$

$\rightarrow$ the minimizer of the left side of HJB

$$u = -G^{-1}(s)M^\top S(s)x$$

$$\rightarrow 0 = \left(\underbrace{a'(s) + tr(\sigma\sigma^\top S)}_{=0} \right) +$$

$$x^\top \left(\underbrace{S'(s) + A(s)^\top S + SA(s) - SM(s)G(s)^{-1}M(s)S - C(s)}_{=0 \quad \text{a Riccati type equation}} \right) x$$

- when situations of partial information occur, linear stochastic control = linear filtering + deterministic control

Filtering

- linear filtering

$$dX_t = AX_t dt + BdW_t$$

$$dY_t = HX_t dt + \Gamma dW_t^*$$

$\longrightarrow$ the Kalman-Bucy filter

The estimate $\hat{X}_t = E(X_t|\mathcal{Y}_t)$ satisfies the SDE

$$d\hat{X}_t = \left(A - SH^\top(\Gamma\Gamma^\top)^{-1}H\right)\hat{X}_t dt + SH^\top(\Gamma\Gamma^\top)^{-1}dY_t$$

where the error covariance $S(t)$ satisfies the matrix Riccati equation

$$\frac{dS}{dt} = AS + SA^\top + BB^\top - SH^\top(\Gamma\Gamma^\top)^{-1}HS$$

- non-linear filtering

the Fokker-Planck equation in operator form

$$\frac{\partial p}{\partial t} = \mathcal{L}^* p$$

and the observation process

$$dY_t = h(X_t)dt + dW_t^*$$

$\longrightarrow$

the conditional probability densities of X_t given $\mathcal{Y}_t$ are given by

$$\bar{p}(t,x) = \frac{Q_t(x)}{\int Q_t(x)dx}$$

where the unnormalized densities $Q_t(x)$ satisfy the Wong-Zakai equation

$$dQ_t(x) = \mathcal{L}Q_t(x)dt + h(x)Q_t(x)dY_t$$

Further Reading: Venkatarama Krishnan, Nonlinear Filtering and Smoothing – An Introduction to Martingales, Stochastic Integrals and Estimation, Dover 2005.