1 Sequences

A function of a positive integer variable, $f(n)$ or f_n is called a sequence: a sequence is thus a set of numbers $f_1,\ f_2\ \ldots$ in a definite order that may be put in correspondence with the natural numbers.

1.1 Limits

A number l is called the limit of an infinite sequence if, for any ϵ we can find a positive number N such that

$$\forall\, n > N, \quad |f_n - l| < \epsilon$$

We then write:

$$\lim_{n\to\infty} f_n = l \qquad (1)$$

If the limit of an infinite sequence exists, it is called convergent, otherwise it is said to be divergent. If, for any positive number M, there exists an N such that,

$$\forall\, n > N, \quad f_n > M,$$

we say

$$\lim_{n\to\infty} f_n \to \infty \qquad (2)$$

A similar meaning is given to

$$\lim_{n\to\infty} f_n \to -\infty$$

The limit (1) has the same kind of properties as for functions, viz:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

etc. and least upper and greatest lower bounds may be defined in the usual way.

1.2 Cauchy's Convergence Criteria

A sequence $\{f_n\}$ converges iff for each $\epsilon > 0$, $\exists$ an N, such that,

$$\forall\, n,\, m > N, \quad |u_n - u_m| < \epsilon \qquad (3)$$

Note: one does not need to know the limit l, e.g. if $f_n = \dfrac{n}{n+1}$, the sequence converges:

$$\lim_{n\to\infty} f_n = 1$$

as may be seen by writing

$$f_n = 1 - \frac{1}{n+1}$$

2 Series

The <u>infinite series</u>,

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 \ldots \qquad (4)$$

converges if the sequence of <u>partial sums</u>

$$S_n = \sum_{k=1}^{n} u_k$$

converges to a limit s. We call s the sum of the infinite series (4). If the sequence S_n diverges, the series is said to diverge etc., e.g.

1. $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

   since the partial sum

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{n} \left(\frac{1}{k} = \frac{1}{k+1}\right) \\ \text{“diagonal cancellation”} \quad &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \ldots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ \Rightarrow \quad S_n &= 1 - \frac{1}{n+1} \end{aligned}$$

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2.

$$
\begin{aligned}
\sum_{k=1}^{\infty} a\, r^{k-1} &= \frac{a}{1-r} \qquad \text{if } |r| < 1 \\
\text{since} \quad S_n &= \sum_{k=1}^{n} a\, r^{k-1} \\
&= \frac{a(1-r^k)}{1-r} \qquad \text{geometric progression.}
\end{aligned}
$$

2.1 Series of Positive (non-negative) Terms

There are three simple tests for discovering whether a series of positive (non-negative) terms converges:

2.1.1 Comparison Test

If a series of positive terms a_n is known to converge and, $\forall\, n > N,\ 0 \leq b_n \leq a_n$, then $\sum b_n$ also converges. If the series a_n is known to diverge and $\forall\, n > N,\ b_n \geq a_n$, then $\sum b_n$ also diverges, e.g.

$$
\begin{aligned}
\sum_k \frac{1}{2^n+1} & \qquad \text{converges since} \\
0 < \frac{1}{2^n+1} < \frac{1}{2^n} & \qquad \forall\, n \\
\text{and} \quad \sum_{n=0} \frac{1}{2^n} &= 2 \qquad \text{G.P.}
\end{aligned}
$$

2.1.2 Quotient Test

If $a_n \geq 0$ and $b_n \geq 0$, and if $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ or ∞, then $\sum a_n$ and $\sum b_n$, either both converge or both diverge, e.g.

$$
\sum \frac{n}{4n^3-2} \quad \text{converges since} \quad \sum \frac{1}{n^2} \quad \text{converges:}
$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (5)$$

Note:

- the sum (5) is a little non-trivial.
- (5) may itself be shown to converge by comparison with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$
- $\sum a\, n\, r^{n-1}$ diverges if $|n| > 1$, since $\sum a\, r^{n-1}$ diverges if $|r| > 1$.

2.1.3 Integral Test

If $f(x)$ is a monotonic decreasing positive function such that

$$f(n) = a_n \quad \text{for} \quad n = N,\ N+1,\ \ldots$$

then $\sum_n a_n$ converges or diverges according to whether $\displaystyle\int_N^{\infty} dx\, f(x)$ converges or diverges, e.g. the series ("p series")

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges for $p \ll 1$ since

$$\int_N^{\infty} \frac{dx}{x^p} = \left[\left(\frac{1}{1-p} \right) x^{1-p} \right]_N^{\infty} \qquad p \neq 1$$

converges for $p > 1$ and diverges for $p \ll 1$.
Note that this result implies that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even 'though the n'th term tends to zero as $n \to \infty$.

2.2 Alternating Series

A series whose terms are alternately positive and negative converges if,

$$\forall\, n > N, \quad \text{(a)} \quad |a_{n+1}| \leq |a_n|, \quad \text{and} \quad \text{(b)} \quad \lim_{n \to \infty} a_n = 0.$$

e.g.

1. The series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots = \log 2$ converges.
2. The series: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}$ converges.

Note: the error in truncating an alternating series at the n'th term is less than $|a_{n+1}|$.

2.3 Absolute Convergence

A series $\sum_n a_n$ is said to be absolutely convergent if $\sum_n |a_n|$ converges. If $\sum_n a_n$ converges, but $\sum_n |a_n|$ does not, the series is said to be conditionally convergent, e.g.

$\displaystyle\sum_{n=1}^{\infty} a\, r^{n-1}$ is absolutely convergent if $|r| < 1$, but $\displaystyle\sum_{n=1}^{p} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

2.4 Theorems

1. The terms of an absolutely convergent series may be rearranged without affecting the sum. A conditionally convergent series cannot be rearranged as the resulting series may diverge or be made to converge to any desired sum.
2. The sum, difference and product of two absolutely convergent series are absolutely convergent, i.e. manipulate as finite series.

3 Tests for Convergence of Series

3.1 Ratio Test

If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then the series $\sum_n a_n$

1. converges absolutely.
2. diverges if $L > 1$.

The test is inconclusive if $L = 1$, e.g.

1. $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges absolutely.

2. The test is inconclusive as $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

3.2 The n'th Root Test

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then the series $\sum a_n$

1. converges absolutely if $L < 1$.

2. diverges if $L > 1$.

The test is inconclusive if $L = 1$, e.g. consider the series $1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \ldots$. The ratio test fails since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist. However:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left\{ \frac{\sqrt[n]{2|r^n|}}{\sqrt[n]{|r^n|}} \right\} \left\{ \frac{\text{even}}{\text{odd}} \right\} \quad \to \quad |r|. \\ &\Rightarrow \quad \text{converges if} \quad |r| < 1. \end{aligned}$$

4 Series of Functions

A <u>sequence of functions</u> $F_n(x)$, say, converges to the limit $F(x)$ in $[a, b]$, if for any $\epsilon > 0$ and for each x in $[a, b]$, $\exists$ an N such that:

$$\forall\, n > N, \quad |F_n(x) - F| < \epsilon$$

We then write:

$$\lim_{n\to\infty} F_n(x) = F(x) \qquad (6)$$

The number N may depend on ϵ and on the particular x chosen in $[a, b]$. However, if N depends only on ϵ but not on x, the sequence is said to be <u>uniformly convergent</u> in $[a, b]$. The <u>infinite series of functions</u>

$$\sum_{k=1}^{\infty} f_k(x) = f_1(x) + f_2(x) + \ldots \qquad (7)$$

is said to be convergent in $[a, b]$, if the sequence of partial sums

$$F_n(x) = \sum_{k=1}^{n} f_k x$$

is convergent in $[a, b]$. We then write the sum:

$$\begin{aligned} F(x) &= \sum_{k=1}^{\infty} f_k(x) \\ & \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \qquad (8) \end{aligned}$$

4.1 Tests for Convergence

4.1.1 Weierstrass M Test

If a sequence of positive constants $M_1,\ M_2,\ \ldots\ M_n$ can be found such that $\forall\ x$ in $[a, b]$

1. $|f_n(x)| \leq M_n \quad n = 1, 2, \ldots$
2. $\sum_n M_n$ converges

then $\sum_n f_n(x)$ is uniformly and absolutely convergent, e.g.

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

is uniformly and absolutely convergent $\forall$ real x, since

$$\left|\frac{\cos(nx)}{n^2}\right| \leq \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{is convergent.}$$

Note:

1. The Weierstrass test, if satisfied is a sufficient condition for uniform convergence, but it is not necessary.
2. Uniform convergence and absolute convergence are independent properties in that a series can be uniformly convergent without being absolutely convergent and vice-versa.

4.2 Theorems

4.2.1 Integration

A uniformly convergent series of continuous functions can be integrated term by term:

$$\int_a^b dx\, F(x) = \sum_{n=1}^{\infty} \int_a^b dx\, f_n(x)$$

4.2.2 Differentiation (a little more complicated)

A convergent series $\sum_{n=1}^{\infty} f_n(x)$ of continuous functions $f_n(x)$ in $[a, b]$ with continuous derivatives $f_n'(x)$ may be differentiated term by term if the series $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent in $[a, b]$.

5 Power Series

A series of the form:

$$\sum_{n=1}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + \dots \qquad (9)$$

where $a_0,\ a_1,\ a_2\ \dots$ are constants is called a power series in x.

5.1 Radius and Range of Convergence

In general, a power series converges for some $|x| < R$ and diverges for $|x| > R$. R is called the radius of convergence of the series. For $|x| = R$ the series may or may not converge.

The concept of radius of convergence becomes most useful when we deal with complex numbers $z = x + iy$ and consider a series

$$\sum_{n=0}^{\infty} a_n\, z^n$$

For real x, the interval $-R < x < R$ is called the interval or range of convergence. One or both of the end points may be included - it depends on the series. If $R \to \infty$, we say the series converges for all x.

5.2 Theorems on Convergence

1. A power series conveges uniformly and absolutely in any interval lying entirely within its interval of convergence.

2. A power series may be differentiated or integrated term by term over any interval lying entirely within its interval of convergence.

3. Within their common interval of convergence, two power series may be

   - added or subtracted term by term.
   - multiplied to obtain a new power series,

   i.e. $\sum_n a_n\, x^n \,.\, \sum_n b_n\, x^n = \sum_n c_n\, x^n$

   where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} \ldots a_n b_0$

4. A power series $\sum_n a_n x^n$ may be divided by a power series $\sum_n b_n x^n$ and, provided $b_0 \neq 0$, the result written as a power series which converges for sufficiently small $|x|$.

5. If $y = \sum_n a_n\, x^n$, then we may write $x = \sum_n b_n\, y^n$ and obtain the coefficients b_n in terms of the a_n by substituting the second series in the first. This is called reversion of a series.

Examples:

(a)
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } -\infty < x < \infty$$

(b)
$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \text{for } -\infty < x < \infty \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \text{for } -\infty < x < \infty
\end{aligned}$$

(c)
$$\begin{aligned}
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad \text{for } -1 < x \leq 1 \\
log\left(\frac{(1+x)}{(1-x)}\right) &= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \quad \text{for } -1 < x < 1
\end{aligned}$$

(d) $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad \text{for } -1 \leq x \leq 1$

(e) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$
in general for $-1 < x < 1$.

Binomial series for any n. If n is a natural number, the series is finite.

6 Asymptotic Series

Power series may be used to define a variety of functions and are very useful in obtaining approximate values of functions for small x, or near some other point x_0 (when we use a power series in $(x - x_0)$).
Similarly, it is useful to have series that can give us information about a function for large x and, if desired, provide approximate values. Asymptotic series of the form:

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots \frac{a_n}{x^n} + \ldots \qquad (10)$$

fill this role.

Example:
The function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \qquad (11)$$

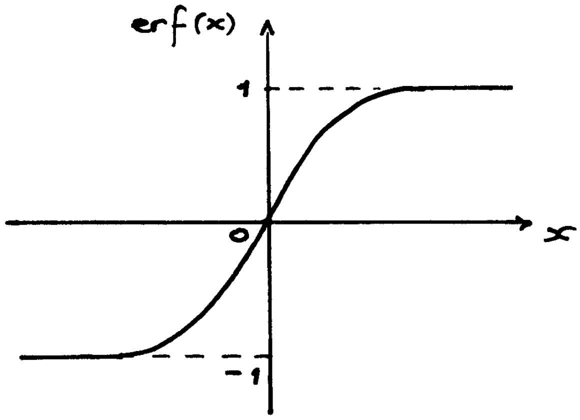
known as the error function, occurs frequently in statistics as it is closely related to the cumulative probability distribution function for a normal or Gauss randon variable. From the definition (11), we see

$$\begin{aligned} erf(-x) &= -erf(x) \\ erf(0) &= 0 \end{aligned}$$

and, since $\int_0^\infty dt\, e^{-t^2} = \frac{\sqrt{\pi}}{2}$,

$$\lim_{x \to \infty} (erf(x)) \quad \to \quad 1$$

The function thus looks like



Note that as $x \to \pm\infty$ the error function is asymptotic to ± 1 respectively. To counter (don't think that is the right word to use - ask Bernard!) this we use the asymptotic series (10).
To obtain an asymptotic series for the error function as $x \to \infty$, we first rewrite (11) as

$$er\,f(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2}$$

and then repeatedly integrate the remaining integral by parts, e.g.

$$I = \int_x^\infty dt\, e^{-t^2}$$

$$\text{let} \qquad u = \tfrac{1}{t} \qquad \frac{dv}{dt} = t\, e^{-t^2}$$

$$\tfrac{du}{dt} = -\tfrac{1}{t^2} \qquad v = -\frac{1}{2}\, e^{-t^2}$$

$$\begin{aligned} \Rightarrow \quad I &= \left[-\frac{1}{2t}\, e^{-t^2}\right]_x^\infty - \int_x^\infty dt \frac{1}{2t^2}\, e^{-t^2} \\ \text{i.e.} \quad I &= \frac{1}{2x}\, e^{-x^2} - \frac{1}{2} \int_x^\infty dt\, \frac{1}{t^2}\, e^{-t^2} \end{aligned}$$

$$\text{Again:} \qquad u = \tfrac{1}{t^3} \qquad \frac{dv}{dt} = t\, e^{-t^2}$$

$$\tfrac{du}{dt} = -\tfrac{3}{t^4} \qquad v = -\frac{1}{2}\, e^{-t^2}$$

$$\begin{aligned}
.\colon \quad I &= \frac{1}{2x}\,e^{-x^2} - \frac{1}{2}\left[\left[-\frac{1}{2t^3}\,e^{-t^2}\right]_x^\infty - \frac{3}{2}\int_x^\infty dt\,\frac{1}{t^4}\,e^{-t^2}\right] \\
\Rightarrow \quad I &= \frac{1}{2x}\,e^{-x^2} - \frac{1}{4x^3}\,e^{-x^2} + \frac{3}{4}\int_x^\infty dt\,\frac{1}{t^4}\,e^{-t^2} \quad \text{etc.} \\
\text{Thus:} \quad I &= \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{3}{4x^4} - \frac{15}{8x^6} + \ldots\right]
\end{aligned}$$

and we see:

$$er\,f(x) = 1 = \frac{2}{\sqrt{\pi}} \cdot \frac{e^{-x^2}}{2x} \cdot \left[1 - \frac{1}{2x^2} + \frac{3}{4x^4} - \frac{15}{8x^6} + \ldots\right] \tag{12}$$

and we have an asymptotic series, useful when x is large.
An asymptotic series like this may actually diverge. In fact, they usually do diverge. For a given large x, the terms get smaller at first, then they begin to grow and the series diverges, e.g.
The general term in the asymptotic series for the error function above is of the form

$$(-1)^n \cdot \frac{(2n-1)!!}{(2x^2)^n}$$

where $(2n-1)!! = (2n-1)(2n-3)(2n-5)\ldots 1$.
We can thus see that for large x, the terms first decrease, but that eventually the factorial in the numerator means that for large n, the terms grow without limit.

6.1 Definition of an Asymptotic Series

If a series $S(x)$ of negative powers of x,

$$S(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots \frac{a_n}{x^n} \ldots \tag{13}$$

has partial sums,

$$S_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots \frac{a_n}{x^n} \tag{14}$$

such that the remainder,

$$R_n(x) = f(x) - S_n(x) \qquad (15)$$

satisfies:

$$\lim_{|x|\to\infty} [x^n R_n(x)] = 0 \qquad \forall\, n,$$

then $S(x)$ is said to be the <u>asymptotic expansion</u> of the function $f(x)$ for large $|x|$.
<u>Note</u>: The crucial property is one that depends on $|x| \to \infty$, <u>not</u> $n \to \infty$.

6.2 Theorem

Asymptotic series may be integrated term by term to yield other asymptotic series.

6.3 Double Series

If we multiply two power series, one in x, the other in y, together, we obtain a double series:

$$\begin{aligned} f(x) &= \sum_n a_n\, x^n \\ g(y) &= \sum_n b_n\, y^n \\ \Rightarrow \quad f(x)\, g(y) &= \sum_n a_n\, x^n \,.\, \sum_m b_m\, y^m \\ &= \sum_n \sum_m a_n\, b_m\, x^n\, y^m \end{aligned}$$

In general, <u>double series</u> such as

$$S = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm}$$

can be defined in a way analogous to that we have discovered for 'ordinary' series above. They also have <u>properties very similar to</u> those of <u>ordinary series</u>, and may be extended to include three or more summations when we have <u>multiple series</u>.