1 Partial Differentiation

1.1 Functions of Several Independent Variables

z is said to be a function of (say) two variables x and y, if for each pair (x, y) within some domain D, we can determine a (unique) value for z.
If you wish, this definition can be based on set theory. It can also be extended to as many independent variables $(x, y, \ldots)$ as we like.

1.2 Dependent and Independent Variables

z is called the dependent variable and we write,

$$z = f(x, y)$$

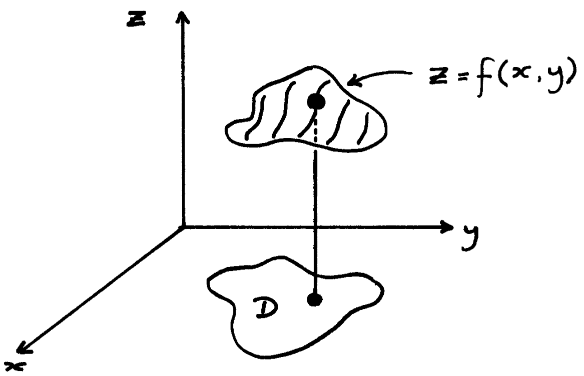
1.3 Single and Multiple Variables

The function is said to be single valued if only one value of z corresponds to each (x, y). Otherwise it is multi-valued.
Examples

1. $z = 3x - 2y + 6$ is single valued
2. $z^2 = 9 - x^2 - y^2$ is multi-valued

1.4 Rectangular Co-ordinate Systems

Just as it is customary to plot $y = f(x)$ in Cartesian co-ordinates in 2-D, $z = f(x, y)$ is often plotted in 3-D Cartesian co-ordinates.
In 2-D, $y = f(x)$ is a curve. In 3-D, $z = f(x, y)$ is a surface.

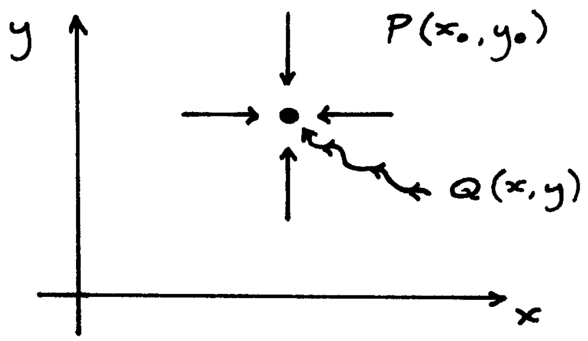


Examples

1. $z = 3x - 2y + 6$ is a plane
2. $z^2 = 9 - x^2 - y^2$ is a sphere
   centre $(0, 0)$, radius 3.

1.5 Limits and Continuity

Limits can be defined analogously to what we did for functions of a single variable. The only difference now is that there are many more ways to approach a point (x_0, y_0) in 2-D than there was to approach x_0 in 1-D:



Thus, a point $Q(x, y)$ may approach a point $P(x_0, y_0)$ along any of an infinite number of curves in the (x, y) plane.
A limit

$$\lim_{x \to x_0 \; y \to y_0} f(x, y) = f_0$$

is then said to converge if exists however $(x, y) \to (x_0, y_0)$ and if its value is independent of how (x, y) approaches (x_0, y_0).
Examples:

$$(1) \quad \lim_{x\to 0\; y\to 0} \left[\frac{\sin(x+y)}{(x+y)}\right] = 1$$

$$(2) \quad \lim_{x\to 0\; y\to 0} \left[\frac{x-y}{x+y}\right] \quad \text{does not exist.}$$

Similarly, $f(x, y)$ can be defined to be continuous if it is defined in some neighbourhood of (x_0, y_0) and

$$\lim_{x\to x_0\; y\to y_0} f(x, y)$$

exists and is equal to $f(x_0, y_0)$.
Note: We introduced the idea of a neighbourhood. See (e.g.) Spiegel, this is just a set of points (x, y), e.g. such that $|x - x_0| < \delta$, $|y - y_0| < \delta$ (rectangular neighbourhood).

2 Partial Derivatives

The derivative of a function $f(x, y, \ldots)$ of several variables w.r.t. one of the independent variables, x say, keeping all others fixed, is called the partial derivative of f w.r.t. x and written: $\frac{\partial f}{\partial x}$.
By definition:

$$\frac{\partial f}{\partial x} = lim_{\delta x\to 0} \left\{\frac{f(x+\delta x, y\ldots) - f(x, y\ldots)}{\delta x}\right\}$$

Partial derivatives w.r.t. the other independent variables, $\frac{\partial f}{\partial y}$, etc. are similarly defined.
Examples:

1. $f(x, y) = 9 - x^2 - y^2$
   $\Rightarrow \frac{\partial f}{\partial x} = -2x, \frac{\partial f}{\partial y} = -2y$

2. $f(x,y) = 3\sin(xy) + 3x^3y^4$

$$\begin{aligned}\Rightarrow \frac{\partial f}{\partial x} &= 3y\cos(xy) + 9x^2y^4 \\ \frac{\partial f}{\partial y} &= 3x\cos(xy) + 12x^3y^3\end{aligned}$$

2.1 Higher Order Partial Derivatives

If $f(x,y)$ has partial derivatives in some region, then as we first saw, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ are themselves functions and we can calculate their derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

in the usual way.
Examples:
Calculate all the second order partial derivatives of the functions in examples 1 and 2 above. Verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Notation
Partial derivatives are often denoted by a suffix x, y, etc.
E.g.

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial f}{\partial y} = f_y$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx} \quad \text{etc.}$$

2.2 Differentials

The increment in the dependent variable $z = f(x, y)$, as we let $\partial x,\ \partial y \to 0$, is called the total differential of z or of f.

$$\delta z = f(x + \delta x,\ y + \delta y) - f(x, y)$$

So in the limit as $\delta x \to 0,\ \delta y \to 0$,

$$dz = df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

2.3 Theorems on Differentials

1. If $f(x, y) = c = constant$, then $\Leftrightarrow df = 0$.
   Note: this means that $x\ y$ cannot be independent, except when f is trivial.

2. The expression $P(x, y)\, dx + Q(x, y)\, dy$ is a total differential of some function $f(x, y)$ iff

   $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

   In such a case, $P\, dx + Q\, dy$ is called an exact differential.
   If $P\, dx + Q\, dy$ is not exact, then in general $\exists$ a function $\mu(x, y)$ [on many such] such that $\mu\, P\, dx + \mu\, Q\, dy$ is exact.

2.4 Chain Rule

If $f = f(x, y)$ and (x, y) are themselves functions of other independent variables $x = x(u, v),\ y = y(u, v)$ say, then:

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \end{aligned}$$

This chain rule remains true if x and y are functions of just one underlying variable t (say) (i.e. are no longer independent)

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

or are functions of several variables $x = x(u, v, w \ldots),\ y = y(u, v, w \ldots)$
Note that when $x = x(t),\ y = y(t)$, the ordinary, total derivatives $\frac{df}{dt}, \frac{dx}{dt}, \frac{dy}{dt}$ appeared above. We can thus use the case where $f = f(x, y)$ and $y = y(x)$, leading to:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Example

$$\begin{aligned} \text{let}\quad f(x,y) &= x^2 + y^2 + 3x^2 y \qquad (1) \\ \text{then}\quad \frac{\partial f}{\partial x} &= 2x + 6xy \\ \frac{\partial f}{\partial y} &= 2y + 3x^2 \end{aligned}$$

Suppose we are now told $y = x^2 \qquad (2)$
then

$$\frac{dy}{dx} = 2x$$

and $f = f(x, y(x))$ is really just a function of x, with:

$$\begin{aligned} \frac{df}{dx} &= 2x + 6xy + (2y + 3x^2) - 2x \\ &= 2x + 6xy + 4xy + 6x^3 \\ &= 2x + 10xy + 6x^3 \\ &= 2x + 10x^3 + 6x^3 \\ \text{i.e.}\quad \frac{df}{dx} &= 2x + 16x^3 \end{aligned}$$

Check: substitute (2) in (1) to find

$$\begin{aligned} f &= x^2 + x^4 + 3x^4 \\ &= x^2 + 4x^4 \\ \frac{df}{dx} &= 2x + 16x^3 \qquad \text{okay!} \end{aligned}$$

<u>Note</u> also the total differential

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Equations for $\frac{\partial f}{\partial u}$, $\frac{df}{dt}$, $\frac{df}{dx}$ all follow from this.

2.5 Functions of Several Variables

All the above (and subsequent results) extend in the obvious manner to functions of several variables, $f = f(x, y, z, \ldots)$. In fact, everything may conveniently be inferred from the total differential

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz + \ldots$$

2.6 Implicit Functions

Differentiating implicit functions such as

$$z^2 = 9 - x^2 - y^2$$

is straightforward and we obtain, for example:

$$2z\frac{\partial z}{\partial x} = -2x \quad \text{etc.}$$

In general, if we define z implicitly as a function of x, y by

$$F(x, y, z) = 0,$$

then

$$dF = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0$$

and we can see, if x, y are independent:

$$
\begin{aligned}
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} &= 0 \\
\Rightarrow \quad \frac{\partial z}{\partial x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \text{etc.}
\end{aligned}
$$

2.7 Taylor's Theorem

This may be extended to functions of several variables.

$$
\begin{aligned}
f(x_0+h, y_0+k) &= f(x_0,y_0) + h\, f_x(x_0,y_0) + k\, f_y(x_0,y_0) \\
&= +\frac{1}{2!}\left\{h^2 f_{xx}(x_0,y_0) + 2hk\, f_{xy}(x_0,y_0) + k^2 f_{yy}(x_0,y_0)\right\} \\
&= +\frac{1}{3!}\{\ldots\} \quad \text{etc.}
\end{aligned}
$$

the higher order terms rapidly yet get complicated, but we can keep track of what is going on by defining the differential operator:

$$
\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)
$$

and noting that the series is a formal expansion of this:

$$
\begin{aligned}
f(x_0+h, y_0+k) &= f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y)|_{x_0,y_0} \\
&\quad +\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y)|_{x_0,y_0} \\
&\quad +\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y)|_{x_0,y_0} \\
&\quad \text{etc.}
\end{aligned}
$$