A Temporal Epistemic Logic with a Reset Operation*

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Abstract. We introduce CTLKR, a temporal epistemic logic extending CTLK with an epistemic operator N_i for n agents, referring to knowledge regarding future states. This modality is defined in terms of the intersection of the transitive reflexive closure of a serial temporal relation and the standard epistemic relation (which is an equivalence relation). We prove that CTLKR has the finite model property, is decidable, and is finitely axiomatisable. Further, we investigate an application of CTLKR to reason about the bit transmission problem.

1 Introduction

Modal logics provide a formal framework to specify computation in distributed and multi-agent systems. In particular, interpreted systems [6] provide a formal Kripke-style semantics to reason about different states of knowledge of agents. On this semantics, different concepts of knowledge have been explored, from implicit knowledge [9] to more sophisticated ones such as distributed, common, deductive, algorithmic knowledge, etc. All these logics are normally seen as formal specification languages for representing agents' knowledge.

A number of works have recently appeared in the literature relating to model checking techniques [4] for verifying automatically that a multi-agent system satisfies a particular temporal epistemic specification [11, 13, 8, 14]. While model checking presents some documented advantages over theorem proving, the core difficulty of the approach is the "state explosion problem", i.e. the fact the model representing the system grows very quickly to a size which is difficult to manage even when coded symbolically.

Bounded model checking [2] and other SAT-based approaches attempt to ease this problem by performing aggressive depth-first search on appropriate

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restricted submodels. This technique has been shown to be quite effective in temporal-epistemic logic as well [13, 15], but a key problem is that an epistemic modality defined on equality of local states (as in interpreted systems) forces to consider states reachable from *any* possible branch from the initial state, thereby limiting the advantages of the techniques. It is sometimes useful though, to reason about epistemic properties of agents that result from a "reset" operation, i.e., to reason about the knowledge regarding the future from the current time, *as if* a pruning of the model was performed at that instant and only the submodel *generated* by that point (as initial state) were to be considered. This happens, for instance, when different instances of the same property are checked a number of times over the same run, such as safe receipt of a stream of bits.

In this paper we introduce an epistemic modality $\mathrm{N}_i, i \in Ag$ (for "reset") that intrinsically incorporates the concept of resetting the model at the point where the modality is considered. This is equivalent to assuming that the agents are able to distinguish (i.e., to remove from their epistemically indistinguishable set of accessible states) the current state from states in the past and from states belonging to a different computational branch from the one that terminated in the state under evaluation. Alternatively, one can see this modality as expressing standard implicit knowledge but under the assumption the system enjoys a particular form of perfect recall [12], in which an agent would be able to recognise states with the same prefix of visited states. Representing perfect recall or even weaker variants of it is particularly costly in terms of model checking as the size of local states grows rapidly with time. The alternative proposed here is to use a standard semantics but evaluate the N_i on the intersection of the epistemic relation for agent i with the reflexive transitive closure of the temporal relation.

The paper is organised as follows. In section 2 we present syntax and semantics. Section 3 is devoted to the construction of the underlying machinery to prove the main results of the paper. Sections 4 and 5 present this main results, namely a decidability theorem and a completeness proof for the logic. In Section 6 we apply the formalism to the bit transmission problem. Section 7 contains conclusions and final remarks.

2 A CTLKR Logic

In this section, we present the syntax and semantics of a new temporal epistemic logic, called CTLKR.

2.1 Syntax

Assume a set of propositional variables $\mathcal{PV}$, and a set of agents $\mathcal{A}g = \{1, \ldots, n\}$ for $n \in \{1, 2, 3, \ldots\}$. The set $\mathcal{WF}$ of well-formed CTLKR formulas is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathrm{EX}\varphi \mid \mathrm{E}(\varphi \mathrm{U} \varphi) \mid \mathrm{A}(\varphi \mathrm{U} \varphi) \mid \mathrm{K}_i\varphi \mid \mathrm{N}_i\varphi,$$

where $p \in \mathcal{PV}$ and $i \in \mathcal{A}g$.

The above syntax extends CTL [3] with standard epistemic modality K_i [6] as well as an operator N_i for *knowledge following a reset operation*. The formula $\mathrm{N}_i\varphi$ is read as "after a reset agent i knows that φ", the formula $\mathrm{K}_i\varphi$, which represents the standard epistemic modality, is read as "agent i knows that φ". The remaining operators can be introduced via abbreviations as usual, i.e., $\alpha \wedge \beta \stackrel{def}{=} \neg(\neg\alpha \vee \neg\beta)$, $\alpha \Rightarrow \beta \stackrel{def}{=} \neg\alpha \vee \beta$, $\alpha \Leftrightarrow \beta \stackrel{def}{=} (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, $\mathrm{AX}\alpha \stackrel{def}{=} \neg\mathrm{EX}\neg\alpha$, $\mathrm{EF}\alpha \stackrel{def}{=} \mathrm{E}(\top\mathrm{U}\alpha)$, $\mathrm{AF}\alpha \stackrel{def}{=} \mathrm{A}(\top\mathrm{U}\alpha)$, $\mathrm{EG}\alpha \stackrel{def}{=} \neg\mathrm{AF}\neg\alpha$, $\mathrm{AG}\alpha \stackrel{def}{=} \neg\mathrm{EF}\neg\alpha$, $\mathrm{A}(\alpha\mathrm{W}\beta) \stackrel{def}{=} \neg\mathrm{E}(\neg\alpha\mathrm{U}\neg\beta)$, $\mathrm{E}(\alpha\mathrm{W}\beta) \stackrel{def}{=} \neg\mathrm{A}(\neg\alpha\mathrm{U}\neg\beta)$, $\overline{\mathrm{K}}_i\alpha \stackrel{def}{=} \neg\mathrm{K}_i(\neg\alpha)$.

Let φ and ψ be CTLKR formulas. We say that ψ is a *sub-formula* of φ if either (a) $\psi = \varphi$; or (b) φ is of the form $\neg\alpha$, $\mathrm{EX}\alpha$, $\mathrm{K}_i\alpha$, or $\mathrm{N}_i\alpha$, and ψ is a sub-formula of α; or (c) φ is of the form $\alpha \vee \beta$, $\mathrm{E}(\alpha\mathrm{U}\beta)$, or $\mathrm{A}(\alpha\mathrm{U}\beta)$ and ψ is a sub-formula of either α or β. The *length* of a CTLKR formula φ is equal to the number of symbols appearing in φ.

2.2 Semantics

Traditionally, the semantics of temporal epistemic logics is given on interpreted systems, defined in the following way [6]. Each agent $i \in \mathcal{A}g$ is associated with a set of local states L_i; the environment is associated with a set of local states L_e. An *interpreted system* is a tuple $IS = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathcal{V})$, where $S \subseteq \prod_{i=1}^{n} L_i \times L_e$ is a set of global states; $T \subseteq S \times S$ is a serial temporal relation on S; $\sim_i \subseteq S \times S$ is an epistemic relation for each agent $i \in \mathcal{A}g$ defined by: $s \sim_i s'$ iff $l_i(s') = l_i(s)$, where $l_i : S \to L_i$ is a function which returns the local state of agent i from a global state; $\mathcal{V} : S \to 2^{\mathcal{PV}}$ is a valuation function that assigns to each state a set of proposition variables that are assumed to be true at that state. The relation T is defined via an evaluation function which applies to states and agents' actions. For more details and further explanations of the notation we refer to [6].

In order to give a semantics to CTLKR we extend the above definition by a adding n relations R_i, one for each agent $i \in \mathcal{A}g$, defined as the intersection between the transitive and reflexive closure of the temporal relation T (denoted by T^*) and the epistemic relation $\sim_i$.

Definition 1 (Model). *Let $\mathcal{A}g$ be a set of agents. A* model *is a tuple $M = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathcal{V}, (R_i)_{i\in\mathcal{A}g})$, where S, T, $\sim_i$, and $\mathcal{V}$ are defined as in the interpreted system above, and $R_i = \sim_i \cap T^*$ for $i \in \mathcal{A}g$.*

A *path* in M is an infinite sequence $\pi = (s_0, s_1, \ldots)$ of states such that $(s_i, s_{i+1}) \in T$ for each $i \in \{0, 1, \ldots\}$. For a path $\pi = (s_0, s_1, \ldots)$, we take $\pi(k) = s_k$. By $\Pi(s)$ we denote the set of all the paths starting at $s \in S$.

Definition 2 (Satisfaction). *Let M be a model, s a state, and α, β* CTLKR formulas. The satisfaction relation $\models$, indicating truth of a formula in model M at state s, is defined inductively as follows:

$$
\begin{array}{ll}
(M,s) \models p & \text{iff } p \in \mathcal{V}(s), \\
(M,s) \models \neg\alpha & \text{iff } (M,s) \not\models \alpha, \\
(M,s) \models \alpha \wedge \beta & \text{iff } (M,s) \models \alpha \text{ and } (M,s) \models \beta, \\
(M,s) \models \mathrm{EX}\alpha & \text{iff } (\exists \pi \in \Pi(s))(M,\pi(1)) \models \alpha, \\
(M,s) \models \mathrm{E}(\alpha \mathrm{U} \beta) & \text{iff } (\exists \pi \in \Pi(s))(\exists m \geq 0)[(M,\pi(m)) \models \beta \\
 & \quad \text{and } (\forall j < m)(M,\pi(j)) \models \alpha], \\
(M,s) \models \mathrm{A}(\alpha \mathrm{U} \beta) & \text{iff } (\forall \pi \in \Pi(s))(\exists m \geq 0)[(M,\pi(m)) \models \beta \\
 & \quad \text{and } (\forall j < m)(M,\pi(j)) \models \alpha], \\
(M,s) \models \mathrm{K}_i\alpha & \text{iff } (\forall s' \in S)\ (s \sim_i s' \text{ implies } (M,s') \models \alpha), \\
(M,s) \models \mathrm{N}_i\alpha & \text{iff } (\forall s' \in S)\ (sR_is' \text{ implies } (M,s') \models \alpha).
\end{array}
$$

Satisfaction for the Boolean and temporal operators as well as the epistemic modality K_i is standard. The formula $\mathrm{N}_i\alpha$ holds at state s in a model M if α holds in all the states that are reachable from s via temporal relation T and they are in the $i-th$ epistemic relation with s. In other words, $(M,s) \models \mathrm{N}_i\alpha$ means that in the state s agent i knows α under assumption that he does not consider as possible states that do not belong to the future of s.

Let M be a model. We say that a CTLKR formula φ is *valid* in M (written $M \models \varphi$), if $M, s \models \varphi$ for all states $s \in S$, and a CTLKR formula φ is *satisfiable* in M, if $M, s \models \varphi$ for some state $s \in S$. We say that a CTLKR formula φ is *valid* (written $\models \varphi$), if φ is valid in all the models M, and that φ is *satisfiable* if it is satisfiable in some model M. In the latter case M is said to be a model for φ.

For another interpretation of N_i, observe that:

Lemma 1. *$(M,s) \models \mathrm{N}_i\phi$ iff $(M^*_s, s) \models K_i\phi$, where M^*_s is the submodel generated by M at s.*

3 Finite Model Property for CTLKR

In this section we prove that CTLKR has the *finite model property* (FMP). A logic has the FMP if any satisfiable formula is also satisfiable in a finite model.

In order to establish the FMP for CTLKR, we follow the construction presented in [5]. Therefore, we begin with providing definitions of two auxiliary structures: a *Hintikka structure* for a given CTLKR formula, and the *quotient construction* for a given model. To define these structures, the following set of formulas is needed.

Let φ be a CTLKR formula. The *Fischer-Ladner closure* of φ, $FL(\varphi)$, is defined by: $FL(\varphi) = CL(\varphi) \cup \{\neg\alpha \mid \alpha \in CL(\varphi)\}$, where $CL(\varphi)$ is the smallest set of formulas that contains φ and satisfy the following condistions:

- if $\neg\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
- if $\alpha \vee \beta \in CL(\varphi)$, then $\alpha, \beta \in CL(\varphi)$,
- if $\mathrm{E}(\alpha \mathrm{U} \beta) \in CL(\varphi)$, then $\alpha, \beta, \mathrm{EXE}(\alpha \mathrm{U} \beta) \in CL(\varphi)$,
- if $\mathrm{A}(\alpha \mathrm{U} \beta) \in CL(\varphi)$, then $\alpha, \beta, \mathrm{AXA}(\alpha \mathrm{U} \beta) \in CL(\varphi)$,
- if $\mathrm{EX}\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,

– if $\mathrm{K}_i\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
– if $\mathrm{N}_i\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$.

Observation. Note that for a given CTLKR formula φ, $FL(\varphi)$ is the set of formulas that are essential to establish the truth of φ in a model. Moreover, this set is finite, and the following lemma holds.

Lemma 2. *Given a* CTLKR *formula* φ, $SizeOf(FL(\varphi)) \leq 2 \cdot |\varphi|$.

Proof. Straightforward by induction on the length of φ.

Definition 3 (Hintikka structure). *Let* φ *be a* CTLKR formula, and $\mathcal{A}g$ a set of agents. A *Hintikka structure* for φ is a tuple $M_H = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathbb{L}, (R_i)_{i\in\mathcal{A}g})$ such that the elements S, T, $\sim_i$, and R_i, for $i \in \mathcal{A}g$, are defined as in Definition 1, and $\mathbb{L} : S \to 2^{FL(\varphi)}$ is a labelling function assigning a set of formulas to each state such that $\varphi \in \mathbb{L}(s)$ for some $s \in S$, and $\mathbb{L}$ satisfies the following conditions:

H.1. if $\neg\alpha \in \mathbb{L}(s)$, then $\alpha \notin \mathbb{L}(s)$
H.2. if $\neg\neg\alpha \in \mathbb{L}(s)$, then $\alpha \in \mathbb{L}(s)$
H.3. if $(\alpha \vee \beta) \in \mathbb{L}(s)$, then $\alpha \in \mathbb{L}(s)$ or $\beta \in \mathbb{L}(s)$
H.4. if $\neg(\alpha \vee \beta) \in \mathbb{L}(s)$, then $\neg\alpha \in \mathbb{L}(s)$ and $\neg\beta \in \mathbb{L}(s)$
H.5. if $\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $\beta \in \mathbb{L}(s)$ or $\alpha \wedge \mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$
H.6. if $\neg\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $\neg\beta \wedge \neg\alpha \in \mathbb{L}(s)$ or $\neg\beta \wedge \neg\mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$
H.7. if $\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $\beta \in \mathbb{L}(s)$ or $\alpha \wedge \neg\mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$
H.8. if $\neg\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $\neg\beta \wedge \neg\alpha \in \mathbb{L}(s)$ or $\neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$
H.9. if $\mathrm{EX}\alpha \in \mathbb{L}(s)$, then $(\exists t \in S)((s,t) \in T$ and $\alpha \in \mathbb{L}(t))$
H.10. if $\neg\mathrm{EX}\alpha \in \mathbb{L}(s)$, then $(\forall t \in S)((s,t) \in T$ implies $\neg\alpha \in \mathbb{L}(t))$
H.11. if $\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $(\exists\pi \in \Pi(s))(\exists n \geq 0)(\beta \in \mathbb{L}(\pi(n))$ and $(\forall j < n)\alpha \in \mathbb{L}(\pi(j)))$
H.12. if $\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then $(\forall\pi \in \Pi(s))(\exists n \geq 0)(\beta \in \mathbb{L}(\pi(n))$ and $(\forall j < n)\alpha \in \mathbb{L}(\pi(j)))$
H.13. if $\mathrm{K}_i\alpha \in \mathbb{L}(s)$, then $\alpha \in \mathbb{L}(s)$
H.14. if $\mathrm{K}_i\alpha \in \mathbb{L}(s)$, then $(\forall t \in S)(s \sim_i t$ implies $\alpha \in \mathbb{L}(t))$
H.15. if $\neg\mathrm{K}_i\alpha \in \mathbb{L}(s)$, then $(\exists t \in S)(s \sim_i t$ and $\neg\alpha \in \mathbb{L}(t))$
H.16. if $\mathrm{N}_i\alpha \in \mathbb{L}(s)$, then $\alpha \in \mathbb{L}(s)$
H.17. if $\mathrm{N}_i\alpha \in \mathbb{L}(s)$, then $(\forall t \in S)(sR_it$ implies $\alpha \in \mathbb{L}(t))$
H.18. if $\neg\mathrm{N}_i\alpha \in \mathbb{L}(s)$, then $(\exists t \in S)(sR_it$ and $\neg\alpha \in \mathbb{L}(t))$
H.19. if $\mathrm{N}_i\alpha \in \mathbb{L}(s)$ and (sR_it), then $\mathrm{N}_i\alpha \in \mathbb{L}(t)$
H.20. if $s \sim_i t$ and $s \sim_i u$ and $\mathrm{K}_i\alpha \in \mathbb{L}(t)$, then $\mathrm{K}_i\alpha \in \mathbb{L}(u)$ and $\alpha \in \mathbb{L}(u)$
H.21. if $\mathrm{K}_i\alpha \in \mathbb{L}(s)$, then $\mathrm{N}_i\alpha \in \mathbb{L}(s)$
H.22. if $\mathrm{AG}\alpha \in \mathbb{L}(s)$, then $\mathrm{N}_i\alpha \in \mathbb{L}(s)$

Note that the labelling rules are of the form "if" and not "if and only if". They provide the requirements that must be satisfied by a valid labelling (i.e., consistent with semantics rules), but they do not require that the formulas belonging to $\mathbb{L}(s)$ form a maximal set of formulas, for any $s \in S$. This means that there are formulas of $FL(\varphi)$ that are satisfied in a given state but they are not included in the label of that state. As usually, we call the rules $H1$-$H8$, $H13$,

$H16$, $H21$ and $H22$ *propositional consistency rules*, the rules $H9$, $H10$, $H14$, $H15$, $H17$-$H20$ *local consistency rules*, and the rules $H11$ and $H12$ *eventuality rules*.

The following lemma holds.

Lemma 3. *Let φ be a* CTLKR formula. The following holds:

(a). If $M = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathcal{V}, (R_i)_{i\in\mathcal{A}g})$ is a model for φ, then $M_H = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathbb{L}, (R_i)_{i\in\mathcal{A}g})$ with $\mathbb{L}$ defined by: for all $s \in S$, $\alpha \in \mathbb{L}(s)$ if $\alpha \in FL(\varphi)$ and $(M, s) \models \alpha$ is a Hintikka structure for φ.

(b). If $M_H = (S, T, (\sim_i)_{i\in\mathcal{A}g}\mathbb{L}, (R_i)_{i\in\mathcal{A}g})$ is a Hintikka structure for φ, then $M = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathcal{V}, (R_i)_{i\in\mathcal{A}g})$ with $\mathcal{V}$ defined by: $\mathcal{V}(s) = \mathbb{L}(s) \cap \mathcal{PV}$, for all $s \in S$, is a model for φ.

Proof.

(a). Straightforward by induction on the length of φ.

(b). By induction on the length of φ. The lemma follows directly for the propositional variables.

Next, assume that the hypothesis holds for all the proper subformulas of φ. Consider φ to be of the following forms:

- $\varphi = \neg\alpha$. Let $\neg\alpha \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H1$), we have that $\alpha \notin \mathbb{L}(s)$. Therefore, by case (a) of the lemma, and the contraposition law, we have that $(M, s) \not\models \alpha$. By the definition of $\models$, we conclude that $(M, s) \models \neg\alpha$.
- $\varphi = \alpha \vee \beta$. Let $\alpha \vee \beta \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H3$), we have that $\alpha \in \mathbb{L}(s)$ or $\beta \in \mathbb{L}(s)$. By the inductive assumption, we have that $(M, s) \models \alpha$ or $(M, s) \models \beta$. By the definition of $\models$, we conclude that $(M, s) \models \alpha \vee \beta$.
- $\varphi = \mathrm{EX}\alpha$. Let $\mathrm{EX}\alpha \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H9$), we have that $(\exists t \in S)$ such that $(s, t) \in T$ and $\alpha \in \mathbb{L}(t)$. Thus, by the inductive assumption, we have that $(\exists t \in S)$ such that $(s, t) \in T$ and $(M, t) \models \alpha$. By the definition of $\models$, we conclude that $(M, s) \models \mathrm{EX}\alpha$.
- $\varphi = \mathrm{AX}\alpha$. Let $\mathrm{AX}\alpha \in \mathbb{L}(s)$ for some $s \in S$. By the definition of AX, we have that $\mathrm{AX}\alpha = \neg\mathrm{EX}(\neg\alpha)$. So, we have that $\neg\mathrm{EX}(\neg\alpha) \in \mathbb{L}(s)$. By the definition of $\mathbb{L}$ (rules $H2$ and $H10$), we have that $(\forall t \in S)$ if $(s, t) \in T$ then $\alpha \in \mathbb{L}(t)$. Thus, by the inductive assumption, we have that $(\forall t \in S)$ if $(s, t) \in T$ then $(M, t) \models \alpha$. By the definition of $\models$, we conclude that $(M, s) \models \mathrm{AX}\alpha$.
- $\varphi = \mathrm{E}(\alpha\mathrm{U}\beta)$. Let $\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H11$), we have that $(\exists\pi \in \Pi(s))$ such that for some $n \geq 0$, $\beta \in \mathbb{L}(\pi(n))$ and $(\forall j < n)\ \alpha \in \mathbb{L}(\pi(j))$. Thus, by the inductive assumption, we have that $(\exists\pi \in \Pi(s))$ such that for some $n \geq 0$, $(M, \pi(n)) \models \beta$ and $(\forall j < n)(M, \pi(j)) \models \alpha$. By the definition of $\models$, we conclude $(M, s) \models \mathrm{E}(\alpha\mathrm{U}\beta)$.
- $\varphi = \mathrm{A}(\alpha\mathrm{U}\beta)$. Let $\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H12$), we have that $(\forall\pi \in \Pi(s))(\exists n \geq 0)(\beta \in \mathbb{L}(\pi(n))$ *and* $(\forall j < n)\alpha \in \mathbb{L}(\pi(j)))$. Thus, by the inductive assumption, we have that

$(\forall \pi \in \Pi(s))(\exists n \geq 0)((M, \pi(n)) \models \beta$ *and* $(\forall j < n)(M, \pi(j)) \models \alpha)$. By the definition of $\models$, we conclude that $(M, s) \models \mathrm{A}(\alpha \mathrm{U} \beta)$.

- $\varphi = \mathrm{K}_i\alpha$. Let $\mathrm{K}_i\alpha \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H14$), we have that $(\forall t \in S)$ if $s \sim_i t$ then $\alpha \in \mathbb{L}(t)$. Thus, by the inductive assumption, we have that $(\forall t \in S)$ if $s \sim_i t$ then $(M, t) \models \alpha$. By the definition of $\models$, we conclude $(M, s) \models \mathrm{K}_i\alpha$.
- $\varphi = \mathrm{N}_i\alpha$. Let $\mathrm{N}_i\alpha \in \mathbb{L}(s)$ for some $s \in S$. By the definition of $\mathbb{L}$ (rule $H17$), we have that $(\forall t \in S)$ if sR_it then $\alpha \in \mathbb{L}(t)$. Thus, by the inductive assumption, we have that $(\forall t \in S)$ if sR_it then $(M, t) \models \alpha$. By the definition of $\models$, we conclude $(M, s) \models \mathrm{N}_i\alpha$.

Definition 4 (Quotient structure). *Let* φ *be a* CTLKR formula, $M = (S, T, (\sim_i)_{i \in \mathcal{A}g}, \mathcal{V}, (R_i)_{i \in \mathcal{A}g})$ a model for φ, and $\leftrightarrow_{FL(\varphi)}$ a binary relation on S defined by $s \leftrightarrow_{FL(\varphi)} s'$ if $(\forall \alpha \in FL(\varphi))((M, s) \models \alpha$ iff $(M, s') \models \alpha)$. Moreover, let $[s]$ denote the set $\{w \in S \mid w \leftrightarrow_{FL(\varphi)} s\}$. The *quotient structure of* M *by* $\leftrightarrow_{FL(\varphi)}$ is defined as $M_{\leftrightarrow_{FL(\varphi)}} = (S', T', (\sim'_i)_{i \in \mathcal{A}g}, \mathbb{L}', (R'_i)_{i \in \mathcal{A}g})$, where $S' = \{[s] \mid s \in S\}$, $T' = \{([s], [s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ s.t. $(w, w') \in T\}$, $\sim'_i = \{([s], [s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ s.t. $(w, w') \in \sim_i\}$, $\mathbb{L}' : S' \to 2^{FL(\varphi)}$ is defined by: $\mathbb{L}'([s]) = \{\alpha \in FL(\varphi) \mid (M, s) \models \alpha\}$, and $R'_i = T'^{*} \cap \sim'_i$.

Observation. Note that as in the CTL case, the resulting quotient structure is finite and it may not be a model. In particular, the following lemma holds; the proof of Theorem 3.6 in [5] can easily be extended to tackle the proof of Lemma 4.

Lemma 4. *The quotient construction does not preserve satisfiability of formulas of the form* $\mathrm{A}(\alpha \mathrm{U} \beta)$*, where* $\alpha, \beta \in \mathcal{WF}$*. In particular, there is a model* M *for* $\mathrm{A}(\top \mathrm{U} p)$ *with* $p \in \mathcal{PV}$ *such that* $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}$ *is not a model for* $\mathrm{A}(\top \mathrm{U} p)$*.*

Proof. Consider a model M for the formula $\mathrm{A}(\top \mathrm{U} p)$ that is shown on Figure 1. We clearly have that for all $0 \leq i \leq n$, $(M, s_i) \models \mathrm{A}(\top \mathrm{U} p)$.

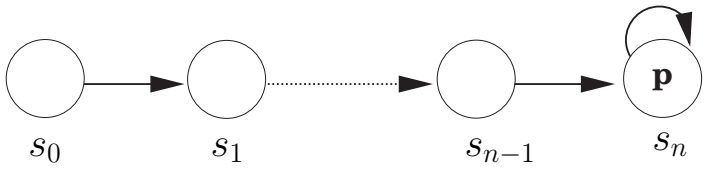

Fig. 1. A model M for the formula $\mathrm{A}(\top \mathrm{U} p)$.

The quotient construction of M by $\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}$, i.e., $M_{\leftrightarrow_{FL(\varphi)}}$, is shown on Figure 2. It is easy to see that the path $\pi = \pi(0), \pi(1), \ldots = (s'_0)^{\omega}$ in $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}$ meets the following condition: for all $i \geq 0$, $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}, \pi(i) \models \neg p$. So, $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}$ is not a model for $\mathrm{A}(\top \mathrm{U} p)$.

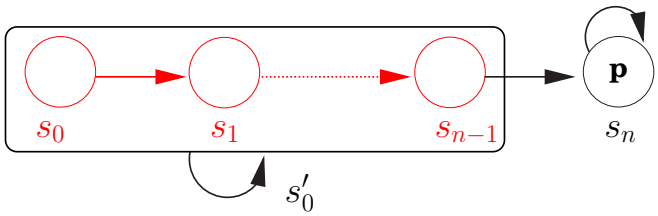

Fig. 2. The quotient construction of M by $\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}$.

Although $M_{\leftrightarrow_{FL(\varphi)}}$ may not be a model, it satisfies another important property, which allows us to view it as a *pseudo-model*; it can be unwound into a proper model that can be used to show that CTLKR has the FMP property. To make this idea precise, we introduce the following auxiliary definitions.

A *directed acyclic graph* is a pair $DAG = (S, T)$, where S is a set of states (nodes) and $T \subseteq S \times S$ is a set of edges (a transition relation). An *interior* (respectively *frontier*) node of a DAG is one which has (respectively does not have) a T-successor. The *root* of a DAG is the node (if it exists) from which all other nodes are reachable. A *fragment* $M' = (S', T', (\sim'_i)_{i\in\mathcal{A}g}, \mathbb{L}', (R'_i)_{i\in\mathcal{A}g})$ of a Hintikka structure $M_H = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathbb{L}, (R_i)_{i\in\mathcal{A}g})$ is a structure such that (S', T') is a finite DAG, in which the interior nodes satisfy $H1$-$H10$ and $H13$-$H22$, and the frontier nodes satisfy $H1$-$H8$, and $H13$, $H16$, $H19$-$H22$. Given $M = (S, T, (\sim_i)_{i\in\mathcal{A}g}, \mathbb{L}, (R_i)_{i\in\mathcal{A}g})$ and $M' = (S', T', (\sim'_i)_{i\in\mathcal{A}g}, \mathbb{L}', (R'_i)_{i\in\mathcal{A}g})$, we say that M is *contained* in M', and write $M \subseteq M'$, if $S \subseteq S'$, $T = T' \cap (S \times S)$, $\sim_i = \sim'_i \cap (S \times S)$, $\mathbb{L} = \mathbb{L}'|S$, $R_i = R'_i \cap (S \times S)$.

The following holds.

Lemma 5. *Let φ be a* CTLKR formula, M a model for φ, and $M' = (S', T', (\sim'_i)_{i\in\mathcal{A}g}, \mathbb{L}', (R'_i)_{i\in\mathcal{A}g})$ the quotient structure of M by $\leftrightarrow_{FL(\varphi)}$. Suppose $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}'([s])$ for some $[s] \in S'$. Then there is a fragment $(S'', T'', (\sim''_i)_{i\in\mathcal{A}g}, \mathbb{L}'', (R''_i)_{i\in\mathcal{A}g}) \subseteq M'$ such that: (a) (S'', T'') is a DAG with root $[s]$; (b) for all the frontier nodes $[t] \in S''$, $\beta \in \mathbb{L}''([t])$; (c) for all the interior nodes $[u] \in S''$, $\alpha \in \mathbb{L}''([u])$.

Proof. Assume that in M each state has a finite number of successors. Then, choose $t \in [s]$. It is easy to see that embedded in M there is a fragment rooted at t of the form claimed by the lemma. Simply take all states on paths that start at t, finish at state t' containing β (i.e., $M, t' \models \beta$ holds), and for all states u between t and t' u contains α. This must be a finite DAG.

If the labels on the states are all distinct (i.e., all the state on the path are distinct), then this fragment is also contained in M' and we are finished. If not, we will systematically eliminate "duplicate" states from this fragment until we finally obtain a fragment which is contained in M'.

We proceed as follows. Define the *depth* of a state t, $d(t)$, in a DAG as the length of the longest path from the root to t. Then suppose that we have two

distinct states t_1 and t_2 such that $d(t_1) \geq d(t_2)$ and $M, t_1 \models \psi$ iff $M, t_2 \models \psi$ for each $\psi \in FL(\varphi)$. We let the deeper state t_1 replace the state t_2 to get a new fragment, i.e. we replace each arc (u, t_2) by the arc (u, t_1) and eliminate all states no longer reachable from the root. Note that t_2 itself is no longer reachable from the root, so it is eliminated.

The resulting graph is easily seen to still be a fragment rooted at s such that for all frontier nodes t', $M, t' \models \beta$ holds, and for interior nodes u, $M, u \models \alpha$ holds. We continue this process until the labels (i.e., the formulas that are true at a given state) on all the states are distinct. This process must terminate after a finite number of steps since the original fragment was finite. The resulting fragment is contained in M' and meets the conditions of the lemma.

If the original model M had one or more states with an infinite number of successors, a structure M'' with no such states is constructed as follows. For each state t and each formula of the form $\mathrm{AX}\alpha$ (or $\mathrm{EX}\alpha$) that belongs to $FL(\varphi)$ and $M, t \models \mathrm{AX}\alpha$ ($M, t \models \mathrm{EX}\alpha$) holds, an arc $(t, u) \in T$ with $M, u \models \alpha$ is chosen. Then the not chosen edges are eliminated. Let the resulting relation be T'' and let $M'' = (S, T'', \sim_1, \ldots, \sim_n, \mathbb{L}, V_1, \ldots, V_n)$. Each node of M'' has a finite number of successors, and it is easy to check that we can carry out the above construction using M'' instead of M since $H12$ still holds (although in general M'' is not a model for φ since the eliminated arcs may have been necessary for fulfillment of formulas such as $\mathrm{E}(\alpha \mathrm{U} \beta)$).

Definition 5 (Pseudo-model). *Let φ be a* CTLKR formula. A *pseudo-model* $M = (S, T, (\sim_i)_{i \in \mathcal{A}g}, \mathbb{L}, (R_i)_{i \in \mathcal{A}g})$ for φ is defined in the same manner as a Hintikka structure for φ in Definition 3, except that condition $H12$ is replaced by the following condition $H'12$: for all $s \in S$, if $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}(s)$, then there is a fragment $(S', T', (\sim'_i)_{i \in \mathcal{A}g}, \mathbb{L}', (R'_i)_{i \in \mathcal{A}g}) \subseteq M$ such that: (a) (S', T') is a DAG with root s; (b) for all frontier nodes $t \in S'$, $\beta \in \mathbb{L}'(t)$; (c) for all interior nodes $u \in S'$, $\alpha \in \mathbb{L}'(u)$.

It can easy be checked that the following lemma holds.

Lemma 6. *Let φ be a* CTLKR formula, $FL(\varphi)$ the Fischer-Ladner closure of φ, M a model for φ, and $M_{\leftrightarrow_{FL(\varphi)}}$ the quotient structure of M by $\leftrightarrow_{FL(\varphi)}$. Then, $M_{\leftrightarrow_{FL(\varphi)}}$ is a pseudo-model for φ.

Proof. Consider φ to be of the following forms:

1. $\varphi = \neg\alpha$. Assume that $(M, s) \models \neg\alpha$ and $\neg\alpha \in \mathbb{L}([s])$. Since $(M, s) \models \neg\alpha$, we have that $(M, s) \not\models \alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we conclude that $\alpha \notin \mathbb{L}([s])$. So, condition $H1$ is fulfilled.
2. $\varphi = \neg\neg\alpha$. Assume that $(M, s) \models \neg\neg\alpha$ and $\neg\neg\alpha \in \mathbb{L}([s])$. Since $(M, s) \models \neg\neg\alpha$, we have that $(M, s) \models \alpha$. Then, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([s])$. So, condition $H2$ is fulfilled.
3. $\varphi = \alpha \vee \beta$. Assume that $(M, s) \models \alpha \vee \beta$ and $\alpha \vee \beta \in \mathbb{L}([s])$. Since $(M, s) \models \alpha \vee \beta$, we have that $(M, s) \models \alpha$ or $(M, s) \models \beta$. Consider the following two cases.

– Assume $(M,s) \models \alpha$. By the definition of $\leftrightarrow_{FL(\varphi)}$, we have that for all $t \in [s]$, $(M,t) \models \alpha$ hold. Thus, by the definition of $\mathbb{L}$, we obtain that $\alpha \in \mathbb{L}([s])$.
– Assume $(M,s) \models \beta$. As above we obtain $\beta \in \mathbb{L}([s])$.

Thus, condition $H3$ is fulfilled.

4. $\varphi = \neg(\alpha \vee \beta)$. Assume that $(M,s) \models \neg(\alpha \vee \beta)$ and $\neg(\alpha \vee \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,s) \models \neg\alpha$ and $(M,s) \models \neg\beta$. By the definition of $\leftrightarrow_{FL(\varphi)}$, we have that $(M,t) \models \neg\alpha$ and $(M,t) \models \neg\beta$ for all $t \in [s]$. Thus, by the definition of $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([s])$ and $\neg\beta \in \mathbb{L}([s])$. So, condition $H4$ is fulfilled.
5. $\varphi = \mathrm{E}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \mathrm{E}(\alpha \mathrm{U} \beta)$ and $\mathrm{E}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $\exists \pi \in \Pi(s)$ such that for some $n \geq 0$, $(M, \pi(n)) \models \beta$ and $(\forall j < n)(M, \pi(j)) \models \alpha$. This implies that either $(M, \pi(0)) \models \beta$ or $\exists n > 0$ such that $(M, \pi(n)) \models \beta$ and $(\forall j < n)(M, \pi(j)) \models \alpha$. Thus, $(M,s) \models \beta$ or $(M,s) \models \alpha \wedge \mathrm{EX}(\alpha \mathrm{EU} \beta)$. Therefore, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we obtain that $\beta \in \mathbb{L}([s])$ or $\alpha \wedge \mathrm{EX}(\alpha \mathrm{EU} \beta) \in \mathbb{L}([s])$. So, condition $H5$ is fulfilled.
6. $\varphi = \neg\mathrm{E}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \neg\mathrm{E}(\alpha \mathrm{U} \beta)$ and $\neg\mathrm{E}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(\forall \pi \in \Pi(s))(\forall n \geq 0)((M, \pi(n)) \models \neg\beta$ or $(\exists j < n)(M, \pi(j)) \models \neg\alpha)$. This implies that either $(M, \pi(0)) \models \neg\alpha \wedge \neg\beta$ or $(\forall \pi \in \Pi(s))(\forall n > 0)((M, \pi(n)) \models \neg\beta$ or $(\exists j < n)(M, \pi(j)) \models \neg\alpha)$. Thus $(M,s) \models \neg\beta \wedge \neg\alpha$ or $(M,s) \models \neg\beta \wedge \neg\mathrm{EXE}(\alpha \mathrm{U} \beta)$. Therefore, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $\neg\beta \wedge \neg\alpha \in \mathbb{L}([s])$ or $\neg\beta \wedge \neg\mathrm{EXE}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. So, condition $H6$ is fulfilled.
7. $\varphi = \mathrm{A}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \mathrm{A}(\alpha \mathrm{U} \beta)$ and $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(\forall \pi \in \Pi(s))(\exists n \geq 0)[(M, \pi(n)) \models \beta$ and $(\forall j < n)\ (M, \pi(j)) \models \alpha]$. This implies that either $(M,s) \models \beta$ or $(\forall \pi \in \Pi(s))(\exists n > 0)\ [(M, \pi(n)) \models \beta$ and $(\forall j < n)(M, \pi(j)) \models \alpha]$. Thus, $(M,s) \models \beta$ or $(M,s) \models \alpha \wedge \mathrm{AXA}(\alpha \mathrm{U} \beta)$, which is equivalent to the fact that $(M,s) \models \beta$ or $(M,s) \models \alpha \wedge \neg\mathrm{EX}(\neg\mathrm{A}(\alpha \mathrm{U} \beta))$. Therefore, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\beta \in \mathbb{L}([s])$ or $\alpha \wedge \neg\mathrm{EX}(\neg\mathrm{A}(\alpha \mathrm{U} \beta)) \in \mathbb{L}([s])$. So, condition $H7$ is fulfilled.
8. $\varphi = \neg\mathrm{A}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \neg\mathrm{A}(\alpha \mathrm{U} \beta)$ and $\neg\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $\exists \pi \in \Pi(s)$ such that $\forall n \geq 0$ either $(M, \pi(n)) \models \neg\beta$ or $\exists j < n$ such that $(M, \pi(j)) \models \neg\alpha$. This implies that either $(M,s) \models \neg\alpha \wedge \neg\beta$ or $\exists \pi \in \Pi(s)$ such that $\forall n > 0$ either $(M, \pi(n)) \models \neg\beta$ or $\exists j < n$ with $(M, \pi(j)) \models \neg\alpha$. Thus, $(M,s) \models \neg\beta \wedge \neg\alpha$ or $(M,s) \models \neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha \mathrm{U} \beta))$. By the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \wedge \neg\beta \in \mathbb{L}([s])$ or $\neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha \mathrm{U} \beta)) \in \mathbb{L}([s])$. So, condition $H8$ is fulfilled.
9. $\varphi = \mathrm{EX}\alpha$. Assume that $(M,s) \models \mathrm{EX}\alpha$ and $\mathrm{EX}\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have $\exists t \in S$ such that $(s,t) \in T$ and $(M,t) \models \alpha$. Thus by definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([t])$. Since $(s,t) \in T$, by the definition of T', we have that $([s],[t]) \in T'$. Therefore, $\exists [t] \in S'$ such that $([s],[t]) \in T'$ and $\alpha \in \mathbb{L}([t])$. So, condition $H9$ is fulfilled.

10. $\varphi = \neg \mathrm{EX}\alpha$. Assume that $(M,s) \models \neg \mathrm{EX}\alpha$ and $\neg \mathrm{EX}\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(\forall t \in S)$ if $(s,t) \in T$ then $(M,t) \models \neg\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([t])$ for all $t \in S$ such that $(s,t) \in T$. Since $(s,t) \in T$, by the definition of T', we have that $([s],[t]) \in T'$. Therefore, we conclude that $\forall [t] \in S'$ if $([s],[t]) \in T'$ then $\neg\alpha \in \mathbb{L}([t])$. So, condition $H10$ is fulfilled.
11. $\varphi = \mathrm{E}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \mathrm{E}(\alpha \mathrm{U} \beta)$ and $\mathrm{E}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By the definition of $\models$, we have that $\exists \pi \in \mathit{\Pi}(s)$ and $\exists n \geq 0$ such that $(M, \pi(n)) \models \beta$ and $(\forall j < n)$, $(M, \pi(j)) \models \alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\beta \in \mathbb{L}([\pi(n)])$ and $(\forall j < n)$, $\alpha \in \mathbb{L}([\pi(j)])$. Moreover, by the definition of T' we have that the path $([\pi(0)],[\pi(1)],\ldots,[\pi(n)],\ldots)$ belongs to $\mathit{\Pi}([s])$. Therefore, $\exists \pi \in \mathit{\Pi}([s])$ and $\exists n \geq 0$ such that $\beta \in \mathbb{L}([\pi(n)])$ and $(\forall j < n)$, $\alpha \in \mathbb{L}([\pi(j)])$. So, condition $H11$ is fulfilled.
12. $\varphi = \mathrm{A}(\alpha \mathrm{U} \beta)$. Assume that $(M,s) \models \mathrm{A}(\alpha \mathrm{U} \beta)$ and $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}([s])$. By Lemma 5, we have that there is a fragment $(S'', T'', (\sim''_i)_{i \in \mathcal{A}g}, \mathbb{L}'', (R''_i)_{i \in \mathcal{A}g}) \subseteq M_{\leftrightarrow_{FL(\varphi)}}$ such that: (a) (S'', T'') is a DAG with root $[s]$; (b) for all the frontier nodes $[t] \in S''$, $\beta \in \mathbb{L}''([t])$; (c) for all the interior nodes $[u] \in S''$, $\alpha \in \mathbb{L}''([u])$. So, condition $H'12$ is fulfilled.
13. $\varphi = \mathrm{K}_i\alpha$. Assume that $(M,s) \models \mathrm{K}_i\alpha$ and $\mathrm{K}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that $s \sim_i t$. So, in particular we have that $(M,s) \models \alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([s])$. So, condition $H13$ is fulfilled.
14. $\varphi = \mathrm{K}_i\alpha$. Assume that $(M,s) \models \mathrm{K}_i\alpha$ and $\mathrm{K}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that $s \sim_i t$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([t])$ for all $t \in S$ such that $s \sim_i t$. Therefore, by the definition of $\sim'_i$ we conclude that $\forall [t] \in S'$ if $[s] \sim'_i [t]$ then $\alpha \in \mathbb{L}([t])$. So, condition $H14$ is fulfilled.
15. $\varphi = \neg\mathrm{K}_i\alpha$. Assume that $(M,s) \models \neg\mathrm{K}_i\alpha$ and $\neg\mathrm{K}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $\exists t \in S$ such that $s \sim_i t$ and $(M,t) \models \neg\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([t])$. Therefore, by the definition of $\sim'_i$ we conclude that $\exists [t] \in S'$ such that $[s] \sim'_i [t]$ and $\neg\alpha \in \mathbb{L}([t])$. So, condition $H15$ is fulfilled.
16. $\varphi = \mathrm{N}_i\alpha$. Assume that $(M,s) \models \mathrm{N}_i\alpha$ and $\mathrm{N}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that sR_it. Since the relation R_i is reflexive, we have that $(M,s) \models \alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([s])$. So, condition $H16$ is fulfilled.
17. $\varphi = \mathrm{N}_i\alpha$. Assume that $(M,s) \models \mathrm{N}_i\alpha$ and $\mathrm{N}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that sR_it. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([t])$ for all $t \in S$ such that sR_it. Therefore, by the definition of R'_i we conclude that for all $[t] \in S'$ if $[s]R'_i[t]$ then $\alpha \in \mathbb{L}([t])$. So, condition $H17$ is fulfilled.
18. $\varphi = \neg\mathrm{N}_i\alpha$. Assume that $(M,s) \models \neg\mathrm{N}_i\alpha$ and $\neg\mathrm{N}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that there exists $t \in S$ such that sR_it and $(M,t) \models \neg\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([t])$. Therefore, by the definition of R'_i we conclude that there exists $[t] \in S'$ such that $[s]R'_i[t]$ and $\neg\alpha \in \mathbb{L}([t])$. So, condition $H18$ is fulfilled.

19. $\varphi = \mathrm{N}_i\alpha$. Assume that $(M,s) \models \mathrm{N}_i\alpha$ and $[s]R'_i[t]$ and $\mathrm{N}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that sR_it. Consider any state t such that sR_it and any state u such that tR_iu. Since the relation R_i is transitive, we have that sR_iu. Thus since $(M,s) \models \mathrm{N}_i\alpha$, it follows that $(M,u) \models \alpha$. Thus, for all t such that sR_it we have that $(M,t) \models \mathrm{N}_i\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\mathrm{N}_i\alpha \in \mathbb{L}([t])$ for all $[t]$ such that $[s]R'_i[t]$. So, condition $H19$ is fulfilled.
20. Assume that $[s] \sim'_i [t]$ and $[s] \sim'_i [u]$ and $\mathrm{K}_i\alpha \in \mathbb{L}([t])$. Since $[s] \sim'_i [t]$ and $[s] \sim'_i [u]$ and $\sim'_i$ is reflexive and transitive, we have that $[t] \sim'_i [u]$. Thus, since $\mathrm{K}_i\alpha \in \mathbb{L}([t])$, by the case 14 of the proof, we have that $\alpha \in \mathbb{L}([u])$. In order to show that $\mathrm{K}_i\alpha \in \mathbb{L}([u])$, consider $[v] \in S'$ such that $[u] \sim'_i [v]$. Since $[t] \sim'_i [u]$ and $\sim'_i$ is transitive, we have that $[t] \sim'_i [v]$. Thus, by the case 14 of the proof, we have that $\alpha \in \mathbb{L}([v])$ for each $[v]$ such that $[u] \sim'_i [v]$. This implies that $\mathrm{K}_i\alpha \in \mathbb{L}([u])$. Therefore, condition $H20$ is fulfilled.
21. $\varphi = \mathrm{K}_i\alpha$. Assume $(M,s) \models \mathrm{K}_i\alpha$ and $\mathrm{K}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M,t) \models \alpha$ for all $t \in S$ such that $s \sim_i t$. Consider the following two sets $K(s,i) = \{t \mid (s \sim_i t)$ and $(M,t) \models \alpha\}$ and $Reach(s) = \{t \mid (sT^*t)\}$. By the definition of $K(s,i)$ and $Reach(s)$, we have that $K(s,i) \cap Reach(s) = \{t \mid (sR_it)$ and $M,t \models \alpha\}$. Therefore, by the definition of $\models$ we have that $(M,s) \models \mathrm{N}_i\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\mathrm{N}_i\alpha \in \mathbb{L}([s])$. So, condition $H21$ is fulfilled.
22. $\varphi = \mathrm{AG}_i\alpha$. Assume $(M,s) \models \mathrm{AG}_i\alpha$ and $\mathrm{AG}\alpha \in \mathbb{L}([s])$. Moreover, let $\Pi(s)$ denote all the paths that start at state s in the model M, and $States(\Pi(s))$ denote the set of states that belong to paths in $\Pi(s)$ (i.e., $States(\Pi(s)) = \{s \mid (\exists \pi \in \Pi(s))(\exists i \geq 0)\pi(i) = s\}$). Then, by the definition of $\models$, we have that for all the states $t \in States(\Pi(s))$, $(M,t) \models \alpha$ holds. Consider the following set: $\mathcal{R}(i,s) = \{t \mid t \in States(\Pi(s))$ and $s \sim_i t\}$. It follows that $\mathcal{R}(i,s)$ defines all the states $t \in S$ such that sR_it. So, since for all $t \in States(\Pi(s))$, $(M,t) \models \alpha$, it follows that $(M,s) \models \mathrm{N}_i\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\mathrm{N}_i\alpha \in \mathbb{L}([s])$. So, condition $H22$ is fulfilled.

Theorem 1. CTLKR has the finite model property.

Proof (sketch). To prove the theorem it is sufficient to show that for a given CTLKR formula φ the following conditions are equivalent: (1) φ is satisfiable; (2) there is a finite pseudo-model for φ; (3) there is a Hintikka structure for φ.

(3) $\Rightarrow$ (1) follows from Lemma 3, part (b). (1) $\Rightarrow$ (2) follows from Lemma 6. To prove (2) $\Rightarrow$ (3) it is enough to construct a Hintikka structure for φ by "unwinding" the pseudo-model for φ. This can be done in the same way as described in [5] for the proof of Theorem 4.1.

4 Decidability for CTLKR

Let φ be a CTLKR formula, and $FL(\varphi)$ the Fischer-Ladner closure of φ. We define $\Delta \subseteq FL(\varphi)$ to be *maximal* if for every formula $\alpha \in FL(\varphi)$, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Theorem 2. *There is an algorithm for deciding whether any* CTLKR formula is satisfiable.

Proof. Given a CTLKR formula φ, we construct a pseudo-model for φ. We proceed as follows.

1. Build an initial pseudo-model $M^0 = (S^0, T^0, (\sim_i^0)_{i\in Ag}, \mathbb{L}^0, (R_i^0)_{i\in Ag})$ for φ with the following constraints:
   - $S^0 = \{\Delta \mid \Delta \subseteq FL(\varphi)$ and Δ is maximal and satisfies all the propositional consistency rules$\}$;
   - $T^0 \subseteq S^0 \times S^0$ is the relation such that $(\Delta_1, \Delta_2) \in T^0$ iff $\neg \mathrm{EX}\alpha \in \Delta_1$ implies that $\neg\alpha \in \Delta_2$;
   - for each agent $i \in \mathcal{A}g$, $\sim_i^0 \subseteq S^0 \times S^0$ is the relation such that $(\Delta_1, \Delta_2) \in \sim_i$ iff $\{\alpha \mid \mathrm{K}_i\alpha \in \Delta_1\} \subseteq \Delta_2$;
   - $\mathbb{L}^0(\Delta) = \Delta$;
   - for each agent $i \in \mathcal{A}g$, $R_i^0 \subseteq S^0 \times S^0$ is the relation such that $(\Delta_1, \Delta_2) \in R_i$ iff $\{\alpha \mid \mathrm{N}_i\alpha \in \Delta_1\} \subseteq \Delta_2$

   Note that the initial pseudo-model satisfies all the propositional consistency properties; property $H10$ (because of the definition of T^0), property $H14$ (because of the definition of $\sim_i^0$), and property $H17$ (because of the definition of R_i^0).
2. Test the initial pseudo-model for fulfilment of the properties $H9$, $H11$, $H'12$, $H15$, and $H18{-}H20$ by repeatedly applying the following deletion rules until no more states in M^0 can be deleted.
   (a) Delete any state which has no T^0-successors.
   (b) Delete any state $\Delta_1 \in S^0$ such that $\mathrm{E}(\alpha \mathrm{U} \beta) \in \Delta_1$ (resp. $\mathrm{A}(\alpha \mathrm{U} \beta) \in \Delta_1$) and there does not exist a fragment $M'' \subseteq M^0$ such that: (i) (S'', T'') is a DAG with root Δ_1; (ii) for all frontier nodes $\Delta_2 \in S''$, $\beta \in \Delta_2$; (iii) for all interior nodes $\Delta_3 \in S''$, $\alpha \in \Delta_3$.
   (c) Delete any state $\Delta_1 \in S^0$ such that $\neg \mathrm{K}_i\alpha \in \Delta_1$, and Δ_1 does not have any $\sim_i$ successor $\Delta_2 \in S^0$ with $\neg\alpha \in \Delta_2$.
   (d) Delete any state $\Delta_1 \in S^0$ such that $\neg \mathrm{N}_i\alpha \in \Delta_1$, and Δ_1 does not have any R_i successor $\Delta_2 \in S^0$ with $\neg\alpha \in \Delta_2$.
   (e) Delete any two states $\Delta_1, \Delta_2 \in S^0$ such that $\Delta_1 R_i \Delta_2$ and $\mathrm{N}_i\alpha \in \Delta_1$ and $\neg \mathrm{N}_i\alpha \in \Delta_2$.
   (f) Delete any three states $\Delta_1, \Delta_2, \Delta_3 \in S^O$ such that $\Delta_1 \sim_i \Delta_2$ and $\Delta_1 \sim_i \Delta_3$ and $\mathrm{K}_i\alpha \in \Delta_2$ and ($\neg \mathrm{K}_i\alpha \in \Delta_3$ or $\neg\alpha \in \Delta_3$).

   Note that this part of the algorithm must terminate, since there are only a finite number of states in the pseudo-model.

We call the algorithm above a *decidability algorithm for* CTLKR, and we can show that the following lemma holds.

Lemma 7. *The decidability algorithm for* CTLKR terminates. Let $M^f = (S^f, T^f, (\sim_i^f)_{i\in Ag}, \mathbb{L}^f, (R_i^f)_{i\in Ag})$ be the resulting structure of the algorithm. The CTLKR formula φ is satisfiable iff $\varphi \in s$, for some $s \in S^f$.

Proof (sketch). Termination is obvious given that the initial set is finite. In order to show the part right-to-left of the satisfaction property, note that either the resulting structure is a pseudo-model for φ, or $S^f = \emptyset$ (this can be shown inductively on the structure of the algorithm). Any pseudo-model for φ is a model for φ (see the proof of Theorem 1).

Conversely, if φ is satisfiable, then there exists a model M such that $M \models \varphi$. Let $M_{\leftrightarrow_{FL(\varphi)}} = (S', T', (\sim'_i)_{i\in\mathcal{A}g}, \mathbb{L}', (R'_i)_{i\in Ag})$ be the quotient structure of M by $\leftrightarrow_{FL(\varphi)}$. $M_{\leftrightarrow_{FL(\varphi)}}$ is a pseudo-model for φ (see the proof of Theorem 1). So, $\mathbb{L}'$ satisfies all the propositional consistency rules, the local consistency rules, and properties $H11$ and $H'12$. Moreover, by the definition of $\mathbb{L}'$ in the quotient structure, $\mathbb{L}'(s)$ is maximal with respect to $FL(\varphi)$ for all $s \in S'$.

Let us consider the following function $f : S' \to S^0$ that is defined by $f(s) = \mathbb{L}'(s)$. It is easy to check that for T^0, $\sim_i^0$, and R_i^0, defined as in step 1 of the decidability algorithm, the following conditions hold:

1. if $(s,t) \in T'$, then $(f(s), f(t)) \in T^0$;
   Proof (via contradiction): Let $(s,t) \in T'$ and $(f(s), f(t)) \notin T^0$. Then, by the definition of T^0 we have that $\neg\mathrm{EX}\alpha \in f(s)$ and $\alpha \in f(t)$. By the definition of f, we have that $\neg\mathrm{EX}\alpha \in \mathbb{L}'(s)$ and $\alpha \in \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M, s \models \neg\mathrm{EX}\alpha$ and $M, t \models \alpha$, which contradict the fact that $(s,t) \in T'$.
2. if $(s,t) \in \sim'_i$, then $(f(s), f(t)) \in \sim_i^0$;
   Proof (via contradiction): Let $(s,t) \in \sim'_i$ and $(f(s), f(t)) \notin \sim_i^0$. Then, by the definition of $\sim_i^0$ we have that $\mathrm{K}_i\alpha \in f(s)$ and $\alpha \notin f(t)$. By the definition of f, we have that $\mathrm{K}_i\alpha \in \mathbb{L}'(s)$ and $\alpha \notin \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M, s \models \mathrm{K}_i\alpha$ and $M, t \models \neg\alpha$, which contradict the fact that $(s,t) \in \sim'_i$.
3. if $(s,t) \in R'_i$, then $(f(s), f(t)) \in R_i^0$;
   Proof (via contradiction): Let $(s,t) \in \sim'_i$ and $(f(s), f(t)) \notin R_i^0$. Then, by the definition of R_i^0 we have that $\mathrm{N}_i\alpha \in f(s)$ and $\alpha \notin f(t)$. By the definition of f, we have that $\mathrm{N}_i\alpha \in \mathbb{L}'(s)$ and $\alpha \notin \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M, s \models \mathrm{N}_i\alpha$ and $M, t \models \neg\alpha$, which contradict the fact that $(s,t) \in R'_i$.

Thus, the image of M' under f is contained in M^f, $M_{\leftrightarrow_{FL(\varphi)}} \subseteq M^f$. It also can be checked that if $s \in S'$, then $f(s) \in S^0$ will not be eliminated via the step 2 of the decidability algorithm. So, in fact, $f(s) \in S^f$. This can be checked by induction on the order in which states of S^0 are eliminated. Therefore, it follows that for some $s \in S^f$ we have $\varphi \in \mathbb{L}^f(s)$.

5 A Complete Axiomatic System for CTLKR

Recall, an *axiomatic system* consists of a collection of *axioms schemes* and *inference rules*. An axiom scheme is a rule for generating an infinite number of axioms, i.e. formulas that are universally valid. An inference rule has the form "from formulas $\varphi_1, \ldots, \varphi_m$ infer formula φ". We say that φ is *provable* (written

$\vdash \varphi$) if there is a sequence of formulas ending with φ, such that each formula is either an instance of an axiom, or follows from other provable formulas by applying an inference rule. We say that a formula φ is *consistent* if $\neg\varphi$ is not provable. A finite set $\{\varphi_1, \ldots, \varphi_m\}$ of formulas is *consistent* exactly if and only if the conjunction $\varphi_1 \wedge \ldots \wedge \varphi_m$ of its members is consistent. A set F of formulas is a *maximally consistent set* if it is consistent and for all $\varphi \notin F$, the set $F \cup \{\varphi\}$ is inconsistent. An axiom system is *sound* (resp. *complete*) with respect to the class of models, if $\vdash \varphi$ implies $\models \varphi$ (resp. if $\models \varphi$ implies $\vdash \varphi$).

Let $i \in \{1, \ldots, n\}$. Consider system CTLKR as defined below:

PC. All substitution instances of classical tautologies.
T1. $\mathrm{EX}\top$
T2. $\mathrm{EX}(\alpha \vee \beta) \Leftrightarrow \mathrm{EX}\alpha \vee \mathrm{EX}\beta$
T3. $\mathrm{E}(\alpha \mathrm{U} \beta) \Leftrightarrow \beta \vee (\alpha \wedge \mathrm{EXE}(\alpha \mathrm{U} \beta))$
T4. $\mathrm{A}(\alpha \mathrm{U} \beta) \Leftrightarrow \beta \vee (\alpha \wedge \mathrm{AXA}(\alpha \mathrm{U} \beta))$
K1. $(\mathrm{K}_i\alpha \wedge \mathrm{K}_i(\alpha \Rightarrow \beta)) \Rightarrow \mathrm{K}_i\beta$
K2. $\neg\mathrm{K}_i\alpha \Rightarrow \mathrm{K}_i\neg\mathrm{K}_i\alpha$
F1. $(\mathrm{N}_i\alpha \wedge \mathrm{N}_i(\alpha \Rightarrow \beta)) \Rightarrow \mathrm{N}_i\beta$
F2. $\mathrm{N}_i\alpha \Rightarrow \alpha$
F3. $\mathrm{N}_i\alpha \Rightarrow \mathrm{N}_i\mathrm{N}_i\alpha$
F4. $\mathrm{K}_i\alpha \Rightarrow \mathrm{N}_i\alpha$
F5. $\mathrm{AG}\alpha \Rightarrow \mathrm{N}_i\alpha$
R1. From α and $\alpha \Rightarrow \beta$ infer β
R2. From α infer $\mathrm{K}_i\alpha$
R3. From $\alpha \Rightarrow \beta$ infer $\mathrm{EX}\alpha \Rightarrow \mathrm{EX}\beta$
R4. From $\gamma \Rightarrow (\neg\beta \wedge \mathrm{EX}\gamma)$ infer $\gamma \Rightarrow \neg\mathrm{A}(\alpha \mathrm{U} \beta)$
R5. From $\gamma \Rightarrow (\neg\beta \wedge \mathrm{AX}(\gamma \vee \neg\mathrm{E}(\alpha \mathrm{U} \beta)))$ infer $\gamma \Rightarrow \neg\mathrm{E}(\alpha \mathrm{U} \beta)$

Theorem 3. *The system* CTLKR is sound and complete with respect to the class of models of Definition 1, i.e. $\models \varphi$ iff $\vdash \varphi$, for any formula $\varphi \in \mathcal{WF}$.

Proof. Soundness can be checked inductively as standard. For completeness, it is sufficient to show that any consistent formula is satisfiable. To do this, first we construct a pseudo-model $M = (S^0, T^0, (\sim_i^0)_{i \in \mathcal{A}g}, \mathbb{L}^0, (R_i^0)_{i \in \mathcal{A}g})$ for φ just as in the decidability algorithm for CTLKR, and for each $s \in S^0$ we define the formula ψ_s as the conjunction of the formulas in s, i.e., $\psi_s = \bigwedge_{\alpha \in s} \alpha$. Next, we show that if a state $s \in S^0$ is eliminated at step 2 of the decidability algorithm for CTLKR, then ψ_s is inconsistent. Once we have shown this, we proceed as follows. It can be checked by propositional reasoning that for any $\alpha \in FL(\varphi)$ we have $\vdash \alpha \Leftrightarrow \bigvee_{\substack{\{s \mid \alpha \in s \text{ and} \\ \psi_s \text{ is consistent}\}}} \psi_s$. In particular, $\vdash \varphi \Leftrightarrow \bigvee_{\substack{\{s \mid \varphi \in s \text{ and} \\ \psi_s \text{ is consistent}\}}} \psi_s$. Thus, if φ is consistent, then ψ_s is consistent for some $s \in S^0$. This particular s will not be eliminated at step 2 of the decidability algorithm for CTLKR. Therefore, a pseudo-model for φ is obtained. So, by Theorem 1, φ is satisfiable.

Claim (1). Let $s \in S^0$ and $\alpha \in FL(\varphi)$. Then, $\alpha \in s$ iff $\vdash \psi_s \Rightarrow \alpha$.

Proof. ('if'). Let $\alpha \in s$. By the definition of S^0, we have that any s in S^0 is maximal. Thus, $\neg\alpha \notin s$. So, $\vdash \psi_s \Rightarrow \alpha$.
('only if'). Let $\vdash \psi_s \Rightarrow \alpha$. So, since s is maximal we have that $\alpha \in s$.

Claim (2). Let $i \in \mathcal{A}g$. If $(s,t) \notin \sim_i$ as constructed in step 1 of the decidability algorithm for CTLKR, then $\psi_t \wedge \psi_s$ is inconsistent.

Proof. Let $(s,t) \notin \sim_i$. Then, by the definition of $\sim_i$, we have that $\mathrm{K}_i\alpha \in s$ and $\alpha \notin t$, for some α. Thus, by maximality $\neg\alpha \in t$. So, we have $\vdash \psi_s \Rightarrow \mathrm{K}_i\alpha$ and $\vdash \psi_t \Rightarrow \neg\alpha$. By axiom $F2$ and $F4$ $\vdash \psi_s \Rightarrow \alpha$. Therefore, $\vdash (\psi_t \wedge \psi_s) \Rightarrow \neg\alpha \wedge \alpha$. Hence, $\psi_t \wedge \psi_s$ is inconsistent.

Claim (3). If $(s,t) \notin T$ as constructed at step 1 of the decidability algorithm for CTLKR, then $\psi_s \wedge \mathrm{EX}\psi_t$ is inconsistent.

Proof. Let $(s,t) \notin T$. By the definition of T we have that $\neg\mathrm{EX}\alpha \in s$ and $\alpha \in t$. Therefore, we have $\vdash \psi_s \Rightarrow \neg\mathrm{EX}\alpha$ and $\vdash \psi_t \Rightarrow \alpha$. By $R3$ we have $\vdash \mathrm{EX}\psi_t \Rightarrow \mathrm{EX}\alpha$. This implies that $\vdash (\psi_s \wedge \mathrm{EX}\psi_t) \Rightarrow (\neg\mathrm{EX}\alpha \wedge \mathrm{EX}\alpha)$. Thus $\vdash (\psi_s \wedge \mathrm{EX}\psi_t) \Rightarrow \bot$, which means that $\psi_s \wedge \mathrm{EX}\psi_t$ is inconsistent.

Claim (4). Let $i \in \mathcal{A}g$. If $(s,t) \notin R_i$ as constructed in step 1 of the decidability algorithm for CTLKR, then $\psi_t \wedge \psi_s$ is inconsistent.

Proof. Let $(s,t) \notin R_i$. Then, by the definition of R_i, we have that $\mathrm{N}_i\alpha \in s$ and $\alpha \notin t$, for some α. Thus, by maximality $\neg\alpha \in t$. So, we have $\vdash \psi_s \Rightarrow \mathrm{N}_i\alpha$ and $\vdash \psi_t \Rightarrow \neg\alpha$. By axiom $F2$ $\vdash \psi_s \Rightarrow \alpha$. Therefore, $\vdash (\psi_t \wedge \psi_s) \Rightarrow \neg\alpha \wedge \alpha$. Hence, $\psi_t \wedge \psi_s$ is inconsistent.

We now show, by induction on the structure of the decidability algorithm for CTLKR, that if a state $s \in S^0$ is eliminated, then $\vdash \neg\psi_s$.

Claim (5). If ψ_s is consistent, then s is not eliminated at step 2 of the decidability algorithm for CTLKR.

Proof. (a). Let $\mathrm{EX}\alpha \in s$ and ψ_s be consistent. By the same reasoning as in the proof of Claim 4(a) in [5], we conclude that s satisfies $H9$. So s is not eliminated.

(b). Let $\mathrm{E}(\alpha\mathrm{U}\beta) \in s$ (resp. $\mathrm{A}(\alpha\mathrm{U}\beta) \in s$) and suppose s is eliminated at step 2 because $H11$ (resp. $H'12$) is not satisfied. Then ψ_s is inconsistent. The proof showing that fact is the same as the proof of Claim 4(c) (resp. Claim 4(d)) in [5].

(c). Let $\neg\mathrm{K}_i\alpha \in s$ and ψ_s be consistent. Consider the set $S_{\neg\alpha} = \{\neg\alpha\} \cup \{\beta \mid \mathrm{K}_i\beta \in s\}$. We will show that $S_{\neg\alpha}$ is consistent. Suppose that $S_{\neg\alpha}$ is inconsistent. Then, $\vdash \beta_1 \wedge \ldots \wedge \beta_m \Rightarrow \alpha$, where $\beta_j \in \{\beta \mid \mathrm{K}_i\beta \in s\}$ for $j \in \{1,\ldots,m\}$. By rule $R2$ we have $\vdash \mathrm{K}_i((\beta_1 \wedge \ldots \wedge \beta_m) \Rightarrow \alpha)$. By axioms $K1$ and PC we have $\vdash (\mathrm{K}_i\beta_1 \wedge \ldots \wedge \mathrm{K}_i\beta_m) \Rightarrow \mathrm{K}_i\alpha$. Since each $\mathrm{K}_i\beta_j \in s$ for $j \in \{1,\ldots,m\}$, we have $\mathrm{K}_i\alpha \in s$. This contradicts the fact that ψ_s is consistent. So, $S_{\neg\alpha}$ is consistent. Now, since each set of formulas can be extended to a maximal one, we have that $S_{\neg\alpha}$ is contained in some maximal set t. Thus $\neg\alpha \in t$, and moreover, by the definition of $\sim_i^0$ in M and the definition of $S_{\neg\alpha}$ we have that $s \sim_i^0 t$. Thus, s satisfies $H15$.

(d). Let $\neg \mathrm{N}_i\alpha \in s$ and ψ_s be consistent. Consider the set $S_{\neg\alpha} = \{\neg\alpha\} \cup \{\beta \mid \mathrm{N}_i\beta \in s\}$. We will show that $S_{\neg\alpha}$ is consistent. Suppose that $S_{\neg\alpha}$ is inconsistent. Then, $\vdash \beta_1 \wedge \ldots \wedge \beta_m \Rightarrow \alpha$, where $\beta_j \in \{\beta \mid \mathrm{N}_i\beta \in s\}$ for $j \in \{1, \ldots, m\}$. By rule $R2$ we have $\vdash \mathrm{K}_i((\beta_1 \wedge \ldots \wedge \beta_m) \Rightarrow \alpha)$. By axioms $K1$ and PC we have $\vdash (\mathrm{K}_i\beta_1 \wedge \ldots \wedge \mathrm{K}_i\beta_m) \Rightarrow \mathrm{K}_i\alpha$. By axiom $F4$ we have that $\vdash (\mathrm{N}_i\beta_1 \wedge \ldots \wedge \mathrm{N}_i\beta_m) \Rightarrow \mathrm{N}_i\alpha$. Since each $\mathrm{N}_i\beta_j \in s$ for $j \in \{1, \ldots, m\}$, we have $\mathrm{N}_i\alpha \in s$. This contradicts the fact that ψ_s is consistent. So, $S_{\neg\alpha}$ is consistent. Now, since each set of formulas can be extended to a maximal one, we have that $S_{\neg\alpha}$ is contained in some maximal set t. Thus $\neg\alpha \in t$, and moreover, by the definition of R_i^0 in M and the definition of $S_{\neg\alpha}$ we have that sR_i^0t. Thus, s satisfies $H18$.

Claim (6). If $\psi_s \wedge \psi_t$ is consistent, then s and t are not eliminated at step 2(e) of the decidability algorithm for CTLKR.

Proof (By contraposition). We show that if s and t are eliminated at step 2.(e) (because $H19$ is not satisfied), then $\psi_s \wedge \psi_t$ is inconsistent. Let s and t be eliminated at step 2.(e). Then, we have that sR_it and $\mathrm{N}_i\alpha \in s$ and $\neg\mathrm{N}_i\alpha \in t$. By Claim 1 we have that $\vdash \psi_s \Rightarrow \mathrm{N}_i\alpha$ and $\vdash \psi_t \Rightarrow \neg\mathrm{N}_i\alpha$. This implies that $\vdash \psi_s \wedge \psi_t \Rightarrow \mathrm{N}_i\alpha \wedge \neg\mathrm{N}_i\alpha$. Therefore, $\vdash \psi_s \wedge \psi_t \Rightarrow \bot$. Thus, $\psi_s \wedge \psi_t$ is inconsistent.

Claim (7). If $\psi_s \wedge \psi_t \wedge \psi_u$ is consistent, then s, t and u are not eliminated at step 2(f) of the decidability algorithm for CTLKR.

Proof (By contraposition). We show that if s, t, and u are eliminated at step 2.(f) (because $H20$ is not satisfied), then $\psi_s \wedge \psi_t \wedge \psi_u$ is inconsistent. Let s, t, and u be eliminated at step 2.(f). Then, we have that $s \sim_i t$ and $s \sim_i u$ and $\mathrm{K}_i\alpha \in t$ and either $\neg\mathrm{K}_i\alpha \in u$, or $\neg\alpha \in u$. Let first assume that $s \sim_i t$ and $s \sim_i u$ and $\mathrm{K}_i\alpha \in t$ and $\neg\mathrm{K}_i\alpha \in u$. By Claim 1 we have that $\vdash \psi_t \Rightarrow \mathrm{K}_i\alpha$ and $\vdash \psi_u \Rightarrow \neg\mathrm{K}_i\alpha$. It follows that $\vdash \psi_t \wedge \psi_u \Rightarrow \mathrm{K}_i\alpha \wedge \neg\mathrm{K}_i\alpha$ holds. This implies that $\vdash \psi_t \wedge \psi_u \Rightarrow \bot$. So, $\vdash \psi_s \wedge \psi_t \wedge \psi_u \Rightarrow \bot$ as well. Therefore $\psi_s \wedge \psi_t \wedge \psi_u$ is inconsistent. The case that $s \sim_i t$ and $s \sim_i u$ and $\mathrm{K}_i\alpha \in t$ and $\neg\alpha \in u$ can be proven similarly.

We have now shown that only states s with ψ_s inconsistent are eliminated. This ends the completeness proof.

6 The Bit Transmission Problem

Imagine we have two processes, a *sender* $\mathfrak{S}$ and a *receiver* $\mathfrak{R}$, which communicate over a possibly faulty communication line. $\mathfrak{S}$ wants to send a finite stream of bits to $\mathfrak{R}$. One way of doing this is the following. $\mathfrak{S}$ immediately starts sending the bit to $\mathfrak{R}$, and continues to do so until it receives an acknowledgement from $\mathfrak{R}$. $\mathfrak{R}$ does nothing until it receives the bit; from then on it sends acknowledgements of receipt to $\mathfrak{S}$. When $\mathfrak{S}$ receives an acknowledgement, it stops sending the "old"

bit to $\mathfrak{R}$, and performs a reset operation, thereby giving a sign to $\mathfrak{R}$ that a new bit will be sent[1]. Then, $\mathfrak{S}$ starts sending the new bit and the cycle repeats.

We apply the CTLKR formalism to model and reason about the above scenario, a variation of the well known bit-transmission problem (BTP) [6]. Let us begin by building a model $M = (S, T, \sim_{\mathfrak{S}},$
$\sim_{\mathfrak{R}}, \mathcal{V}, R_{\mathfrak{S}}, R_{\mathfrak{R}})$ for BTP.

There are three active components in the scenario: agents $\mathfrak{S}$ and $\mathfrak{R}$, and a communication channel represented by an environment $\mathfrak{E}$. Each of these can be modelled by considering their local states. For $\mathfrak{S}$, it is enough to consider five possible local states. They represent the value of the bit that $\mathfrak{S}$ is attempting to transmit, and whether or not $\mathfrak{S}$ has received an acknowledgement or an end signal from $\mathfrak{R}$. We thus have:

$$L_{\mathfrak{S}} = \{0,\ 1,\ 0\text{-}ack,\ 1\text{-}ack, end\}.$$

Let $n > 0$ be the maximal length of the stream of beats to be sent. We consider

$$L_{\mathfrak{R}} = \{X^j Y \mid X \in \{0\text{-}ack, 1\text{-}ack\}, Y \in \{\epsilon, 0, 1\}, 0 \leq j \leq n\}.$$

$\mathfrak{R}$'s local state is ϵ if $\mathfrak{R}$ has received no bits from $\mathfrak{S}$. $\mathfrak{R}$'s local state is 0 (resp. 1) if the received bit is 0 (resp. 1). $\mathfrak{R}$'s local state is $k_1\text{-}ack \ldots k_j\text{-}ack$ (resp. $k_1\text{-}ack \ldots k_j\text{-}ack\ k$) for $k_1, \ldots, k_j, k \in \{0, 1\}$ and $j \leq n$, if the stream of bits he received is $k_1 \ldots k_j$ (resp. $k_1 \ldots k_j k$) and $\mathfrak{S}$ has performed j reset actions. It remains to model the local states of the environment $\mathfrak{E}$. For $\mathfrak{E}$ it is enough to consider a singleton: $L_{\mathfrak{E}} = \{\cdot\}$.

The following sets of actions are available to the agents: $Act_{\mathfrak{S}} = \{sendbit,$ $reset, \lambda\}$; $Act_{\mathfrak{R}} = \{sendack, sendend,\ \lambda\}$, where λ stands for no action. The actions $Act_{\mathfrak{E}}$ for the environment correspond to the transmission of messages between $\mathfrak{S}$ and $\mathfrak{R}$ on the unreliable communication channel. We will assume that the communication channel can transmit messages in both directions simultaneously and independently. The set of actions for $\mathfrak{E}$ is $Act_{\mathfrak{E}} = \{\leftrightarrow, \rightarrow, \leftarrow, -\}$, where $\leftrightarrow$ represents the action in which the channel transmits any message successfully in both directions, $\rightarrow$ that it transmits successfully from $\mathfrak{S}$ to $\mathfrak{R}$ but loses any message from $\mathfrak{R}$ to $\mathfrak{S}$, $\leftarrow$ that it transmits successfully from $\mathfrak{R}$ to $\mathfrak{S}$ but loses any message from $\mathfrak{S}$ to $\mathfrak{R}$, and $-$ that it loses any messages sent in either direction.

The protocols the agents are running are defined as follows:

- $P_{\mathfrak{S}}(0) = \{sendbit\}$,
- $P_{\mathfrak{S}}(1) = \{sendbit\}$,
- $P_{\mathfrak{S}}(0\text{-}ack) = \{reset\}$,
- $P_{\mathfrak{S}}(1\text{-}ack) = \{reset\}$,
- $P_{\mathfrak{S}}(end) = \{\lambda\}$,
- $P_{\mathfrak{R}}(\epsilon) = \{\lambda\}$,

[1] For simplicity we assume that resets are communicated with no faults; an acknowledgement protocol could be introduced for resets without violating the properties we show below

– $P_{\mathfrak{R}}(k_1\text{-}ack \dots k_j\text{-}ack) = \{\lambda\}$,
– $P_{\mathfrak{R}}(k_1\text{-}ack \dots k_{j-1}\text{-}ack\ k_j) = \{sendack\}$,
– $P_{\mathfrak{R}}(k_1\text{-}ack \dots k_{n-1}\text{-}ack\ k_n) = \{sendend\}$,
– $P_{\mathfrak{E}}(\cdot) = \{\leftrightarrow, \rightarrow, \leftarrow, -\}$,

where $k_1, \dots, k_j, k_n \in \{0, 1\}$ and $j < n$.

The evolution of BTP is defined by means of an evolution function $t : (L_{\mathfrak{S}} \times L_{\mathfrak{R}} \times L_{\mathfrak{E}}) \times Act \rightarrow 2^{L_{\mathfrak{S}} \times L_{\mathfrak{R}} \times L_{\mathfrak{E}}}$, where Act is a subset of $Act_{\mathfrak{S}} \times Act_{\mathfrak{R}} \times Act_{\mathfrak{E}}$. It is straightforward to infer a definition of this function from the informal description of the scenario we considered above together with the local states and protocols defined above; the function t determines not only the set of reachable global states $S \subseteq L_{\mathfrak{S}} \times L_{\mathfrak{R}} \times L_{\mathfrak{E}}$, but also gives us the transition relation T. Namely, for all the $s,\ s' \in S$, $(s, s') \in T$ iff there exists $act \in Act$ such that $t(s, act) = s'$.

To complete the description of M for BTP, we introduce the set of propositional variables: $\mathcal{PV} = \{ack\} \cup \{jBit = 0, jBit = 1, reset_j \mid 0 < j \leq n\}$, and we define the valuation function $\mathcal{V} : S \rightarrow 2^{\mathcal{PV}}$ as:

– $ack \in \mathcal{V}(s)$ if $l_{\mathfrak{S}}(s) = 0\text{-}ack$ or $l_{\mathfrak{S}}(s) = 1\text{-}ack$,
– $jBit = 0 \in \mathcal{V}(s)$ if $l_{\mathfrak{R}}(s) = (k_1 \dots k_j)$ and $(k_j = 0$ or $k_j = 0\text{-}ack)$ and $k_1, \dots, k_{j-1} \in \{0\text{-}ack, 1\text{-}ack\}$, for $0 < j \leq n$,
– $jBit = 1 \in \mathcal{V}(s)$ if $l_{\mathfrak{R}}(s) = (k_1 \dots k_j)$ and $(k_j = 1$ or $k_j = 1\text{-}ack)$ and $k_1, \dots, k_{j-1} \in \{0\text{-}ack, 1\text{-}ack\}$, for $0 < j \leq n$,
– $reset_j \in \mathcal{V}(s)$ if $(l_{\mathfrak{R}}(s) = k_1 \dots k_j$ and $k_1, \dots, k_j \in \{0\text{-}ack, 1\text{-}ack\}$ and $0 < j \leq n)$ or $(l_{\mathfrak{R}}(s) = \epsilon$ and $j = 0)$.

Let us consider the following property ($\star$):"*whenever a fresh bit has been sent,* $\mathfrak{S}$ *knows that whenever he receives an acknowledgement,* $\mathfrak{R}$ *knows the value of the bit*". One can try to express this property with the following CTLK formula, for $0 < j \leq n$:

$$\mathrm{AG}[reset_j \Rightarrow \mathrm{AGK}_{\mathfrak{S}}(ack \Rightarrow (\mathrm{K}_{\mathfrak{R}}(jBit = 0) \vee \mathrm{K}_{\mathfrak{R}}(jBit = 1)))] \qquad (1)$$

But one can check that Formula (1) is not valid in M. The problem is that a bit received before a reset may account for the receiver's knowledge about the current bit. What we need to do is to express explicitly that past states should not be considered in $\mathfrak{S}$'s accessible states, following the reset operation. The epistemic modality N_i enable us not to include past states, and can be used to capture this intuition. Indeed, property ($\star$) can be formalised by the following CTLKR formula:

$$\mathrm{AG}[reset_j \Rightarrow \mathrm{AGN}_{\mathfrak{S}}(ack \Rightarrow (\mathrm{K}_{\mathfrak{R}}(jBit = 0) \vee \mathrm{K}_{\mathfrak{R}}(jBit = 1)))] \qquad (2)$$

It is easy to check manually that Formula (2) is valid in M.

7 Conclusion and Final Remarks

In the paper we have proposed a new notion of knowledge, called *knowledge from now on*, that is interpreted over intersection of the standard epistemic relation for

agents and reflexive, transitive closure of the temporal relation. We have shown that CTLKR in which this notion of knowledge is expressible, is decidable, and can be axiomatised. In the following we give a short review of existing work on intersection.

Providing axiomatisations for modalities that are defined on intersections of relations is non trivial, because the intersection of two relations is not modally definable. One of the cases, well known in the literature, is the case of distributed knowledge [7, 10]. Namely, if an epistemic language (say $S5D_n$) with distributed knowledge D is considered, then the axiom system for $S5D_n$ is defined by taking $S5_n$ axioms and adding axioms that say that D acts like a knowledge operator (i.e., all the axioms for K_i replaced with D holds) and adding the following additional axiom: $\mathrm{K}_i\alpha \Rightarrow D\alpha$ for $i = 1 \ldots n$. Furthermore, if $n = 1$, i.e., there is only one agent, then the following axiom is added $D\alpha \Rightarrow K_1\alpha$. The proof technique used in [7] consists in a reduction to equivalence Kripke trees, and the authors assume that the relations from which the intersection is taken have the same properties. So given this, obviously the technique can not be apply to prove the intersection between T^* and $\sim_i$, what it is done in that paper. Other case is given in [1], where a complete axiomatisation of a relative modal logic with composition and intersection is given.

Finally, we would like to stress that a logic defined by the same grammar as the logic CTLKR, but interpreted over the models with the relation R_i defined by: for any agent $i \in Ag$, sR_it if $s \sim_i t$ and sTt, is also decidable and complete. This can be shown via the same technique as the one applied in the paper.

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