The Class of Representable Ordered Monoids has a Recursively Enumerable, Universal Axiomatisation but it is Not Finitely Axiomatisable

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Abstract

An ordered monoid is a structure with an identity element ($1'$), a binary composition operator (;) and an antisymmetric partial order ($\leq$), satisfying certain axioms. A representation of an ordered monoid is a 1-1 map which maps elements of an ordered monoid to binary relations in such a way that $1'$ is mapped to the identity relation, ; corresponds to composition of binary relations and $\leq$ corresponds to inclusion of binary relations.

We devize a two player game that tests the representability of an ordered monoid n times and show that these games characterise representability. From this we obtain a recursively enumerable, universal axiomatisation of the class of all representable ordered monoids.

For each $n < \omega$ we construct an unrepresentable ordered monoid $\mathcal{A}_n$ and show that the second player has a winning strategy in a game of length n. Hence we prove that the class of all representable ordered monoids is not finitely axiomatisable.

Relation Algebras are badly behaved in a number of ways. The class of representable relation algebras cannot be defined by finitely many axioms [Mon64], nor by any set of equations using a finite number of variables [Jón91], nor by any Sahlqvist theory [Ven97], the equational theory of relation algebras and the equational theory of representable relation algebras is undecidable [Tar41], the problem of determining whether a finite relation algebra is representable or not is itself undecidable [HH01].

An important line of research is to consider reducts of relation algebras, by dropping some of the operators from the signature. We aim to find out exactly what causes this "bad behaviour" and how it can be avoided. Mikulás has surveyed much of this research [Mik03].

In the current paper we consider algebras in the reduced signature $\{\leq, 1', ;\}$. Such an algebra is representable if its elements can be interpretted as binary relations over some domain in such a way that $\leq$ is represented as inclusion of binary relations, $1'$ is the identity over the domain and ; is composition of binary relations. These algebras have better behaviour, in some ways at least: the class of representable algebras is defined by a universal, recursively enumerable theory (see below) and the equational theory of this class is decidable [And90]; in a subsequent paper we will show that any finite, representable algebra in this class has a finite representation. But in this paper we will show that there can be no finite axiomatisation of the representable algebras in this signature. Oddly, it seems that the representation class for the signature $\{\leq, 1', \smile, ;\}$, with converse also included, *is* finitely axiomatisable, as we shall show in a later paper.

The signature $\{., 1', ;\}$ (sometimes called the *Jerry-Fragment*) is more expressive than $\{\leq, 1', ;\}$, but nevertheless the proof that the class of representable algebras for this signature is not finitely axiomatisable seems to be more intricate. Hirsch and Mikulás intend to prove this result in a subsequent paper.

Ordered monoids and representations

DEFINITION 1 *An ordered monoid* $\mathcal{A} = (A, \leq, 1', ;)$ *consists of a set* A *(called the* domain *of* $\mathcal{A}$*), a binary relation* $\leq$ *over* A*, a constant* $1' \in A$ *(the identity) and a binary operator* ; *(composition), satisfying*

- $\leq$ *is an antisymmetric partial order,*
- ; *is associative,*
- $1'$ *is an identity for* ;*,*
- ; *is monotonic.*

A representation h *of an ordered monoid* $\mathcal{A}$ *is a 1-1 map* $h : \mathcal{A} \to \wp(D \times D)$ *(for some set* D*,* $\wp(\mathcal{A})$ *denotes the power set of the domain of* $\mathcal{A}$*)) such that*

- $a \leq b \iff h(a) \leq h(b)$,
- $h(1') = \{(d, d) : d \in D\}$,
- $h(a; b) = h(a)|h(b)$,

for all $a, b \in \mathcal{A}$*, where '*$|$*' denotes composition of binary relations.*

Are all ordered monoids representable? Is the class of representable ordered monoids finitely axiomatisable? No.

Networks

DEFINITION 2 *Let* $\mathcal{A}$ *be an ordered monoid. A* prenetwork (D, N) *over* $\mathcal{A}$ *consists of a set of nodes* D *and a map* $N : D \times D \to \wp(\mathcal{A})$*. If the prenetwork* (D, N) *satisfies*

- $\exists e(e \leq 1' \wedge e \in N(x, y)) \iff x = y$,

- *if $\alpha \in N(x,y)$ and $\beta \in N(y,z)$ then there is $\gamma \in N(x,z)$ with $\gamma \leq \alpha;\beta$,*

for all $x,y,z \in D$, then (D,N) is called a network *over $\mathcal{A}$.*

Henceforth we will stretch the notation and allow N to denote the set of nodes, the labelling function and the prenetwork itself and distinguish these different meanings by context. Thus $x \in N$ means that x is one of the nodes of the prenetwork N and $N(x,y)$ denotes the value of the labelling function on the edge (x,y). When there is ambiguity we may write $\mathsf{nodes}(N)$ to denote the set of nodes of the prenetwork N.

DEFINITION 3 *A prenetwork is called* finitary *if it has finitely many nodes and each edge is labelled by a finite set of elements.*

For any prenetworks M, N, we write $M \subseteq N$, and we say that M is a subnetwork *of N, if $\mathsf{nodes}(M) \subseteq \mathsf{nodes}(N)$ and for all $x,y \in \mathsf{nodes}(M)$ and all $\alpha \in M(x,y)$ there is $\alpha^- \in N(x,y)$ with $\alpha^- \leq \alpha$.*

We write $M \leq N$, and we say that M is an induced subnetwork *of N, if $\mathsf{nodes}(M) \subseteq \mathsf{nodes}(N)$ and for all $x,y \in \mathsf{nodes}(M)$ we have $M(x,y) = N(x,y)$. The same notation is used for networks and prenetworks.*

If $N_\lambda : \lambda \in \Lambda$ are prenetworks then $N = \bigcup_{\lambda \in \Lambda} N_\lambda$ is the prenetwork satisfying

$$\begin{aligned}\mathsf{nodes}(N) &= \bigcup_{\lambda \in \Lambda} \mathsf{nodes}(N_\lambda) \\ N(x,y) &= \bigcup_{\lambda : x,y \in N_\lambda} N_\lambda(x,y)\end{aligned}$$

for all $x,y \in \mathsf{nodes}(N)$. As a special case we write $M \cup N$ for $\bigcup_{i \in 2} N_i$, where $2 = \{0,1\}$, $N_0 = M$ and $N_1 = N$.

Note that $M \subseteq (M \cup N)$ and $M \leq N$ implies $M \subseteq N$.

Games and Representability

We define a two player game $G_n(\mathcal{A})$ to test the representability of the ordered monoid $\mathcal{A}$ n times. The players are $\forall$ and $\exists$. $\forall$ will demand information about certain witnesses which ought to exist if $\mathcal{A}$ really is representable. $\exists$'s job, essentially, is to decide whether the witnesses are distinct from each other or not.

DEFINITION 4 *Let $n \leq \omega$ and let $\mathcal{A}$ be an ordered monoid. A play of the game $G_n(\mathcal{A})$ has n rounds and consists of a sequence of n finitary pre-networks. In the initial round (round 0) $\forall$ picks any two elements $\alpha_1, \alpha_0 \in \mathcal{A}$ such that $\alpha_1 \not\leq \alpha_0$. This move for $\forall$ is denoted (α_1, α_0). $\exists$ has two choices for her response: she can play $N_0 = I(\alpha_1)$ or $N_0 = D(\alpha_1)$, defined as follows. In either* *case* $\mathsf{nodes}(N_0) = \{x_0, x_1\}$ *but* $x_0 = x_1$ *in* $I(\alpha_1)$ *and* $x_0 \neq x_1$ *in* $D(\alpha_1)$. *For the labelling,*

$$I(\alpha_1)(x_0, x_0) = \{1', \alpha_1\}$$

$$\begin{aligned} D(\alpha_1)(x_0, x_0) = D(\alpha_1)(x_1, x_1) &= \{1'\} \\ D(\alpha_1)(x_0, x_1) &= \{\alpha_1\} \\ D(\alpha_1)(x_1, x_0) &= \emptyset \end{aligned}$$

In a later round (round i, $0 < i < n$*) suppose the finitary prenetwork* $N_{i-1} \supseteq N_0$ *has just been played. Note that* $x_0, x_1 \in N_{i-1}$. $\forall$ *has the choice of two kinds of move.*

Composition Move *He can pick* $x, y, z \in N_{i-1}$ *and* $\rho \in N_{i-1}(x, y)$, $\sigma \in N_{i-1}(y, z)$. *This move is denoted* $(N_{i-1}, x, y, z, \rho, \sigma)$. $\exists$ *has no choice for her response: she must play* $N_i \supseteq N_{i-1}$ *where* N_i *is identical to* N_{i-1} *except that* $\rho;\sigma \in N_i(x, z)$ — *i.e.* $N_i(x, z) = N_{i-1}(x, z) \cup \{\rho;\sigma\}$ *and* $N_i(u, v) = N_{i-1}(u, v)$ *whenever* $(u, v) \neq (x, z)$.

Witness Move *Alternatively,* $\forall$ *picks any two nodes* $x, y \in N_{i-1}$ *and any* ρ, σ *such that there is* $\gamma \leq \rho;\sigma$ *with* $\gamma \in N_{i-1}(x, y)$. *This move for* $\forall$ *is denoted* $(N_{i-1}, x, y, \rho, \sigma)$. $\exists$ *has* $|\mathsf{nodes}(N_{i-1})|+1$ *choices for her response. She must find a witness* w *for this move and she must choose whether* $w \in N_{i-1}$ *(there are* $|\mathsf{nodes}(N_{i-1})|$ *choices here) or* w *is a new node, not in* N_{i-1} *(essentially 1 choice, since the name of the new node does not matter). If she chooses* $w \notin N_{i-1}$ *she lets* w *be the first unused node name in some fixed and infinite enumeration. To define these choices, let* $T = T(x, y, w, \rho, \sigma)$ *be a prenetwork with* $\mathsf{nodes}(T) = \{x, y, w\}$. *We do not assume that* x, y, w *are distinct. If they are distinct, the labelling is defined by*

$$\begin{aligned} T(w, w) &= \{1'\} \\ T(x, w) &= \{\rho\} \\ T(w, y) &= \{\sigma\} \end{aligned}$$

and otherwise, take unions — e.g. if $w = x \neq y$ *let* $T(x, x) = \{1', \rho\}$, *etc. Now define* $N^+(N_{i-1}, x, y, w, \rho, \sigma) = N_{i-1} \cup T$ *(see definition 3). [Note that the definition looks different according to whether* $w \in N_{i-1}$ *or not.] For her response,* $\exists$ *chooses a node* $w \in N_{i-1}$ *or she lets* w *be a new node. Then she plays* $N_i = N^+(N_{i-1}, x, y, w, \rho, \sigma)$.

$\forall$ *wins the play if there is* $i < n$ *and either*

- *There are* $u \neq v \in N_i$ *and* $e \leq 1'$ *with* $e \in N_i(u, v)$, *or*
- *There is* $\alpha^- \leq \alpha_0$ *and* $\alpha^- \in N_i(x_0, x_1)$.

Otherwise $\exists$ *wins the play.*

For any finitary prenetwork N *where* $x_0, x_1 \in N$*, we define a variant* $G_n(N, \mathcal{A}, \alpha_0)$ *of the game* $G_n(\mathcal{A})$*. The only difference is that in the initial round,* $N_0 = N$ *is played. The rules for playing in later rounds are unchanged. As before,* $\forall$ *wins if there is* $i < n$ *and* $\alpha^- \leq \alpha_0$ *with* $\alpha^- \in N_i(x_0, x_1)$ *or* $x \neq y \in N_i$ *and* $e \leq 1'$ *with* $e \in N_i(x, y)$*.*

A (deterministic) strategy *for* $\exists$ *in a game* G *determines a unique next move for her, given any possible initial segment of a play of the game. Such a strategy is a* winning strategy *for* $\exists$ *if, no matter what moves* $\forall$ *makes, she will always win a play of* G *when she plays according to the strategy.*

PROPOSITION 5 *Let* $\mathcal{A}$ *be an ordered monoid.*

1. *If* $\mathcal{A}$ *is representable then* $\exists$ *has a winning strategy in* $G_\omega(\mathcal{A})$*.*
2. *If* $\mathcal{A}$ *is countable and* $\exists$ *has a winning strategy in* $G_\omega(\mathcal{A})$ *then* $\mathcal{A}$ *is representable.*
3. *For each* $n < \omega$ *there is a first-order formula* σ_n *in the signature of ordered monoids such that* $\exists$ *has a winning strategy in* $G_n(\mathcal{A})$ *iff* $\mathcal{A} \models \sigma_n$*.*
4. *If, for each* $n < \omega$*, there is an unrepresentable ordered monoid* $\mathcal{A}_n$ *where* $\exists$ *has a winning strategy in* $G_n(\mathcal{A}_n)$ *then the class of representable ordered monoids cannot be defined by finitely many axioms.*

PROOF SKETCH:

1. Let h be a representation of $\mathcal{A}$ over the domain D. Let $\forall$ play (α_1, α_0) where $\alpha_1 \not\leq \alpha_0$ in the initial round. To win the game $\exists$ maintains an embedding $' : \mathsf{nodes}(N) \to D$, where N is any network played, such that $(x_0', x_1') \in h(\alpha_1) \setminus h(\alpha_0)$ and for all $x, y \in \mathsf{nodes}(N)$ and all $\alpha \in N(x, y)$ we have $(x', y') \in h(\alpha)$.
2. Take any α_1, α_0 with $\alpha_1 \not\leq \alpha_0$. Consider a play of the game in which $\forall$ plays (α_1, α_0) in the initial round and then plays $(N, x, y, z, \rho, \sigma)$ for all networks N occurring in the play, all $x, y, z \in N$ and all $\rho \in N(x, y)$, $\sigma \in N(y, z)$, and $\forall$ also plays (N, x, y, ρ, σ) for all networks N occurring in the play, all $x, y, z \in N$ and all ρ, σ such that there is $\gamma \leq \rho;\sigma$ with $\gamma \in N(x, y)$. Since $\mathcal{A}$ is countable, there is a way of scheduling all these moves in a play of the game. Let

   $$N_0 \subseteq N_1 \subseteq \ldots$$

   be a play of such a game in which $\exists$ uses her winning strategy. Let $N^*(\alpha_1, \alpha_0) = \bigcup_{i \in \omega} N_i$ (see definition 3). Since $\forall$ makes all possible composition moves (definition 4) whenever

$\sigma \in N^*(x,y)$ and $\tau \in N^*(y,z)$ we have $\sigma;\tau \in N^*(x,z)$. Witness moves ensure that if $\gamma \in N^*(x,y)$ and $\gamma \leq \sigma;\tau$ then there is $z \in N^*$ such that $\sigma \in N^*(x,z)$ and $\tau \in N^*(z,y)$.

By construction of the prenetworks, we have $1' \in N(x,x)$ for all prenetworks played and for all $x \in N$. Since $\exists$ wins the play we see that there is $e \leq 1'$ with $e \in N^*(x,y)$ iff $x = y$. Hence N^* is a network, though not necessarily a finitary one. Furthermore, since $\exists$ wins the play, there is no $\alpha^- \leq \alpha_0$ with $\alpha^- \in N^*(x_0, x_1)$.

By renaming, we can arrange that the nodes of $N^*(\alpha_1, \alpha_0)$ are disjoint from the nodes of $N^*(\alpha_1', \alpha_0')$ if $(\alpha_1, \alpha_0) \neq (\alpha_1', \alpha_0')$. Let

$$N = \bigcup_{\alpha_1 \not\leq \alpha_0} N^*(\alpha_1, \alpha_0)$$

Since this is a disjoint union of networks, it is clearly a network. Also, if $\alpha_1 \not\leq \alpha_0$ then there are $x, y \in N^*(\alpha_1, \alpha_0) \leq N$ such that $\alpha_1 \in N(x,y)$ but there is no $\alpha^- \leq \alpha_0$ with $\alpha^- \in N(x,y)$. Now define a representation h of $\mathcal{A}$ with domain $\mathsf{nodes}(N)$:

$$h(\rho) = \{(x,y) : \exists \rho^- \leq \rho,\ \rho^- \in N(x,y)\}.$$

3. See [HH02, theorem 9.28]. First we need some syntax. A *term* is anything of the form $s_0; s_1; \ldots; s_k$ (some $k \in \mathbb{N}$) where s_i is either a variable or the identity $1'$, for $i \leq k$. T denotes the set of all terms. A *term network* N consists of a finite set of nodes, $\mathsf{nodes}(N)$, and a labelling function (also denoted N) $N : \mathsf{nodes}(N) \times \mathsf{nodes}(N) \to \wp(T)$, where $N(x,y)$ is a finite set of terms, for all $x, y \in \mathsf{nodes}(N)$. If A is any assignment of variables to elements of the ordered monoid $\mathcal{A}$ and N is any term network, then N^A denotes the finitary prenetwork with the same nodes as N and with labelling $N^A(x,y) = \{\tau^A : \tau \in N(x,y)\}$, where $\tau^A \in \mathcal{A}$ denotes the value of the term τ under the variable assignment A. If M, N are term networks, $M \cup N$ denotes the term network with nodes $\mathsf{nodes}(M) \cup \mathsf{nodes}(N)$ and with labelling defined as in definition 3.

   Let N be a term network. We define two extensions.

   - If $x, y \in N$ and σ is a term we let $N^1(N, x, y, \sigma)$ denote the term network where $\mathsf{nodes}(N^1) = \mathsf{nodes}(N)$ and $N^1(x,y) = N(x,y) \cup \{\sigma\}$ and $N^1(u,v) = N(u,v)$ for all $u, v \in N$ with $(u,v) \neq (x,y)$.
   - For any $x, y \in N$, any node z (both $z \in N$ and $z \notin N$ are allowed) and any terms σ, τ, let $T(x,y,z,\sigma,\tau)$ be the term network where $\mathsf{nodes}(T) = \{x, y, z\}$ with labelling $T(z,z) = \{1'\}$, $T(x,z) = \{\sigma\}$ and $T(z,y) = \{\tau\}$ and $T(u,v) = \emptyset$ whenever $(u,v) \neq (z,z), (x,z)$ or (z,y). [Strictly,

as before, we have just defined T in the case where x, y, z are distinct. For other cases, we take unions, e.g. if $x = z \neq y$ then $T(x, x) = \{1', \sigma\}$, etc.] Now let $N^2(N, x, y, z, \sigma, \tau) = N \cup T$.

For any term network N containing the nodes x_0, x_1, and any $k \in \mathbb{N}$ we define a formula $\rho_k(N, v_0)$ such that for any variable assignment A,

$$\exists \text{ has a w.s. in } G_k(N^A, \mathcal{A}, A(v_0)) \iff \mathcal{A}, A \models \rho_k(N, v_0) \tag{1}$$

The formulas are defined recursively. $\rho_0(N, v_0) =$

$$\bigwedge_{x \neq y \in N,\ \tau \in N(x,y)} \neg(\tau \leq 1') \wedge \bigwedge_{\tau \in N(x_0, x_1)} \neg(\tau \leq v_0)$$

ρ_0 merely states, in first-order logic, the winning conditions for $G_0(N, \mathcal{A}, \alpha_0)$ — more precisely (1) holds with $k = 0$. Now suppose $\rho_k(M, v_0)$ is defined (some $k \geq 0$), for all term networks M, and (1) holds for this value of k. We define ρ_{k+1}.

$$\rho_{k+1}(N, v_0) = \bigwedge_{x,y,z \in N,\ \sigma \in N(x,y),\ \tau \in N(y,z)} \rho_k(N^1(N, x, z, \sigma; \tau), v_0) \wedge$$

$$\bigwedge_{x,y \in N, \tau \in N(x,y)} \forall \alpha \forall \beta [\alpha; \beta \geq \tau \to (\bigvee_{w \in \mathsf{nodes}(N) \cup \{z\}} \rho_k(N^2(N, x, y, w, \alpha, \beta), v_0)$$

where z is any node not in N, in the last line. ρ_{k+1} translates, roughly, to the statement 'for any composition move by $\forall$, ρ_k holds on the resulting prenetwork, and for any witness move by $\forall$ at least one of the $\exists$ moves leads to a prenetwork where ρ_k holds'. Thus (1) holds.

Finally, for any variable v_1, let $I(v_1)$ denote the term network with a single node $x_0 = x_1$ labelled $I(v_1)(x_0, x_0) = \{1', v_1\}$ and let $D(v_1)$ denote the term network with two distinct nodes x_0, x_1 with labelling

$$\begin{aligned} D(v_1)(x_0, x_0) = D(v_1)(x_1, x_1) &= \{1'\} \\ D(v_1)(x_0, x_1) &= \{v_1\} \\ D(v_1)(x_1, x_0) &= \emptyset \end{aligned}$$

Let

$$\sigma_k = \forall v_0 \forall v_1 [\neg(v_1 \leq v_0) \to (\rho_k(I(v_1), v_0) \vee \rho_k(D(v_1), v_0)$$

4. Let $\mathcal{B} = \Pi_D \mathcal{A}_n$, where D is a non-principal ultrafilter over ω. We are given that $\exists$ has a winning strategy in $G_n(\mathcal{A}_n)$ hence she has a winning strategy in $G_n(\mathcal{A}_m)$ whenever $m \geq n$. By the

previous part, $\mathcal{A}_n \models \sigma_m$ whenever $m \geq n$. By Łoś' theorem, $\mathcal{B} \models \sigma_n$ and hence $\exists$ has a winning strategy in $G_n(\mathcal{B})$, for each $n < \omega$. Now, as in [HH02, theorem 10.12, proposition 10.13], there is a countable $\mathcal{B}'$ elementarily equivalent to $\mathcal{B}$ such that $\exists$ has a winning strategy in $G_\omega(\mathcal{B}')$. By part 2 of this proposition, $\mathcal{B}'$ is representable.

Now suppose for contradiction that a single formula θ defines the class of representable ordered monoids, i.e. for any structure $\mathcal{A}$ of the type of ordered monoids we have $\mathcal{A} \models \theta$ iff $\mathcal{A}$ is a representable ordered monoid. Since $\mathcal{B}'$ is representable, $\mathcal{B}' \models \theta$. By elementary equivalence, $\mathcal{B} \models \theta$. By Łoś' theorem the set S of all $n < \omega$ for which $\mathcal{A}_n \models \theta$ should be large, i.e. in the ultrafilter. But by theorem 12, for each $n < \omega$ we have $\mathcal{A}_n \not\models \theta$ and so there are no values of n for which $\mathcal{A}_n \models \theta$. Hence $S = \emptyset$, a contradiction.

$\Box$

Thus, to prove that the class of representable ordered monoids cannot be defined by finitely many axioms, it remains to define an unrepresentable ordered monoid $\mathcal{A}_n$, for each $n < \omega$, such that $\exists$ has a winning strategy in $G_n(\mathcal{A}_n)$.

THEOREM 6 *An ordered monoid $\mathcal{A}$ (whether countable or not) is representable if and only if $\mathcal{A} \models \sigma_n$ for all $n < \omega$, where σ_n is defined in the proof sketch of the previous proposition. Further, each σ_n is equivalent to a universal sentence.*

PROOF:

Show that the class of representable ordered monoids is a pseudo-universal class (just define a two sorted language with one sort for the elements of an ordered monoid and the other sort for points in the domain of a representation of it). Then use [HH02, theorem 9.28]. For universality, just bring all the universal quantifiers to the front in the definition of σ_n. $\Box$

An unrepresentable ordered monoid

DEFINITION 7

1. *Define an alphabet $\Sigma = \{b, f, g, \overline{f}, \overline{g}\}$ and define a binary relation $\prec$ over Σ^* as follows.*

$$\begin{aligned} \Lambda &\prec f\overline{f} \\ \Lambda &\prec \overline{g}g \\ \Lambda &\prec \overline{f}f \prec g\overline{g} \prec \Lambda \\ b &\prec (fg)^n \end{aligned}$$

where Λ is the empty string.

2. *For arbitrary $s, t \in \Sigma^*$ we let $s \leq_1 t$ iff there are s_0, s_1, t_0, t_1, u, v such that $s = s_0 u s_1,\ t = t_0 v t_1$ and $u \prec v$.*

3. *Let $\leq$ be the reflexive transitive closure of $\leq_1$. So $s \leq t$ iff there is a finite chain $s = s_0, s_1, \ldots, s_k = t$ (some $k \in \mathbb{N}$) where for each $i < k$ we have $s_i \leq_1 s_{i+1}$.*

4. *We write $s \equiv t$ iff $s \leq t$ and $t \leq s$ and we write $s < t$ if $s \leq t$ but $t \not\leq s$.*

5. *For any $s \in \{f, g, \overline{f}, \overline{g}\}^*$, $\overline{s}$ denotes the string obtained by reversing the order of s and replacing each occurrence of $f, g, \overline{f}, \overline{g}$ by $\overline{f}, \overline{g}, f, g$ respectively.*

DEFINITION 8 *Let $x \in \Sigma^*$. Define $\widehat{x}$ from x by repeatedly deleting any substrings $\overline{f}f$ or $g\overline{g}$ until no further deletions are possible.*
Let $A_n = \{\widehat{x} : x \in \Sigma^*\} \subset \Sigma^*$.

It is easy to check that the definition of $\widehat{x}$ does not depend on the order chosen to do the deletions.

LEMMA 9 *Let $\widehat{x} = x$ and let $\phi \in \{f, g, \overline{f}, \overline{g}\}$. Either $\widehat{x\phi} = x\phi$, or $\widehat{x\phi}\ \overline{\phi} = x$ and $\overline{\phi}\phi \equiv 1'$. Either $\widehat{\phi x} = \phi x$, or $\overline{\phi}\ \widehat{\phi x} = x$ and $\phi\overline{\phi} \equiv 1'$.*

LEMMA 10 *The following are equivalent.*

- $\widehat{x} = \widehat{y}$
- $x \equiv y$.

DEFINITION 11 *Let $n \geq 1$. We define the structure $\mathcal{A}_n = (A_n, \leq, 1', ;)$, where A_n is defined in definition 8, $\leq$ is defined in definition 7, $1'$ is the empty string and $;$ is defined by string concatenation i.e. $x; y =_{def} \widehat{xy}$.*

THEOREM 12 *Let $n \geq 1$. $\mathcal{A}_n$ is not a representable ordered monoid.*

PROOF:

Suppose for contradiction that h is a representation of $\mathcal{A}_n$ over some domain D.

Note that if $x, y \in D$ and $(x, y) \in h(f)$ then since $(x, x) \in h(1') \subseteq h(f\overline{f})$, there is $z \in D$ such that $(x, z) \in h(f)$ and $(z, x) \in h(\overline{f})$. But then $(z, y) \in h(\overline{f})|h(f) = h(\widehat{\overline{f}f}) = h(1')$ and therefore $z = y$. So if $(x, y) \in h(f)$ then $(y, x) \in h(\overline{f})$. Similarly, for any $\phi \in \{f, g, \overline{f}, \overline{g}\}$, if $(x, y) \in h(\phi)$ then $(y, x) \in h(\overline{\phi})$.

Observe that $b(\overline{g}\overline{f})^n b \not\geq b$. So there are $x, y \in D$ with $(x, y) \in h(b) \setminus h(b(\overline{g}\overline{f})^n b)$. But then, since $b \leq (fg)^n$ and since h respects the composition operator, there are $z_0, z_1, \ldots, z_{2n} \in D$ such that $x = z_0,\ y = z_{2n},\ (z_i, z_{i+1}) \in h(f)$ for even $i < 2n$ and $(z_i, z_{i+1}) \in h(g)$ for odd $i < 2n$. For even $i < 2n$, $(z_i, z_{i+1}) \in h(f)$ so, by

the previous paragraph, $(z_{i+1}, z_i) \in h(\overline{f})$. Similarly, for odd $i < 2n$ we have $(z_{i+1}, z_i) \in h(\overline{g})$. Since $(x, y) \in \{(x, y)\}|\{(y, x)\}|\{(x, y)\}$ it follows that $(x, y) \in h(b(\overline{g}\,\overline{f})^n b)$, contrary to assumption.
□

DEFINITION 13 *Let $k \in \mathbb{N}$. A string $\alpha \in \{f, \overline{f}, g, \overline{g}\}^*$ is said to be* k-short *if $\widehat{\alpha} = A_0B_0A_1B_1 \ldots A_{k-1}B_{k-1}$ for some $A_i \in \{f, \overline{g}\}^*$, $B_i \in \{\overline{f}, g\}^*$ (each $i < k$).*

LEMMA 14 *Let $k > 0$, $\alpha \in \mathcal{A}_n$ and $\phi \in \{f, g, \overline{f}, \overline{g}\}$.*

1. *If $\alpha\phi$ is k-short then so is α.*
2. *If $\phi\alpha$ is k-short then so is α.*
3. *If α is k-short and $\phi \in \{\overline{f}, g\}$ then $\alpha\phi$ and $\overline{\phi}\alpha$ are also k-short.*

PROOF:

1. Suppose $\alpha\phi$ is k-short, so $\widehat{\alpha} = A_0B_0 \ldots A_{k-1}B_{k-1}$ for some $A_i \in \{f, \overline{g}\}^*$, $B_i \in \{\overline{f}, g\}^*$, for $i < k$. Either $\widehat{\alpha\phi} = \widehat{\alpha}\ \phi$ or $\widehat{\alpha\phi}\ \overline{\phi} = \widehat{\alpha}$ (by lemma 9). With the first alternative, $\widehat{\alpha}$ is a substring of $\widehat{\alpha\phi} = A_0B_0 \ldots B_{k-1}$, so clearly α is k-short. With the second alternative, $\widehat{\alpha} = A_0 \ldots A_{k-1}B_{k-1}\overline{\phi}$, and $\overline{\phi}\phi \equiv 1'$ implies $\phi \in \{\overline{f}, g\}$. So $B_{k-1}\phi \in \{\overline{f}, g\}^*$ and hence α is k-short.
2. Similar.
3. Let $\widehat{\alpha} = A_0 \ldots B_{k-1}$ be k-short and $\phi \in \{\overline{f}, g\}$. Since $B_{k-1}\phi \in \{\overline{f}, g\}^*$ and $\widehat{\alpha\phi} = A_0 \ldots A_{k-1}; (\widehat{B_{k-1}\phi})$, we see that $\alpha\phi$ is also k-short. Similarly $\overline{\phi}\alpha$ is also k-short.

□

Game Strategy

Let $n \geq 1$ and $m > 2^n$. We define a strategy for $\exists$ in the game $G_n(\mathcal{A}_m)$. Recall from definition 4 that $\exists$ is required to play a sequence of finitary prenetworks $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{m-1}$ in a play of this game. To help her play the game, $\exists$ will calculate a sequence of networks $N'_0 \leq N'_1 \leq \ldots \leq N'_{m-1}$ where $N_i \subseteq N'_i$ for $i < m$. These networks N'_i will satisfy certain other properties.

DEFINITION 15 *Let N be a prenetwork and let $k \in \mathbb{N}$. We say that N is* k-good *if*

I. *N is a finitary network.*

II. *If st is k-short and $st \in N(x, y)$ then $\overline{(st)} \in N(y, x)$ and there is $z \in N$ such that $s \in N(x, z)$ and $t \in N(z, y)$.*

LEMMA 16 *Let $k \geq 1$, let N be k-good, let $x \in N$, let $\psi \in \{f, g, \overline{f}, \overline{g}\}$ and suppose there is no node $w \in N$ such that $N(x, w) = \psi$. Define a network $N^* \geq N$ with exactly one extra node, z, and labelling of edges incident with z defined by,*

$$\begin{aligned} N^*(z,z) &= \{1'\} \\ N^*(u,z) &= \{\rho;\psi : \rho \in N(u,x)\} \\ N^*(z,u) &= \{\overline{\psi};\lambda : \lambda \in N(x,u)\} \end{aligned}$$

where $u \in N$ is arbitrary. Then N^* is also k-good.

PROOF:

First we must check that N^* is a finitary network. It is clearly finitary. To show that N^* is a network, since N is known to be a network, we need only check the consistency of triangles and edges incident with the extra node z. By definition, $1' \in N^*(z, z)$. We show that $1' \notin N^*(u, z), N^*(z, u)$ for any $u \in N$. If $1' \in N^*(u, z)$ then $1' = \rho;\psi = \widehat{\rho\psi}$ for some $\rho \in N(u, x)$. Hence $\rho = \overline{\psi}$ (and $\overline{\psi}\psi \equiv 1'$), and by condition II for N, $\psi \in N(x, u)$. But this contradicts the assumption in the lemma. Similarly, $1' \notin N^*(z, u)$. Observe that $1'$ is minimal with respect to $<$, so N^* satisfies the first condition in definition 2.

Now we check that N^* is consistent with respect to composition. Let $u, v \in N$. If $\alpha \in N^*(u, z)$ and $\beta \in N^*(z, v)$ we require an element in $N^*(u, v)$ below $\alpha;\beta$. Well, since $\alpha \in N^*(u, z)$ we have $\alpha = \rho;\psi = \widehat{\rho\psi}$ for some $\rho \in N(u, x)$ and similarly $\beta = \widehat{\overline{\psi}\lambda}$ for some $\lambda \in N(x, v)$. By consistency of N, there is $\delta \in N(u, v)$ with $\delta \leq \rho;\lambda$. Hence $\delta \leq \rho;\lambda \leq \rho\psi;\overline{\psi}\lambda \equiv \alpha;\beta$, as required. Similarly, if $\alpha \in N^*(u, v)$ and $\beta \in N^*(v, z)$ then there is $\delta \in N^*(u, z)$ with $\delta \leq \alpha;\beta$, and if $\alpha \in N^*(z, u)$ and $\beta \in N^*(u, v)$ then there is $\delta \in N^*(z, v)$ with $\delta \leq \alpha;\beta$. This proves that N^* is a network.

Finally, we check condition II. Suppose $st \in N^*(u, z)$ is k-short. Then $st = \widehat{\rho\psi}$ for some $\rho \in N(u, x)$. By lemma 14, ρ is also k-short. By condition II for N, $\overline{\rho} \in N(x, u)$ and so $\widehat{\overline{\psi}\overline{\rho}} = \overline{\rho\psi} \in N^*(z, u)$. We seek a witness $w \in N^*$ with $s \in N^*(u, w)$ and $t \in N^*(w, z)$. If $t = 1'$ then trivially the required witness is $w = z$. Suppose $t \neq 1'$. By lemma 9, either $st = \rho\psi$ or $\rho = st\,\overline{\psi}$. In the former case, since $t \neq 1'$, we must have $t = t'\psi$ for some t' and $\rho = st' \in N(u, x)$. By condition II for N there is $w \in N$ with $s \in N(u, w)$ and $t' \in N(w, x)$. Therefore $t = t'\psi \in N^*(w, z)$, so w is the required witness in N^*. In the latter case $st\overline{\psi} \in N(u, x)$ so, inductively, there is $w \in N$ with $N(u, w) = s$ and $N(w, x) = t\overline{\psi}$ and so $t = \widehat{t\overline{\psi}\psi} \in N^*(w, z)$ and w is the required witness in N^*. Thus N^* satisfies condition II and is k-good.

□

LEMMA 17 *Let $k \geq 1$, let N be a k-good network and let $x, y \in N$. Let $\phi \in \{f, g, \bar{f}, \bar{g}\}$, suppose $\beta \in \mathcal{A}_m$ is not k-short and suppose there is $\gamma \in N(x, y)$ with $\gamma \leq \phi; \beta$. Also suppose that there is no node $w \in N$ and $\beta^- \leq \beta$ such that $\phi \in N(x, w)$ and $\beta^- \in N(w, y)$.*

Define $N^* \geq N$ with exactly one extra node, z say, and let the edges incident with z be labelled,

$$\begin{aligned} N^*(z,z) &= \{1'\} \\ N^*(u,z) &= \{\rho;\phi : \rho \in N(u,x)\} \\ N^*(z,u) &= \{\overline{\phi};\mu : \mu \in N(x,u)\} \cup \{\beta;\lambda : \lambda \in N(y,u)\} \end{aligned}$$

where $u \in N$ is arbitrary. Then N^* is also k-good, and $\phi \in N^*(x, z)$, $\beta \in N^*(z, y)$.

PROOF:

We check that N^* is a network. Since N is consistent we need only check the consistency of edges and triangles involving the new node z. We check the rule for the identity first. We'll show that $1' \notin N^*(u, z)$. For this, suppose instead that $1' \in N^*(u, z) = \{\rho; \phi : \rho \in N(u, x)\}$. Then $1' = \rho; \phi = \widehat{\rho\phi}$ for some $\rho \in N(u, x)$ so $\rho = \overline{\phi}$ and $\overline{\phi}; \phi \equiv 1'$. By condition II of definition 15 for N, $\phi \in N(x, u)$. By consistency of N (condition I) there is $\beta^- \in N(u, y)$ with $\beta^- \leq \overline{\phi}; \gamma \leq \overline{\phi}; \phi; \beta \equiv \beta$. But this contradicts the assumption in the lemma, that no witness node w exists in N.

Similarly, suppose $1' \in N^*(z, u)$. $1' \in \{\beta; \mu : \mu \in N(y, u)\}$ is impossible since, by lemma 14, $\beta\mu$ is not k-short, for any μ. If $1' \in \{\overline{\phi}; \lambda : \lambda \in N(x, u)\}$ then, as above, we derive a contradiction to our assumption that there is no witness node in N.

Now we check the rule of composition for N^*. Let $u, v \in N$ be arbitrary and let $\sigma \in N^*(u, z)$, $\tau \in N^*(z, v)$. We seek an element below $\sigma; \tau$ in $N^*(u, v)$. Since $\sigma \in N^*(u, z)$ we have $\sigma = \rho; \phi$ for some $\rho \in N(u, x)$. Since $\tau \in N^*(z, v)$ we have either $\tau = \overline{\phi}; \mu$ for some $\mu \in N(x, v)$ or $\tau = \beta; \lambda$ for some $\lambda \in N(y, v)$. Since N is known to be a network, either $\sigma; \tau = \widehat{\rho\phi\overline{\phi}\mu} \geq \rho\mu \geq \delta$ for some $\delta \in N(u, v)$, or $\sigma; \tau = \widehat{\rho\phi\beta\lambda} \geq \rho\gamma\lambda \geq \delta$, for some $\delta \in N(u, v)$, where $\gamma \in N(x, y)$. Similarly, we can check that if $\sigma \in N^*(z, u)$ and $\rho \in N^*(u, v)$ then there is something below $\sigma; \rho$ in $N^*(z, v)$, and if $\sigma \in N^*(u, v)$ and $\rho \in N^*(v, z)$ then there is something below $\sigma; \rho$ in $N^*(u, z)$. This proves condition I — N^* is a network.

Now we check condition II. Suppose $st \in N^*(u, z)$ is k-short. Then $st = \widehat{\rho\phi}$ for some $\rho \in N(u, x)$. By lemma 14 ρ is k-short so by condition II for N $\overline{\rho} \in N(x, u)$. Therefore $\widehat{\overline{\phi}\,\overline{\rho}} = \overline{st} \in N^*(z, u)$.

We also seek a witness $w \in N^*$ with $s \in N^*(u, w)$ and $t \in N^*(w, z)$. If $t = 1'$ then the required witness is $w = z$. Suppose

$t \neq 1'$. By definition of $\mathcal{A}_m$, $\widehat{\lambda} = \lambda$. Either $st = \widehat{\lambda\phi} = \lambda\phi$, or $\lambda = st\,\overline{\phi}$ and $\phi\overline{\phi} \equiv 1'$, by lemma 9. In the former case we have $st = \lambda\phi$ and since $t \neq 1'$ we have $t = t'\phi$ (some t') and $st' = \lambda$. By condition II for N, there is $w \in N$ with $s \in N(u, w)$ and $t' \in N(w, x)$. Hence $t = \widehat{t'\phi} \in N^*(w, z)$ so w is the required witness in N^* in this case. In the latter case, we have $\phi \in \{f, \overline{g}\}$ and by lemma 14, $st\overline{\phi}$ is also k-short, so there is $w \in N$ with $s \in N(u, w)$ and $t\overline{\phi} \in N(w, x)$. Therefore $t\overline{\phi}; \phi = t \in N^*(w, z)$, so w is the required witness in N^*.

The case where $st \in N^*(z, u)$ is k-short is handled similarly, bearing in mind that no element of the form $\beta; \mu$ can be k-short, by lemma 14.

Since N is a network and $1'$ is minimal with respect to $<$, $1' \in N(x, x)$ and $1' \in N(y, y)$, so $\phi \in N^*(x, z)$ and $\beta \in N^*(z, y)$.

□

Similarly,

LEMMA 18 *Let $k \geq 1$, N be k-good, $x, y \in N$, $\phi \in \{f, g, \overline{f}, \overline{g}\}$ and $\alpha \in \mathcal{A}_m$ and suppose (a) α is not k-short, (b) there is $\gamma \in N(x, y)$ with $\gamma \leq \alpha; \phi$ and (c) there is no witness $w \in N$ and $\alpha^- \leq \alpha$ such that $\alpha^- \in N(x, w)$ and $\phi \in N(w, y)$.*

Then there is a k-good network $N^* \geq N$ with a node z such that $\alpha \in N^*(x, z)$ and $\phi \in N^*(z, y)$.

THEOREM 19 *Let $n \geq 1$ and $m > 2^n$. Player $\exists$ has a winning strategy in $G_n(\mathcal{A}_m)$.*

Notation: If $\alpha = a_0 a_1 \ldots a_{|\alpha|-1}$ is a string and $i \leq j < |\alpha|$ then $\alpha[i, j]$ denotes the substring $a_i a_{i+1} \ldots a_{j-1}$.

PROOF:

Recall that a play of $G_n(\mathcal{A}_m)$ is a sequence of prenetworks $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_{m-1}$. To help her play this game, $\exists$ will calculate a sequence $N'_0 \leq N'_1 \leq \ldots \leq N'_{m-1}$ of networks satisfying

- $N_i \subseteq N'_i$,
- N'_i is 2^{n-i}-good,

for $i \leq m$.

In round zero let $\forall$ play (α_1, α_0) where $\alpha_1 \not\leq \alpha_0$. If α_1 is not 2^n-short then $N_0 = N'_0 = D(\alpha_1)$ has exactly two nodes, x_0 and x_1, as defined in definition 4. Otherwise, let α_1 be 2^n-short. In this case N'_0 has $|\alpha_1| + 1$ nodes, $x^0, x^1, \ldots, x^{|\alpha_1|}$ and every edge is labelled by a singleton. To define this labelling, let $N'_0(x^i, x^j) = \{\alpha_1[i, j]\}$ when $i \leq j \leq |\alpha_1|$ and $N'_0(x^i, x^j) = \{\overline{\alpha_1[j, i]}\}$ when $j < i \leq |\alpha_1|$. It can easily be checked that N'_0 is 2^n-good. Also there are $x_0, x_1 \in N'_0$ with $N'_0(x_0, x_1) = \{\alpha_1\}$ (if α_1 is 2^n short let $x_0 = x^0$ and $x_1 = x^{|\alpha_1|}$). If $x_0 = x_1$ (this happens iff $\alpha_1 = 1'$) then $N_0 = I(\alpha_1)$ else $N_0 = D(\alpha_1)$. Either way, $N_0 \subseteq N'_0$ and $N'_0(x_0, x_1) = \{\alpha_1\}$.

Now let $0 < i < n$ and consider round i. Let $r = n - i$ be the number of rounds left in the play of the game. Let the last prenetwork played in the game be N_{i-1} and suppose $\exists$ has calculated a 2^{r+1}-good network $N'_{i-1} \supseteq N_{i-1}$ and $N'_{i-1} \geq N'_0$. If $\forall$ plays $(N_{i-1}, x, y, z, \rho, \sigma)$ (where $x, y, z \in N_{i-1}$, $\rho \in N_{i-1}(x, y)$, $\sigma \in N_{i-1}(y, z)$) then, by the rules of the game, $\exists$ must let N_i be identical to N_{i-1} except that $N_i(x, z) = N_{i-1}(x, z) \cup \{\rho; \sigma\}$. She lets $N'_i = N'_{i-1}$. Since N'_i is a network it follows that $N_i \subseteq N'_i$ and trivially $N'_{i-1} \leq N'_i$.

Now suppose $\forall$ picks $x, y \in N_{i-1}$ and α^+, β^+ such that there is $\gamma \leq \alpha^+; \beta^+$ with $\gamma \in N_{i-1}$. In short, $\forall$ plays $(N_{i-1}, x, y, \alpha^+, \beta^+)$. First, $\exists$ finds minimal $\alpha, \beta \in \mathcal{A}_m$ such that $\alpha \leq \alpha^+$, $\beta \leq \beta^+$ and $\alpha; \beta \geq \gamma$. By 'minimal' we mean that if $\alpha^- \leq \alpha$, $\beta^- \leq \beta$ and $\alpha^-; \beta^- \geq \gamma$ then $\alpha^- = \alpha$, $\beta^- = \beta$. Such minimal elements exist in $\mathcal{A}_m$, since $\leq$ is clearly well-founded in $\mathcal{A}_m$.

Then $\exists$ will construct a 2^r-good network $N'_i \geq N'_{i-1}$ containing a node z such that $\alpha \in N'_i(x, z)$ and $\beta \in N'_i(z, y)$. The remainder of this proof shows how she can find such a network N'_i. Having done that she lets $N_i = N^+(N_{i-1}, x, y, z, \alpha^+, \beta^+)$, defined in definition 4. Since $N'_{i-1} \leq N'_i$, $\alpha \leq \alpha^+$ and $\beta \leq \beta^+$ it follows that $N_i \subseteq N'_i$. Since N'_i is a network it follows that there cannot be $u \neq v \in N_i$ and $\tau \in N_i(u, v)$ such that $\tau \leq 1'$. Also, since $N'_0 \leq N'_i$ and $N'_0(x_0, x_1) = \{\alpha_1\}$ there cannot be $\alpha^- \leq \alpha_0$ with $\alpha^- \in N_i(x_0, x_1)$. This will show that $\forall$ does not win round i.

Thus it suffices to find a 2^r-good $N'_i \geq N'_{i-1}$ with a node z such that $\alpha \in N'_i(x, z)$ and $\beta \in N'_i(z, y)$. There are four cases to consider.

α, β **both long** If neither α nor β is 2^r-short then $\exists$ lets $N'_i \geq N_{i-1}$ extend N'_{i-1} with a single node, z say. For the labelling of edges incident with z,

$$\begin{aligned} N'_i(z, z) &= \{1'\} \\ N'_i(u, z) &= \{\rho; \alpha : \rho \in N'_{i-1}(u, x)\} \\ N'_i(z, u) &= \{\beta; \lambda : \lambda \in N'_{i-1}(y, u)\} \end{aligned}$$

where $u \in N'_{i-1}$ is arbitrary. Note, by lemma 14, that no element of $N'_i(u, z)$ or $N'_i(z, u)$ is 2^r-short. So it is easy to check that N'_i is 2^r-good.

α **is short,** β **is long** Let $\alpha = a_1 a_2 \ldots a_j$, for some $j \in \mathbb{N}$ and some characters $a_i : i \leq j$. By lemma 17 there is a 2^{r+1}-good $N^1 \geq N'_i$ with a node z_1 such that $a_1 \in N^1(x, z_1)$ and $a_2 \ldots a_j \beta \in N^1(z_1, y)$. Iterating this process, there is a 2^{r+1}-good N^* with a node z_j such that $a_1 \in N^*(x, z_1)$, $a_2 \in N^*(z_1, z_2), \ldots, a_j \in N^*(z_{j-1}, z_j)$ and $\beta \in N^*(z_j, y)$. Since N^* is a network it follows that there is $\alpha^- \leq a_1 a_2 \ldots a_j = \alpha \in N^*(x, z_j)$. By minimality, $\alpha^- = \alpha$. $\exists$ lets $N'_i = N^*$ and $z = z_j$. Since N'_i is 2^{r+1}-good it is certainly 2^r-good.

α **is long,** β **is short** The case where β is 2^r-short but α is not, is proved similarly.

α, β **both short** If both α and β are 2^r-short then $\gamma \leq \alpha; \beta$ is 2^{r+1}-short. In this case $\exists$ can actually find a 2^{r+1}-good network N'_i satisfying the required conditions. Since a 2^{r+1}-good network is certainly 2^r-good, the inductive conditions are maintained.

By minimality of α, β, either $\gamma = \alpha\beta$ or $\alpha = \alpha'\psi$, $\beta = \overline{\psi}\beta'$ and $\alpha'; \beta' \geq \gamma$ (some $\psi \in \{f, \overline{f}, g, \overline{g}\}$, some α', β'). In the former case, since N'_{i-1} is 2^{r+1}-good, there is already a node $z \in N'_{i-1}$ such that $\alpha \in N'_{i-1}(x, z)$, $\beta \in N'_{i-1}(z, y)$, so $\exists$ can let $N'_i = N_i$. In the latter case, by a suitable induction on $|\alpha| + |\beta|$, there is a 2^{r+1}-good network $N_1 \geq N'_{i-1}$ with a node $w \in N_1$ such that $\alpha^- \in N_1(x, w)$ and $\beta^- \in N_1(w, y)$ for some $\alpha^- \leq \alpha'$ and $\beta^- \leq \beta'$. By lemma 16 there is a 2^{r+1}-good $N'_i \geq N_1$ with a node z such that $\psi \in N'_i(w, z)$ and $\overline{\psi} \in N'_i(z, w)$. Then, since N'_i is a network, there is $\alpha^* \leq \alpha^-; \psi$ and $\beta^* \leq \overline{\psi}; \beta^-$ with $\alpha^* \in N'_i(x, z)$ and $\beta^* \in N'_i(z, y)$. By minimality, $\alpha^* = \alpha$ and $\beta^* = \beta$.

□

PROBLEM 1 *We have been somewhat wasteful in the way we constructed a winning strategy for* $\exists$ *in* $G_n(\mathcal{A}_{2^n+1})$. *A rough calculation shows that a winning strategy for* $\forall$ *in* $G_n(\mathcal{A}_m)$ *along the lines of theorem 12 would take at least* $6m$ *rounds. Show that* $\exists$ *has a winning strategy in* $G_n(\mathcal{A}_n)$, *for* $n \geq 1$.

THEOREM 20 *There class of representable ordered monoids cannot be defined by finitely many axioms.*

PROOF:

By proposition 5, theorem 12 and theorem 19. □

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