Undecidability of algebras of binary relations

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Abstract Let S be a signature of operations and relations definable in relation algebra (e.g. converse, composition, containment, union, identity, etc.), let $R(S)$ be the class of all S-structures isomorphic to concrete algebras of binary relations with concrete interpretations for symbols in S, and let $F(S)$ be the class of S-structures isomorphic to concrete algebras of binary relations over a finite base. To prove that membership of $R(S)$ or $F(S)$ for finite S-structures is undecidable, we reduce from a known undecidable problem — here we use the tiling problem, the partial group embedding problem and the partial group finite embedding problem to prove undecidability of finite membership of $R(S)$ or $F(S)$ for various signatures S. It follows that the equational theory of $R(S)$ is undecidable whenever S includes the boolean operators and composition. We give an exposition of the reduction from the tiling problem and the reduction from the group embedding problem, and summarize what we know about the undecidability of finite membership of $R(S)$ and of $F(S)$ for different signatures S.

Introduction

It has been known for some time that various problems involving binary relations are undecidable: for example Tarski proved that the set of equations valid over all representable relation algebras is undecidable. Here, a representation of a relation algebra is an isomorphism to an algebra of binary relations, and the isomorphism must respect all relation algebra operations. If we weaken the notion of representation so that only a specified set of relation algebra operations must be preserved, it might be that the equational theory becomes decidable (for example, if just the booleans must be preserved, or if just identity, composition and inclusion must be preserved, then the equational theory is decidable). So where is the boundary? How

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much of the signature of relation algebra is needed in order to prove undecidability? How far can Tarski's result be extended?

In order to prove that a decision problem is undecidable, the obvious way is to *reduce* a known undecidable problem to it. For known undecidable problems, there are many to choose from: Turing machines, Minsky machines, Post's correspondence problem, tiling problems and so on. We make no claim to survey all the work on undecidability of binary relation problems, but in broadest outline, the first batch of results to do with undecidability of binary relations were based on the undecidability of parts of set theory, the second main collection of results were based on the undecidability of the word problem for semigroups, and the third and fourth sets of cases are studied in more detail here and are based on the tiling problem and the partial group embedding problem, respectively.

The undecidability results are of two related types: the first kind is where we prove the undecidability of some logical theory (e.g. the set of equations valid over relation algebra); the second kind is where we prove the undecidability of some embedding problems, between certain classes of relation algebras and their reducts (e.g. the set of finite, representable relation algebras). The undecidability of an embedding problem normally entails the undecidability of a related logical theory. We study some of the key constructions, proofs and results.

Definitions

Definition 1. A *relation algebra* $\mathscr{A} = (A, +, -, 0, 1, 1', \smile, ;)$ consists of a boolean algebra $(A, +, -, 0, 1)$ together with additive, normal operators $1'$, $\smile$, ;, respectively 0-ary, 1-ary and 2-ary, such that $(A, 1', ;)$ is a monoid, $\smile$ is an involution (i.e. $(a^\smile)^\smile = a$ and $(a;b)^\smile = b^\smile;a^\smile$), satisfying $a;b \cdot c = 0 \iff c;b^\smile \cdot a = 0$. A relation algebra $\mathscr{A}$ is a *set* relation algebra, or a *proper relation algebra*, if there is a base set X say, an equivalence relation U over X, and each element of A is a subset of U and the operators are set union, complement in U, empty set, U, the identity over X, the converse and composition operators over binary relations, respectively. A *representation* is an isomorphism from a relation algebra to a proper relation algebra. A *finite* representation is a representation to a proper relation algebra on a finite base.

An atom of a relation algebra is an atom of the boolean reduct, i.e. a minimal non-zero element, and a relation algebra is atomic if every non-zero element is above some atom. A *unit* is an atom below the identity.

A *relation algebra term* is built from variables and constants $0, 1, 1'$ using operators $+, -, \smile, ;$. An *equation* has the form $s = t$, for two relation algebra terms s, t, and a *quasi-equation* has the form $(e_1 \wedge e_2 \wedge \ldots \wedge e_k) \to e$, where e, e_i $(i \leq k)$ are equations. For any class $\mathscr{K}$ of relation algebras, the equational theory $Eq(\mathscr{K})$ is the set of all equations valid on each relation algebra in $\mathscr{K}$, and the quasi-equational theory of $\mathscr{K}$ is the set of quasi-equations valid over $\mathscr{K}$.

By additivity and the identity law, if a is an atom of a relation algebra $\mathscr{A}$ then there are unique units $st(a), end(a)$ such that $a = st(a);a;end(a)$ [22, theorem 3.5].

Other relations and operators are definable: for example, the set inclusion relation is definable by $a \leq b \iff a+b=b$, and $a \cdot b = -(-a+-b)$, $\mathsf{dom}(a) = 1' \cdot (a;1)$, and $\mathsf{rng}(a) = \mathsf{dom}(a^{\smile})$. A *signature* S is any subset of $\{0,1,\leq,\cdot,+,-,1',\mathsf{dom},\mathsf{rng},{}^{\smile},;\}$. The additional relations and operators play a significant role when they are included in S, but not all the symbols used in their definition are included in S, for example consider a signature including the relation $\leq$ but not including $+$ or $\cdot$. An *S-structure* consists of a set with an interpretation of relations and functions in S, it is proper if it consists of binary relations on some base set X, and symbols in S have natural set-theoretically defined interpretations, i.e. non-negative booleans are interpreted as $\emptyset$, U, $\subseteq$, $\cap$, $\cup$ for some maximal binary relation $U \subseteq X \times X$, negation is complement in U, and

$$1' = \{(x,x) : x \in X\} \qquad a^{\smile} = \{(y,x) : (x,y) \in a\}$$
$$\mathsf{dom}(a) = \{(x,x) : \exists y\,(x,y) \in a\} \qquad \mathsf{rng}(a) = \{(y,y) : \exists x\,(x,y) \in a\}$$
$$a;b = \{(x,y) : \exists z\,((x,z) \in a \wedge (z,y)) \in b\}.$$

An *S-representation* is an isomorphism from an S-structure to a proper S-structure. The class of all S-representable S-structures is denoted $R(S)$. A *symmetric S-representation* is a representation in which the maximal relation U is symmetric. Let $R_{\leftrightarrow}(S)$ be the class of S-structures with symmetric S-representations. If converse is included in S then clearly $R_{\leftrightarrow}(S) = R(S)$, but also for the signature $S = \{-,+,1',;\}$ it is known [15, lemma 1.1] that $\mathscr{A} \in R(\{-,+,1',;\}) \iff \mathscr{A} \in R_{\leftrightarrow}(\{-,+,1',;\})$ for any S-structure $\mathscr{A}$ provided it satisfies the normality laws $D(x);x = x = x;R(x)$ where $D(x) = 1' \cdot x;1$ and $R(x) = 1' \cdot 1;x$. The class of all S-structures with an S-representation on a finite base is denoted $F(S)$. An instance of the *S-representation problem* is a finite S-structure $\mathscr{A}$: it is a yes-instance if it has an S-representation (i.e. it belongs to $R(S)$), it is a no-instance otherwise. An instance of the *finite S-representation problem* is also an arbitrary finite S-structure $\mathscr{A}$: it is a yes-instance if it has an S-representation on a finite base (i.e. it belongs to $F(S)$), it is a no-instance otherwise. Given any class $\mathscr{K}$ of S-structures, the *equational theory* $Eq(\mathscr{K})$ is the set of equations $s = t$ valid over $\mathscr{K}$, where s,t are terms built from variables, constants and functions in S and the *quasi-equational theory* is the set of implications $(e_1 \wedge e_2 \wedge \ldots \wedge e_k) \to e$ valid over K, where e, e_i $(i \leq k)$ are equations.

Main Results

Theorem 2. *For any relation algebra signature S where $S \supseteq \{\cdot,+,;\}$ or $S \supseteq \{\cdot,{}^{\smile},;\}$, it is undecidable whether $\mathscr{A} \in R(S)$ for arbitrary finite S-structures $\mathscr{A}$. If ${}^{\smile} \notin S \supseteq \{\leq,-,;\}$ then it is undecidable whether $\mathscr{A} \in R_{\leftrightarrow}(S)$, for finite S-structures $\mathscr{A}$.*

If converse is not in S and S contains $\{\cdot,+,;\}$ or $\{-,\leq,;\}$ then for an arbitrary finite S-structure $\mathscr{A}$ it is undecidable whether $\mathscr{A} \in F(S)$, and undecidable whether $\mathscr{A}$ is the S-reduct of a finite relation algebra.

Corollary 3. *The quasi-equational theory of $R(S)$, $R_{\leftrightarrow}(S)$ or $F(S)$, as appropriate, is undecidable for any of the signatures S mentioned in the previous theorem.*

The equational theory of $R(S)$ is undecidable when $S \supseteq \{+,-,;\}$, the equational theory of $F(S)$ is undecidable when $\smile \notin S \supseteq \{+,-,;\}$.

Some earlier results

The first approach to undecidability was based on set theory. In their formalisation of set theory without variables, Tarski and Givant consider proper relation algebras $\mathscr{P}$ containing some designated element ε. See [30, 8, 9] for the details; we sketch the briefest outline. They define an equational language $\mathscr{L}^{\times}$ of equations between binary predicates. The basic predicates are $\mathring{1}, \mathsf{E}$, and compound predicate terms are built from these using relation algebra operators. The equations of the language are equations between predicate terms. Each term may be interpreted in $\mathscr{P}$ by mapping $\mathring{1}$ to the identity element of $\mathscr{P}$, mapping E to ε, and extending to compound terms using the (concrete) operators of $\mathscr{P}$. An equation holds in $\mathscr{P}$ if and only if the two terms denote the same binary relation. Tarski and Givant also define a finite schema of equations and write $\vdash^{\times} A = B$ if the equation $A = B$ follows from instances of these axioms by the rule of replacement. The theory of consequences of the empty set axiom together with a sentence that requires a set $x \cup \{y\}$, given any sets x, y, is known to be undecidable [31, Statements 1.6, 6.1]. By a translation into $\mathscr{L}^{\times}$ they deduce that both the set of equations valid in proper relation algebras and the set of $\vdash^{\times}$-deducible equations are undecidable. From the former result they deduce that the equational theory of all representable relation algebras is undecidable, and from the latter that the equational theory of the axiomatically defined class of all relation algebras is also undecidable.

Let Σ be a finite alphabet. Observe that Σ^* with concatenation forms a semigroup. An instance of the word problem for semigroups is a finite set E of equations between elements of Σ^* and a single equation $s = t$. It is a yes instance if there is a finite sequence of substitutions $u_0u_1u_2 \mapsto u_0v_1u_2$ where $(u_1, v_1) \in E$ starting from s and reaching t; it is a no instance otherwise. This problem was proved undecidable in [26]. Maddux used this problem to prove the undecidability of the equational theory of the classes of three-dimensional cylindric algebras and three-dimensional diagonal-free algebras [21]. These undecidability results were also proved, via the undecidability of first-order logic, in [24, 4]. From the undecidability of the word problem for semigroups, [1] were able to prove the undecidability of the equational theory of any class of relation algebras containing for each $n \in \omega$ either (i) an algebra with at least n elements below the identity, (ii) a group relation algebra on a symmetric group with at least n elements, or (iii) a group relation algebra on the group reduct of a vector space with at least n dimensions.

Tiling

The previously mentioned approaches prove the undecidability of various equational theories but do not consider the problem of determining representability or finite representability of relation algebras, or of their reducts to weaker signatures. Our main focus here will be on extending these results to cover these representation problems. First we use the undecidability of the tiling problem to prove that membership of $R(S)$ is undecidable even for weak signatures, but this leaves open the decidability of $F(S)$ for such signatures. Then we use the undecidability of the partial group embedding property for an alternative proof of the undecidability of membership of $R(S)$, and also the undecidability of membership of $F(S)$, provided converse is excluded.

The problem of whether representability of finite relation algebras is decidable was discussed by Roger Maddux and Ralph McKenzie in the early 1980s, Maddux suggesting a solution by tiling. It was raised again by McKenzie at a conference on universal algebra and lattice theory in Szeged in 1996. The problem is listed in [3, page 730, open problem 3] (credited to Maddux) and [23, problem 14, page 463].

Theorem 4. *It is undecidable whether a finite relation algebra is representable.*

Proof. For a full rigorous proof see [14, chapter 18]. We will take a brisk look at some of the issues involved in the proof.

We start from the beginning. We want to show that there is no algorithm to determine whether a finite relation algebra is representable, and we aim to do it by reducing a known undecidable problem to the representability problem for relation algebras. The undecidable problem that we select here is the *tiling problem*, as Maddux suggested: given a finite set τ of square tiles, each of whose edges is coloured, can the integer plane $\mathbb{Z}\times\mathbb{Z}$ be tiled with copies of tiles from τ in such a way that adjacent tiles have the same colours on the edges where they touch? Formally, τ is a yes-instance of the tiling problem iff there is a 'tiling' map $f:\mathbb{Z}\times\mathbb{Z}\to\tau$ such that for every $x,y\in\mathbb{Z}$, the colour of the right-hand (respectively, top) edge of the tile $f(x,y)$ is the same as the colour of the left-hand (respectively, bottom) edge of the tile $f(x+1,y)$ (respectively, $f(x,y+1)$). This problem, and many variants of it,[1] are known to be undecidable [5].

So given a finite set τ of tiles, we would like to construct a finite relation algebra $RA(\tau)$ that is representable just when τ can tile the plane. Note that the algebra $RA(\tau)$ cannot be constructed from a tiling of the plane by tiles from τ, because there may not be such a tiling! Instead, the algebra should directly encode the tiles and their colours in some way, so that we can read off a tiling from any representation, and conversely, if there is a tiling then we can somehow construct a representation.

The tiling problem is appropriate because of the obvious (as it seems to us) resemblance of a tiling to a representation of a relation algebra. In a tiling $f:\mathbb{Z}\times\mathbb{Z}\to$

[1] One such variant is when (i) the instances are restricted to those τ that contain a special tile whose four sides all have a fixed 'white' colour that occurs only on this one tile, and (ii) τ is a yes-instance iff each tile in τ occurs in some tiling of the plane. This is the variant actually used in the proof, but these extra particulars will not concern us in this outline.

τ, *each tile* $t \in \tau$ *'is' already a binary relation on* $\mathbb{Z}$, namely $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : f(x,y) = t\}$. This is a relation between the 'x-axis' and the 'y-axis' of the plane, and it will clarify things if we separate out the two axes — make them disjoint. Viewing f in this way, the tiles run between the axes, as shown in figure 1. A line

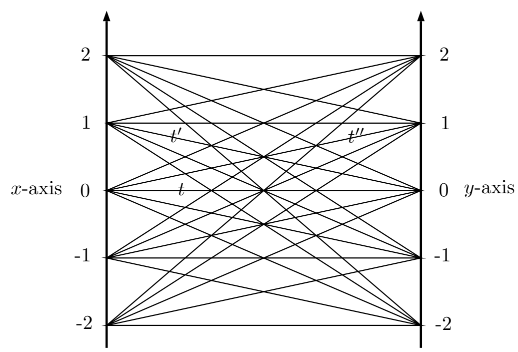

Fig. 1 A tiling of the plane $\mathbb{Z} \times \mathbb{Z}$ with x- and y-axes made disjoint

from x to y represents the location (x,y) and is 'labelled' by the tile $f(x,y)$. In the figure, we have supposed that $f(0,0) = t$, $f(1,0) = t'$, and $f(0,1) = t''$.

Figure 1 starts to suggest a possible design of $RA(\tau)$. It looks so far like a non-integral[2] relation algebra with two subidentity atoms: e_1 for x-axis points and e_2 for y-axis points. The base of any representation will be partitioned into the set of points x_1 of sort 1 (the 'x-axis' — those points x_1 with $(x_1,x_1) \in e_1$) and the points y_2 of sort 2 (the 'y-axis' — those y_2 with $(y_2,y_2) \in e_2$). We use indices $-_1$ and $-_2$ to indicate the sorts of points in representations. Similarly, we write atoms in the form a_{ij}, indicating that the atom relates points of sort i to points of sort j. (Formally, $e_{i'} \,;a_{ij} \,;e_{j'} \neq 0$ iff $i = i'$ and $j = j'$.) We write a_{ii} simply as a_i, as with e_1, e_2.

For each tile $t \in \tau$, we will include an atom t_{12} of $RA(\tau)$, relating points of sort 1 to points of sort 2. We also add its converse, t_{21}. Hoping to recover the $\mathbb{Z}$-structure of the axes, we add an atom $+1_1$ relating points on the x-axis, with intended interpretation $\{(x_1, x_1+1) : x_1 \in x\text{-axis}\}$, and a similar atom $+1_2$ for the y-axis. We also add their converses, -1_1 and -1_2. For tiles $t, t' \in \tau$, we will define composition in $RA(\tau)$ so that $t_{12} \,;t'_{21} \geq +1_1$ just when the colours of the right of t and the left of t' are the same, so we could put t' directly to the right of t in a tiling. Similarly, we will define $t_{21} \,;t''_{12} \geq +1_2$ when the top of t and the bottom of t'' have the same colour. See figure 2. (Sometimes we leave indices of atoms in diagrams to be determined by context.)

Let us now have a closer look at the atoms a_1 relating points of sort 1. These atoms, currently e_1, $+1_1$ and -1_1, should form the atoms of a finite representable relation algebra. The 1-sort of a representation of $RA(\tau)$ will be a representation of

[2] A relation algebra is integral if its identity element $1'$ is an atom.

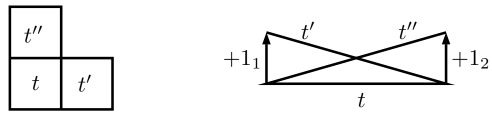

Fig. 2 Matching of colours of tile edges reflected in relation algebra composition

this algebra. So a representation of it should contain a copy of $\mathbb{Z}$, or something like it, for our x-axis.

What kind of relation algebra could it be? One danger is that $+1_1\,;+1_1$ might be a new atom, say $+2_1$, relating a point x_1 to 'x_1+2'; then $+1_1\,;+2_1 = +3_1$, another new atom; and so on — we end up with infinitely many atoms, whereas $RA(\tau)$ should be finite. But there is a well-known solution here: add a *flexible (self-converse) atom* w_1 with $w_1 \leq a_1\,;b_1$ for all non-identity atoms a_1, b_1 of sort 1. So the sort-1 atoms will be e_1, $+1_1$, -1_1, and w_1. It is well known through work of Comer and Maddux [6, 22] that every finite relation algebra with a flexible atom is representable. And starting at any point p^0 in (the base of) a representation of this algebra, we can find a point p^1 related to it by $+1_1$, because $(p^0, p^0) \in e_1 \leq +1_1\,;-1_1$. We can then find a point p^2 related to p^1 by $+1_1$ in the same way. And because $(p^0, p^0) \in e_1 \leq -1_1\,;+1_1$, we can find a point p^{-1} related by $+1_1$ to p^0. Continuing this 'in both directions', we can find points p^x for $x \in \mathbb{Z}$ with (p^x, p^{x+1}) related by $+1_1$ for each x. We expect that $(p^x, p^{x'})$ will be related by w_1 when $|x - x'| > 1$. The points p^x might not all be distinct, and there will be other points around as well, but this won't matter, because the overall structure of $RA(\tau)$ should allow us to read off a tiling from a representation, anyway.

Now, in any representation of $RA(\tau)$, the sort-1 points will form a representation of the relation algebra discussed above. And if we include a similar flexible atom w_2 of sort 2 in $RA(\tau)$, then the sort-2 points will form a representation of the relation algebra with atoms $e_2, +1_2, -1_2, w_2$. So we can find points p_1^x $(x \in \mathbb{Z})$ of sort 1, and similarly, points q_2^y $(y \in \mathbb{Z})$ of sort 2. Each pair (p_1^x, q_2^y) will lie in a unique tile atom $f(x,y)_{12}$, say, and the resulting map $f : \mathbb{Z} \times \mathbb{Z} \to \tau$ will be a tiling. So we can read off a tiling from any representation of our algebra.

For the converse, we need to construct a representation of $RA(\tau)$ from a tiling of the plane by τ. We intend to do it by a 'step-by-step' construction, or by a two-player game. But there are all kinds of problems.

Here is one. When building our representation, suppose we have constructed points p_1, q_1 of sort 1 and related by w_1. We will put $w_1 \leq t_{12}\,;t'_{21}$ for any tiles t, t', since our idea is that w_1 will relate all points x, x' that lie far apart on the x-axis, and the tiling could in principle place any tiles at all at (x,y), (x',y). Also, $w_1 \leq +1_1\,;-1_1$ since it is flexible. These compositions will force the existence of points s_2, r_1 in the representation as shown in figure 3.

Now r_1 is of sort 1 and s_2 of sort 2, so (r_1, s_2) will be related by a tile atom, say t^*_{12}. But then, the triangles $p_1 s_2 r_1$ and $q_1 s_2 r_1$ entail that $+1_1 \leq t_{12}\,;t^*_{21}$ and $+1_1 \leq t'_{12}\,;t^*_{21}$,

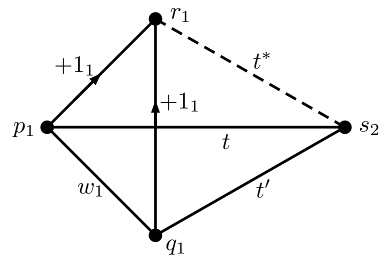

Fig. 3 First problem

so t^* fits to the right of both t,t' — and there is no reason why such a t^* should exist for arbitrary tiles t,t', even if τ can tile the plane.

Related instances of the same problem arise when some or all of the $+1_1$ atoms in figure 3 are replaced by -1_1. They raise no new issues and we will skip over them, here and below.

It seems that we need another flexible atom w_{12}, to relate r_1 to s_2 in cases like this. It certainly solves the 'figure 3 problem', *and we do add such an atom,* and its converse w_{21}.

But of course we have now created a second problem. When we try to read off a tiling from a representation of $RA(\tau)$, we may find w_{12} atoms where we expected tile atoms. We will fail to construct the tiling.

The key to getting out of this difficulty is to realise that r_1 in figure 3 arises as part of a *triangle* with base p_1q_1. The point r_1 witnessed the composition $(p_1,q_1) \in w_1 \leq +1_1\,;-1_1$, and p_1,q_1 are distinct. When reading off a tiling from a representation, as above, we used only points $p^0,p^1,\ldots$ that each arose essentially from a *single* preceding point. For example, p^6 witnessed the composition $(p^5,p^5) \in e_1 \leq +1_1\,;-1_1$ and so arises from the single point p^5. Points 'generated' by a single point do need to be related to sort-2 points by tiles, so that we can read off a tiling; but points like r_1 'generated' by pairs of points need not be.

This prompts a radical redesign of our algebra $RA(\tau)$. *We introduce a third sort of points: sort* 0. There will now be three subidentity atoms, e_0,e_1,e_2. We introduce a *'green'* atom g_{01} from sort 0 to sort 1, a similar green atom g_{02} from sort 0 to sort 2, and their converses g_{10},g_{20}. We stipulate that $w_{12} \not\leq g_{10}\,;g_{02}$ — so w_{12} is no longer completely flexible. Now, if p_1,q_2 are sort 1 and 2 points connected by a *'green path'* — there is some 'source' sort 0 point s_0 related to p_1 and q_2 by green atoms — then (p_1,q_2) cannot be in w_{12}, so must lie in a tile atom.

Now, in figure 3, the fact that r_1 witnesses the composition $(p_1,q_1) \in w_1 \leq +1_1\,;-1_1$ *does not mean* that there has to be a green path from r_1 to s_2. In relation algebra representations built step by step, we cannot in general demand existence of a point with specified relations to three other points (-1_1 to p_1 and q_1, and green to some third point). And if there isn't a green path, then (r_1,s_2) could be in w_{12} and not in any tile atom t^*_{12} at all. The figure 3 problem is solved, at least potentially (to show it conclusively, see the full proof).

The second problem is also solved, because we can still read off a tiling from a representation of $RA(\tau)$ as before, by ensuring that the p_1^x and q_2^y are all related by green to a fixed source point s_0. Given p_1^x such that $(s_0, p_1^x) \in g_{01}$, we ensure that $g_{01} \leq g_{01} ; -1_1$, and if p_1^{x+1} witnesses this composition then not only $(p_1^x, p_1^{x+1}) \in +1_1$ but also $(s_0, p_1^{x+1}) \in g_{01}$, so we can continue. Similarly for the q_2^y. Now there is a green path via s_0 from every p_1^x to every q_2^y, so they must be related by tile atoms $f(x,y)_{12}$ as desired. See figure 4.

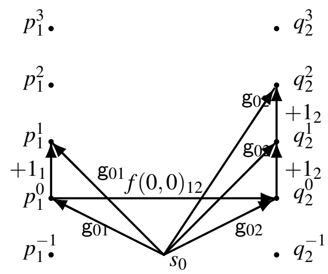

Fig. 4 Reading off a tiling from a representation of $RA(\tau)$

To fill out the design, we add more flexible atoms w_{ij} so that we have one for every pair of sorts i, j. (The converse of w_{ij} is w_{ji}. We had w_1, w_2, w_{12}, w_{21} already.) Our atoms are now e_0, e_1, e_2 (subidentity) and the w_{ij}, plus the new green atoms g_{ij} for $\{i, j\} \in \{\{0,1\},\{0,2\}\}$, the ± 1 atoms, and the tile atoms t_{12}, t_{21}.

We assure despondent readers that we are genuinely making good progress. Unfortunately, the figure 3 problem reappears in a more subtle form. Consider figure 5. The left-hand diagram in figure 5 is similar to figure 3, but we now have a source point s_0 for p_1, q_1 (so (s_0, p_1) and (s_0, q_1) are green). These points p_1, q_1 are related by w_1 and there is a chain of points connecting them, related successively by $+1_1$, just as we would expect to see in a copy of $\mathbb{Z}$. Each point in the chain is related by green to s_0. The point r_1 appears much as before, as a witness to the composition $(p_1, q_1) \in w_1 \leq -1_1 ; -1_1$.

Now by our recent innovations, r_1 should be related to s_0 by w_{10}, not by g_{10}. So the old problem of figure 3 seems to have gone away. But for any tile t^*, we have $(r_1, s_0) \in w_{10} \leq t^*_{12} ; g_{20}$, and on the right-hand side of figure 5 we have added a witness u_2 to this demonic composition. There is a green path (via s_0) from u_2 to p_1, q_1 and all the points in between. So all these points must be related to u_2 by tile atoms, shown as $t^1, \ldots, t^6$ in figure 5. Unlike the earlier situation, these tile atoms are not specified in advance, but there is a fixed number of them (here, six, but it can be arbitrarily large) and they all lie in triangles whose third side is $+1_1$, so they must fit with t^* and each other as shown in figure 6. Whether τ can tile the plane or not, there is no reason whatever to suppose that, for arbitrary t^*, tiles $t^1, \ldots, t^6$ with these properties exist.

There is also a kind of dual or mirror image of this problem, where the tiles arise before the ± 1s, and it is shown in figure 7. On the left-hand side of figure 7, the point

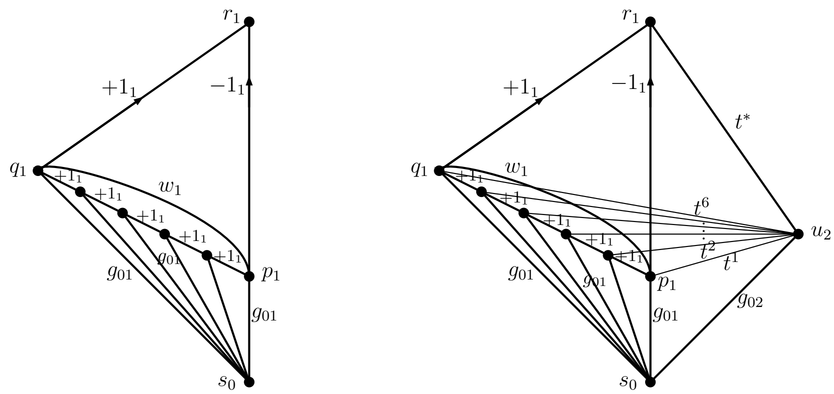

Fig. 5 First problem (subtle form)

t^*	t^1	t^2	t^3	t^4	t^5	t^6	t^*

Fig. 6 Required tile pattern

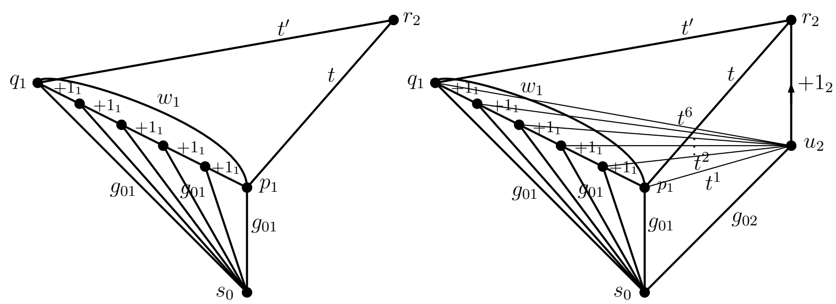

Fig. 7 Mirror image problem

r_2 witnesses the composition $(p_1,q_1) \in w_1 \leq t_{12}\,;t'_{21}$, as in figure 3, and once again the tiles t,t' are arbitrary. Then (s_0,r_2) will lie in w_{02}, so there need be no green path to r_2 from the points between p_1 and q_1, and no need for tile atoms to relate them. But $(s_0,r_2) \in w_{02} \leq g_{02}\,;+1_2$, and on the right of figure 7 we see a witness, u_2, to this composition. There is a green path to u_2 (via s_0) from p_1,q_1, and all points between, so all the resulting pairs will lie in tile atoms, again say $t^1_{12},\ldots,t^6_{12}$. And the triangles with a $+1_1$ or $+1_2$ edge force that $t,t',t^1,\ldots,t^6$ can be arranged as in figure 8. Again, for arbitrary t,t' there is no reason at all to suppose that suitable tiles $t^1,\ldots,t^6 \in \tau$ can be found.

These are critical problems and they demand action. At the heart of them is the need to witness the compositions $(r_1,s_0) \in w_{10} \leq t^*_{12}\,;g_{20}$ in figure 5 and $(s_0,r_2) \in w_{02} \leq g_{02}\,;+1_2$ in figure 7. So during the construction of a representation of $RA(\tau)$,

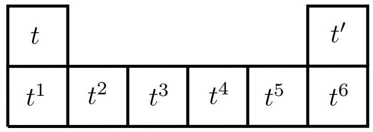

Fig. 8 Required tile pattern in mirror image problem

we arrange that whenever a configuration like the left-hand diagram in figure 5 appears, (r_1, s_0) does *not* lie in w_{10} but in a special new 'blocking atom' called u_{10}. We require that for every tile t we have $u_{10} \not\leq t_{12}\,;g_{20}$. A dual atom u_{20} for similar configurations involving sort 2 is also added, plus of course the converses u_{01}, u_{02} of these atoms. Blocking atoms like these are distinctive features of *rainbow constructions,* introduced in [12] and developed in several later theorems, including this one.

Now, in the left-hand diagram in figure 5, we will have $(r_1, s_0) \in u_{10}$. This means that the right-hand configuration will not occur, because no witness u_2 to the composition $(r_1, s_0) \in u_{10} \leq t^*_{12}\,;g_{20}$ is required or indeed possible, since $u_{10} \not\leq t^*_{12}\,;g_{20}$.

For the mirror image problem of figure 7, we introduce similar new blocking atoms v_{02}, v_{01}, and their converses, with $v_{01} \not\leq g_{01}\,;+1_1$ and $v_{01} \not\leq g_{01}\,;-1_1$ (and dually for v_{02}). We arrange that whenever a configuration as on the left of figure 7 arises, then $(s_0, r_2) \in v_{02}$. Then, on the right of the figure, no witness u_2 to the composition $(s_0, r_2) \in v_{02} \leq g_{02}\,;+1_2$ is required or possible, since $v_{02} \not\leq g_{02}\,;+1_2$.

Here is a full list of the atoms that we have introduced. If τ is a tiling instance with k tiles $t_0, \ldots, t_{k-1}$ then $RA(\tau)$ has $2k+28$ atoms. They are

start	end	atoms
0	0	e_0, w_0
0	1	$g_{01}, u_{01}, v_{01}, w_{01}$
0	2	$g_{02}, u_{02}, v_{02}, w_{02}$
1	1	$e_1, +1_1, -1_1, w_1$
2	2	$e_2, +1_2, -1_2, w_2$
1	2	$t^i_{12}\,(i < k),\ w_{12}$

plus the converses of the $(0-1), (0-2)$ and $(1-2)$ atoms (g_{10}, etc., as described above).

Perhaps faintly surprisingly, it turns out that there are no more problems, and from now on everything goes extremely well. We can read off a tiling in the way we explained from any representation of the relation algebra $RA(\tau)$ that we have now built. Conversely, if τ tiles the plane, we can construct a representation of $RA(\tau)$, carefully using the tiling to pick the correct atom to relate points of sorts 1 and 2 connected by a green path, and ensuring that the atoms u, v are used as prescribed to prevent the problems we saw in figures 5 and 7. Used in this way, they ensure that every edge labelled by a tile atom is associated with a tiling f of the plane and a pair of coordinates (x,y) such that the atom labelling the edge is $f(x,y)_{12}$. For the full, delicate details see [14, chapter 18].

We should mention that $RA(\tau)$ might not be a relation algebra for certain no-instances τ of the tiling problem (because composition might not be associative). But clearly it always will be for yes-instances, and being a relation algebra is a decidable property. So the map ρ given by $\rho(\tau) = RA(\tau)$ if $RA(\tau)$ is a relation algebra, and any fixed finite non-representable relation algebra, otherwise, is a computable reduction of the tiling problem to representability of finite *relation algebras.*

We also mention that although $RA(\tau)$ is not integral (it has three subidentity atoms), it is simple, and so the problem of determining representability of finite *simple* relation algebras is also undecidable. □

Later we will use a version of the tiling problem to extend this undecidability result to weaker subsignatures, however since the representations we constructed for our tiling algebras have infinite bases, the undecidability of finite representability for finite algebras is not addressed by this.

Partial Group Embedding

For converse-free signatures there is a completely distinct approach [15], and the issue of decidability of finite representability can be resolved—negatively for many signatures. The starting point for this second approach lies in problems of embedding compositional objects into objects with converse-like structure, which has played a prominent role in embedding problems in the world of semigroup theory. The idea stems originally from the efforts of Evans [7], who showed that the decidability of the uniform word problem in a class of algebras is equivalent to the problem of extending a partial algebra to a complete algebra in the class. The undecidability of the uniform word problem for groups and for finite groups [29] then imply that it is undecidable to determine if a finite partial Cayley table extends to a full group table, or to a finite group table. It was Kublanovsky who first saw how to successfully encode partial groups (as instances of the partial group extension problem) into semigroups, with totally defined operations. This was fully implemented in [11], where it was shown that the class of finite subsemigroups (i.e. finite subsets, closed under the semigroup operation) of some of the most familiar (and ostensibly well-behaved) families of structured semigroups (such as Brandt semigroups and completely 0-simple semigroups) were nonrecursive. Other undecidability results followed, including the embeddability of semigroup amalgams into semigroups and of ring amalgams into rings [17], as well as other problems associated with the embeddability of semigroups into larger semigroups [16, 20, 28].

How does this relate to binary relations? One of the most tightly structured forms of semigroups are the so-called Brandt semigroups, which posses a natural representation as injective functions, and the embeddability of finite semigroups in these is one of the problems shown undecidable in [11] (note that a relatively minor error in the argument led to a global fault in the proof of this fact; see [18] for correct statement and proof). A concrete description of a Brandt semigroup is to take a group $\mathscr{G}$, a non-empty set K and define a multiplication on the set $\{g_{i,j} \mid i,j \in K,\ g \in \mathscr{G}\} \cup \{0\}$ by setting $g_{i,j}\,;h_{j,k} := (gh)_{i,k}$ and letting all other products be 0. This construction, which we denote by $\mathsf{Br}^0_K(\mathscr{G})$, obviously embeds the group $\mathscr{G}$ as a subsemigroup (via $g \mapsto g_{ii}$, any fixed $i \in K$). The underlying Brandt groupoid (category with inverse) $\mathsf{Br}_K(\mathscr{G})$ is obtained by deleting zero and letting zero products in $\mathsf{Br}^0_K(\mathscr{G})$ be undefined, and the complex algebra of this Brandt groupoid is always a representable relation algebra with $|K|$ sub-identity atoms. It may be represented over a base $K \times \mathscr{G}$, by representing a typical singleton $\{g_{i,j}\}$ as the bijection $\{((i,f),(j,fg) : f \in \mathscr{G}\}$ from the sub-identity domain $i \times \mathscr{G}$ to the sub-identity domain $j \times \mathscr{G}$. Jónsson and Tarski showed that a relation algebra is representable if and only if it is a subalgebra of the complex algebra of a Brandt groupoid [19, Theorem 5.8]. A Brandt groupoid may be defined equivalently as the atom structure of a relation algebra in which every atom a satisfies $a;a^{\smile} + a^{\smile}a \leq 1'$. (There are generalisations of Brandt groupoids encompassing a broader range of atom structures: see, e.g., [10, 2].) The goal is to modify the semigroups used to show undecidability of embedding into Brandt semigroups, modelled on how they would appear as sub boolean monoids of the complex algebra of the target Brandt groupoid.

Consider $\mathsf{Br}^0_3(\mathscr{G})$ (that is, with $K = \{0,1,2\}$) and then restrict to the set $\{g_{i,j} \mid i < j < 3,\ g \in \mathscr{G}\} \cup \{0\}$. Now we have kept some shadow of the group $\mathscr{G}$, but only that part that was "stretched" across the three indices. Indeed, the only remaining non-zero products are those of the form $g_{0,1}\,;h_{1,2} = (gh)_{0,2}$: we cannot input this output into any further non-zero product, and there is no obvious remnant of the group inverse operation. We can make this shadow fainter still by replacing the full group $\mathscr{G}$ by any subset $P \subseteq G$ and restricting to the set

$$P_{0,1} \cup P_{1,2} \cup (P\,;P)_{0,2} \cup \{0\}.$$

Now on this subsemigroup of $\mathsf{Br}^0_3(\mathscr{G})$ (call it $S(P)$) we have found something approaching a genuinely partial group (namely that part of $\mathscr{G}$ on the arbitrary subset P), albeit stretched across the indices $0,1,2$ and with an added zero element to make the operation ; total. Perhaps too much has been discarded? After all, any partial Cayley table on a set P (not necessarily restricting a group, and not necessarily even associative) will produce a totally defined semigroup $S(P)$ via this construction. Perhaps surprisingly, it turns out that with only a little more care, there is enough of the remnant shadow of group structure to guarantee that embeddability of this construction (or at least a small variation of it) into a Brandt semigroup is equivalent to extendability of the partial Cayley table P to a group.

To make this into a representability problem, we need to turn the $S(P)$ construction into a boolean monoid (the converse-free fragment of relation algebra). We have already seen that the complex algebra of a full Brandt groupoid $\mathsf{Br}_3(\mathscr{G})$ is a representable relation algebra, whose converse-free reduct is a boolean monoid. Assuming that P is a small enough subset of the group $\mathscr{G}$, then it is relatively straightforward to describe the structure of the sub boolean monoid generated by $S(P)$. This structure can then be used as a template for the case of an arbitrary partial group P.

When P embeds into a group G, the generated structure is easily represented as binary relations over $3 \times G$. Conversely, given any $\{-,+,1',;\}$-representation of this structure, the base of the representation is partitioned into three parts and the set of permutations of one of these parts forms a group into which P embeds.

Now let us give proper details. A partial group $*: P \times P \to A$ is a binary, total, surjective function. (The set P is described in the discussion above, while A is the set of products of two elements from P). Let K be any class of groups. An instance of the K-embedding problem is a finite partial group $*: P \times P \to A$: it is a yes instance if there is $G \in K$ and an injection from A into G preserving all defined products, it is a no instance otherwise. The following is essentially due to Slobodskoiĭ (see also [27, §7.4.3]).

Proposition 5. *If K is any class of groups containing all finite groups then the K-embedding problem is undecidable.*

We will be particularly interested in the cases where K is the class of all groups, and where K is the class of all finite groups. We will reduce these problems to the representation problem and the finite representation problem (respectively) for finite boolean monoids.

Given a finite, partial group $*: P \times P \to A$ let $M(*)$ be the finite boolean monoid whose atoms are

$$\{\mathsf{e}_{ii} : i < 3\} \cup \{\mathsf{w}_{ij} : i, j < 3\} \cup \{\mathsf{a}_{01}, \mathsf{a}_{12} : a \in P\} \cup \{\mathsf{b}_{02} : b \in A\}.$$

The identity is $1' = \mathsf{e}_{00} + \mathsf{e}_{11} + \mathsf{e}_{22}$, converse is not defined in boolean monoids. We define composition on atoms first, by letting $a;b = \sum_{(a,b,c)\notin F} c$ where F is the set of forbidden triples of atoms, consisting of:

$$\begin{aligned}
(x_{ij}, y_{j',k'}, z_{i^*,k^*}) &\text{ unless } i = i^*,\ j = j' \text{ and } k' = k^* \\
(\mathsf{e}_{ii}, x_{ij}, y_{ij}),\ (x_{ij}, \mathsf{e}_{jj}, y_{ij}) &\text{ where } x \neq y \\
(\mathsf{a}_{01}, \mathsf{b}_{12}, z_{02}) &\text{ where } \mathsf{a}, \mathsf{b} \in P,\ z \neq (\mathsf{a} * \mathsf{b}) \\
(\mathsf{a}_{01}, \mathsf{w}_{12}, (\mathsf{a} * \mathsf{b})_{02}),\ (\mathsf{w}_{01}, \mathsf{b}_{12}, (\mathsf{a} * \mathsf{b})_{02}) &\text{ where } \mathsf{a}, \mathsf{b} \in P.
\end{aligned}$$

Equivalently, $x_{ij}; y_{j',k'} = 0$ for $j \neq j'$ and

$$\begin{aligned}
\mathsf{e}_{ii}; x_{ij} = x_{ij} = x_{ij}\mathsf{e}_{jj} \qquad & \mathsf{a}_{01}; \mathsf{b}_{12} = (ab)_{12} \\
\mathsf{a}_{01}; \mathsf{w}_{12} = \sum_{\neg\exists b \in P: ab = y} y_{02} \qquad & \mathsf{w}_{01}; \mathsf{b}_{12} = \sum_{\neg\exists a \in P: ab = y} y_{02} \\
u_{ij}; v_{ji} = \mathsf{w}_{ii}\ (u \neq v) \qquad & u_{ij}; v_{jk} = 1_{ik}\ (= \sum_{x_{ij} \in M(*)} x_{ij}), \quad \text{otherwise}
\end{aligned}$$

where $a, b \in P$, $i, j, k < 3$ and y_{02} ranges over all atoms of $M(*)$ with indices $0, 2$ other than those excluded from the sum. In the case when $*$ embeds into a group G and $i, j < 3$ then the embedding carries the white atom w_{ij} to the co-finite sum of all atoms g_{ij} for $\mathsf{g} \in G$ where g_{ij} is *not* in $M(*)$.

Let $\mathscr{G} = (G, e, \circ)$ be a group. The three dimensional brandt groupoid $\mathsf{Br}_3(\mathscr{G})$ has elements $\{g_{ij} : g \in G,\ i, j < 3\}$ and a partial binary operator $\bullet$, only defined when subscripts match, and where $f_{ij} \bullet g_{jk} = (f \circ g)_{ik},$ for $f, g \in G,\ i, j, k < 3$. The base of the complex algebra $\mathfrak{Cm}(\mathsf{Br}_3(\mathscr{G}))$ is the power set of $\mathsf{Br}_3(\mathscr{G})$ with boolean operators $\cup, \setminus$, together with relation algebra operators

$$\begin{aligned} 1' &= \{e_{00}, e_{11}, e_{22}\} \\ a^{\smile} &= \{g_{ij} : g_{ji}^{-1} \in a\} \\ a;b &= \{(f \circ g)_{ij} : \exists k < 3,\ f_{ik} \in a \wedge g_{kj} \in b\}. \end{aligned}$$

Lemma 6. *For any group $\mathscr{G}$, $\mathfrak{Cm}(\mathsf{Br}_3(\mathscr{G}))$ is a relation algebra, with a representation on the base $3 \times \mathscr{G}$ defined by*

$$g_{ij}^{\theta} = \{((i, f), (j, f \circ g)) : f \in \mathscr{G}\}.$$

Theorem 7. *Let $* : P \times P \to A$ be a finite, partial group. The following are equivalent.*

1. $*$ embeds into a group,
2. $M(*)$ embeds into a relation algebra,
3. $M(*)$ embeds into a representable relation algebra,
4. $M(*)$ is a representable boolean monoid.

Furthermore, if we replace 'group' by 'finite group' in the first, and 'relation algebra' by 'relation algebra with a representation on a finite base' in the second and 'representable' by 'finitely representable' in the third and fourth, we get another quartet of equivalent statements.

Proof. (1) $\Rightarrow$ (3). Suppose $* : P \times P \to A$ embeds into a group $\mathscr{G} = (G, e, \circ)$, without loss $A \subseteq G$. By replacing $\mathscr{G}$ by $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ (if necessary) we may assume that $|\mathscr{G}| > 2|A|$. Define a map $\phi : M(A) \to \mathfrak{Cm}(\mathsf{Br}_3(G))$ by letting $x^{\phi} = \{x\}$ for any atom $x \notin \{\mathsf{w}_{ij} : i, j < 3\}$, $\mathsf{w}_{ij}^{\phi} = \{g_{ij} : g \in G\} \setminus \{\mathsf{a}_{ij} : \mathsf{a} \in A,\ \mathsf{a}_{ij} \text{ is defined}\}$, and extending to non-atomic elements of $M(*)$ by taking unions. ϕ is clearly an injection and can be shown to be an embedding of $M(*)$ into $\mathfrak{Cm}(\mathsf{Br}_3(\mathscr{G}))$. When checking that composition is preserved, one slightly tricky case is to show that $(\mathsf{w}_{ij}; \mathsf{w}_{jk})^{\phi} = \mathsf{w}_{ij}^{\phi}; \mathsf{w}_{jk}^{\phi}$. For this case, since $(\mathsf{w}_{ij}, \mathsf{w}_{jk}, z_{ik})$ is never forbidden (any atom $z_{ik} \le 1_{ik}$ of $M(*)$) we have $\mathsf{w}_{ij}; \mathsf{w}_{jk} = 1_{ik}$, for $i, j, k < 3$. Since $|\mathscr{G}| > 2|A|$ we know that $|\{g \in \mathscr{G} : g_{ij} \in \mathsf{w}_{ij}^{\phi}\}|, |\{g^{-1} * f : g_{jk} \in \mathsf{w}_{jk}^{\phi}\}| > \frac{1}{2}|\mathscr{G}|$ (for any $f \in \mathscr{G}$), so there are $g, g' \in \mathscr{G}$ such that $g_{ij} \in \mathsf{w}_{ij}^{\phi} \wedge g'_{jk} \in \mathsf{w}_{jk}^{\phi} \wedge g = g'^{-1} * f$, hence $\mathsf{w}_{ij}^{\phi}; \mathsf{w}_{jk}^{\phi} \ge f_{ik}^{\phi}$, for all $f \in \mathscr{G}$, so $\mathsf{w}_{ij}^{\phi}; \mathsf{w}_{jk}^{\phi} = 1_{ik}^{\phi}$, as required.

(3) $\Rightarrow$ (2) is trivial.

For (2) $\Rightarrow$ (1), suppose $M(*)$ embeds in a relation algebra $\mathscr{R}$, without loss the embedding is the inclusion map. Let $G \subseteq \mathscr{R}$ be defined by

$$G = \{a \in \mathscr{R} : \mathsf{e}_{00}; a = a; \mathsf{e}_{00} = a \wedge a; 0' \cdot a = 0'; a \cdot a = 0\}. \tag{1}$$

[We are not assuming that $\mathscr{R}$ is representable, but it may help the understanding of (1) and some of its consequences to observe for any representation θ over a base X say, that since $1' = \mathsf{e}_{00} + \mathsf{e}_{11} + \mathsf{e}_{22}$ we may partition the base into three parts $X = X_0 \cup X_1 \cup X_2$ where $x \in X_i \iff (x,x) \in \mathsf{e}_{ii}^{\theta}$, for $i < 3$. (1) ensures that each element of G is represented as a permutation of X_0.] Regardless of whether $\mathscr{R}$ is representable or not, G includes e_{00} and is closed under converse and composition, hence $\mathscr{G} = (G, \mathsf{e}_{00}, ;)$ is a group. Since $\mathsf{e}_{ij} ; \mathsf{e}_{jk} = \mathsf{e}_{ik}$ in $M(*)$ for $i \le j \le k$, it follows by the triangle law that $\mathsf{e}_{ij} = \mathsf{e}_{ik} ; \mathsf{e}_{jk}^{\smile}$ and $\mathsf{e}_{jk} = \mathsf{e}_{ij}^{\smile} ; \mathsf{e}_{ik}$ in $\mathscr{R}$. Let $\phi : A \to \mathscr{G}$ be defined by $\mathsf{a}^{\phi} = \mathsf{a}_{02} ; (\mathsf{e}_{02})^{\smile}$ (it is not hard to check that $a^{\phi} \in G$). We claim that ϕ is an embedding of $*$ into $\mathscr{G}$. To prove the claim, let $\mathsf{a}, \mathsf{b} \in P$. Then

$$\begin{aligned}
(\mathsf{a} * \mathsf{b})^{\phi} &= (\mathsf{a} * \mathsf{b})_{02} ; \mathsf{e}_{02}^{\smile} \\
&= (\mathsf{a}_{01} ; \mathsf{b}_{12}) ; \mathsf{e}_{02}^{\smile} \\
&= \mathsf{a}_{01} ; \mathsf{e}_{11} ; \mathsf{b}_{12} ; \mathsf{e}_{02}^{\smile} \\
&= \mathsf{a}_{01} ; (\mathsf{e}_{12} ; \mathsf{e}_{02}^{\smile} ; \mathsf{e}_{01}) ; \mathsf{b}_{12} ; \mathsf{e}_{02}^{\smile} \\
&= (\mathsf{a}_{02} ; \mathsf{e}_{02}^{\smile}) ; (\mathsf{b}_{02} ; \mathsf{e}_{02}^{\smile}) \\
&= \mathsf{a}^{\phi} ; \mathsf{b}^{\phi}
\end{aligned}$$

(3) $\Rightarrow$ (4) is trivial. For the converse implication (4) $\Rightarrow$ (3), suppose $M(*)$ has a boolean monoid representation θ over some base set X. The representation of the unit 1^{θ} must be a reflexive, transitive relation, but need not be symmetric, creating a problem if we are to embed $M(*)$ in a representable relation algebra. However, $M(*)$ happens to be a *normal* boolean monoid since it satisfies $\mathsf{dom}(x) ; x = x = x ; \mathsf{rng} x$. We may restrict the boolean monoid representation of a normal boolean monoid to the symmetric interior $1^{\theta} \cap (1^{\theta})^{\smile}$ of the unit to obtain a boolean monoid representation θ° where the unit $1^{\theta^{\circ}}$ is an equivalence relation over some base X [15, lemma 2.1]. But then $M(*)$ embeds into the proper relation algebra of all subsets of $1^{\theta^{\circ}}$, proving (3).

This proof of the equivalence of (1)–(4) goes through with only minor alterations to prove the equivalence of the second quartet of statements (concerning finite representation and finite groups). □

Extending the results

We now extend these results to weaker signatures. In [13] the reduction of the tiling problem to the representation problem for relation algebras was extended to cover all relation algebra signatures S with $S \supseteq \{\cdot, +, ;\}$. When $S \supseteq \{\cdot, ^{\smile}, ;\}$ a version of the *deterministic tiling problem*, where an instance consists of a set τ of tiles with adjacencies and a specified start tile $t_0 \in \tau$, was shown to be undecidable and used to prove the undecidability of membership of $R(S)$ for finite S-structures [15].

Theorem 8. *Let S be a relation algebra signature containing $\{\cdot, +, ;\}$. τ is a yes-instance of the tiling problem if and only if $RA(\tau)$ is S-representable. If instead*

$S \supseteq \{\cdot, \smile, ;\}$ then (τ, t_0) is a yes-instance of the deterministic tiling problem if and only if $RA(\tau, t_0)$ is S-representable.

See [13, lemma 12] and [15, theorem 8.6] for the definitions and proofs.

By a different method, [25] used the results on partial group embeddings and extended them to weaker signatures.

Lemma 9. *Let $\mathscr{A}$ be a finite, simple boolean monoid (so the signature of $\mathscr{A}$ is the whole relation algebra signature minus the converse operator), let $\{\cdot, +, ;\} \subseteq S$, $\smile \notin S$ be a subsignature of relation algebra. If θ is an S-representation of the S-reduct of $\mathscr{A}$ over a finite base X then $\mathscr{A}$ has a boolean monoid representation over a finite base.*

Proof. We need to modify θ so that boolean negation and the identity are correctly represented. Since S contains the lattice operators, whenever $(x,y) \in 1^\theta \setminus 0^\theta$ there is a unique atom a of $\mathscr{A}$ such that $(x,y) \in a^\theta$ and by faithfulness of θ, pairs $(x,y) \in (1')^\theta \setminus 0^\theta$ exist. Since $1' = 1';1'$ there is z_0 with $(x,z_0),(z_0,y) \in (1')^\theta$ and since $0;1' = 1';0 = 0$ we have $(x,z_0),(z_0,y) \notin 0^\theta$. We can iterate that to obtain an infinite sequence $z_0, z_1, \ldots$ where $(z_i, z_{i+1}) \in (1')^\theta \setminus 0^\theta$. Since X is finite we have $z_i = z_j$ for some $i < j$ and $(z_i, z_i) \in (1')^\theta \setminus 0^\theta$. So suppose $(x,x) \in (1')^\theta \setminus 0^\theta$. Now let $Y = \{y \in X : (x,y),(y,x),(y,y) \in 1^\theta \setminus 0^\theta\}$. By simplicity of $\mathscr{A}$, the restriction of θ to the base Y is still an S-representation of the S-reduct of $\mathscr{A}$, but now we have $1^{\theta\restriction_Y} = Y \times Y$, a *square* S-representation on a finite base $Y \subseteq X$, moreover $\theta\restriction_Y$ also respects negation since θ respects the lattice operators and $0^\theta \cap (Y \times Y) = \emptyset$. It remains to fix the identity, but this is straightforward as the relation $(1')^{\theta\restriction_Y}$ is a *congruence* relation $\sim$ over Y, and $\theta\restriction_Y$ induces a finite representation $\hat{\theta} : \mathscr{A} \to \wp(Y/\sim \times Y/\sim)$, now respecting also the identity. $\square$

By proposition 5, theorem 7 and lemma 9,

Lemma 10. *Membership of $F(S)$ is undecidable for $S \supseteq \{\cdot, +, ;\}$ where $\smile \notin S$.*

We also consider signatures without lattice operators but with boolean ordering and negation. Such a signature S might omit the boolean unit 1, but even so, in any S-representation θ of $\mathscr{A}$ over a base X there is a binary relation U defined by $(x,y) \in U \iff \exists a \in \mathscr{A},\ (x,y) \in a^\theta$. For the next result we need to assume that U is symmetric. This result is from [25] where other restrictions on U are also considered.

Lemma 11. *Let $S \supseteq \{-, \leq, ;\}$, $\smile \notin S$, let $* : P \times P \to A$ be a partial group. If θ is a S-representation of the S-reduct of $M(*)$ over a base X where 1^θ is symmetric then $*$ is a yes-instance of the partial group embedding problem, moreover if X is finite then $*$ is a yes-instance of the group finite embedding problem, for an arbitrary representation θ.*

Proof. 0 may not be in the signature S, but there is an element $0 \in M(*)$ and since $0 \leq -0 \in M(*)$ we know that $0^\theta = \emptyset$. Although 1 may not be in the signature S, the element $1 \in M(*)$ is maximal with respect to $\leq$ and satisfies $1;1 = 1$ so must be represented as a transitive relation, by assumption it is symmetric. For $i < 3$, by faithfulness of θ, e_i^θ is non-empty and e_i^θ is transitive, since $\mathsf{e}_i;\mathsf{e}_i = \mathsf{e}_i$. Since 1^θ is symmetric, if $(x,y) \in \mathsf{e}_i^\theta$ then either $(y,x) \in \mathsf{e}_i^\theta$ or $(y,x) \in (-\mathsf{e}_i)^\theta$. The latter case yields a contradiction $(x,y) \in (\mathsf{e}_i;(-\mathsf{e}_i);\mathsf{e}_i)^\theta \subseteq (-\mathsf{e}_i)^\theta$, so $(y,x) \in \mathsf{e}_i^\theta$ and e_i^θ is symmetric, hence reflexive over its domain. It follows from the faithfulness of θ that $X_0 = \{x \in X : (x,x) \in \mathsf{e}_0^\theta\} \subseteq X$ is non-empty. If $(x,y) \in \mathsf{e}_i^\theta$ then $(y,z) \in a^\theta \leftrightarrow (x,z) \in a^\theta$.

Let Y be the set of maximal non-empty cliques of $(1')^\theta$. Let $\hat{\theta} : M(*) \to \wp(Y \times Y)$ be defined by $a^{\hat{\theta}} = \{([x],[y]) \in Y \times Y : (x,y) \in a^\theta\}$, where $[x]$ is the $(1')^\theta$-clique of $x \in X$ (undefined if $(x,x) \notin (1')^\theta$). Since $1';a;1' = a$ is valid in $M(*)$, $\hat{\theta}$ is still an S-representation of $M(*)$. Observe that $\hat{\theta}$ also respects the identity, $(1')^{\hat{\theta}} = \{(y,y) : y \in Y\}$. Hence, if $a \in M(*)$ satisfies $a;(-1') \leq -a$ and $(-1');a \leq -a$ then $\hat{\theta}$ represents a as a partial injective function. Although converse is not in the signature, $(a^{\hat{\theta}})^\smile$ is concretely defined as the inverse of the partial injection $a^{\hat{\theta}}$, clearly a partial injection.

Then $\mathsf{a}_{02}^{\hat{\theta}}|(\mathsf{e}_{02}^{\hat{\theta}})^\smile$ is a permutation of X_0 so the map that sends a to $\mathsf{a}_{02}^{\hat{\theta}}|(\mathsf{e}_{02}^{\hat{\theta}})^\smile$ is an embedding of A into the group of all permutations of X_0, so $*$ is a yes-instance. Clearly, if X is finite then the group of permutations of X_0 is also finite. □

We can now prove theorem 2.

Proof. For the part concerning the undecidability of $R(S)$ for finite S-structures, both cases $S \supseteq \{\cdot,+,\}$, $S \supseteq \{\cdot,\smile,;\}$ are covered by theorem 8, for the part concerning the undecidability of $R_{\leftrightarrow}(S)$ the case $\smile\ \notin S \supseteq \{-,\leq,;\}$ follows from proposition 5, lemma 11. For the part concerning $F(S)$, the case $\smile\ \notin S \supseteq \{\cdot,+,;\}$ is lemma 10 and the case $\smile\ \notin S \supseteq \{-,\leq,;\}$ follows from proposition 5, theorem 7 and lemma 11. □

To prove corollary 3 let X be R or F and consider the representation class $X(S)$. We reduce membership of $X(S)$ for finite S-structures to the quai-equational theory of $X(S)$. The *diagram* $\Delta(\mathscr{A})$ of a finite S-structure $\mathscr{A}$ is the conjunction of all true equations $\Omega(\bar{a}) = b$ or negations of false equations $\Omega(\bar{a}) \neq b$ where Ω is an n-ary operator in S, $\bar{a}$ is an n-tuple over $\mathscr{A}$ and $b \in \mathscr{A}$. Then $\mathscr{A} \in X(S)$ if and only if $\neg\Delta(\mathscr{A})$ is not valid over $X(S)$. Since $\neg\Delta(\mathscr{A})$ is equivalent to a quasi-equation we have a reduction from finite membership of $X(S)$ to non-validity over $X(S)$. If S includes the booleans $+,-$ then a quasi-equation is equivalent to a formula built from equations using only negation and conjunction. Each equation $s = t$ may be replaced by the equivalent $(s \cdot -t + -s \cdot t) = 0$. A conjunction of equations $s = 0 \wedge t = 0$ is equivalent to $s + t = 0$. Since $1;x;1$ is a discriminator term, a negated equation $\neg s = 0$ is equivalent to the equation $1 - (1;x;1) = 0.$, hence the quasi-equation equivalent to $\neg\Delta(\mathscr{A})$ can effectively be replaced by an equivalent equation.

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