Axiomatising various classes of relation and cylindric algebras

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Abstract

We outline a simple approach to axiomatising the class of representable relation algebras, using games. We discuss generalisations of the method to cylindric algebras, homogeneous and complete representations, and atom structures of relation algebras.

1 Introduction

Relation algebras are to binary relations what boolean algebras are to unary ones. They are used in artificial intelligence, where, for example, the Allen–Koomen temporal planning system checks the consistency of given relations between time intervals. In mathematics, they form a part of *algebraic logic.* The history of this goes back to the nineteenth century, the early workers including Boole, de Morgan, Peirce, and Schröder; it was studied intensively by Tarski's group (including, at various times, Chin, Givant, Henkin, Jónsson, Lyndon, Maddux, Monk, Németi) from around the 1950s, and currently we know of active groups in Amsterdam, Budapest, Rio de Janeiro, South Africa, and the U.S., among other places.

Abstract relation algebras have the boolean operations on binary relations (regarded as sets of pairs), and also composition, converse, and a constant for identity (equality). Their basic theory is more intricate than that of boolean algebras, as no finite set of axioms picks out the relation algebras that are isomorphic to genuine fields of binary relations. In fact, a central problem of the area (for 'cylindric algebras', due to Henkin, Monk, and Tarski) is to find a 'simple intrinsic characterization' for these 'representable' relation algebras. (There do exist intrinsic characterisations, in the form of first-order axiomatisations — see [L2,Mo2,HMT], for example.)

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We approach this problem using model-theoretic forcing by games. It attempts to construct a field of binary relations for a given relation algebra. The conditions for success can be easily expressed by first-order axioms, and so we obtain our characterisation.

There are certainly antecedents of this approach. It can be seen as a variation on the argument of Lyndon [L1]. In so far as the game construction is a 'step-by-step' argument, it has affinities with the work of many authors, such as Andréka, Burgess, Henkin, Maddux (notably [Ma1, page 159], for example), Reynolds, Robinson, Venema: see [HH1] for a short survey. The method of games is well-known in model theory: see [Hodg1, Hodg2], for example. For the topological connections, via the Banach–Mazur theorem, see [Ox]. There are also precedents, such as [E,K,Sc,Sv,V] (to take just a few), for axiomatising the conditions for success in a game, still more sophisticated instances of which can be found in descriptive set theory.

The advantage of using this approach to characterise the representable relation algebras is that, in our view, the proofs are much simpler than those used in previous characterisations. Perhaps more importantly, it is easy to generalise the method. For example, the class of representable *cylindric algebras* (n-ary relations) can be axiomatised. Moreover, higher-order properties can be dealt with, such as *homogeneity* and *completeness* of representations; in the latter case, we obtain the negative result that the class of algebras of fields of binary (or n-ary) relations respecting arbitrary unions is not elementary. We may also study the *atom structures* of atomic relation algebras using this approach. These results will be discussed below.

The article begins with definitions of relation algebras and representations. Some examples are given. Then, in section 5, we describe the method of games and apply it to axiomatise the representable relation algebras. In sections 6 to 9 we discuss generalisations to algebras with homogeneous representations, cylindric algebras, algebras with complete representations, and atom structures.

This article has some of the character of an extended abstract. The interested reader will find fuller details of our arguments, some further work, and also some more references, in [Hi, HH1, HH2, Hodk]. Some of these are currently available on the world wide web at
`http://www.cs.ucl.ac.uk/staff/R.Hirsch/publications.html`
and at `http://theory.doc.ic.ac.uk:80/tfm/papers/HodkinsonIM/`

2 Relation algebras

A *relation algebra* is a structure $\mathcal{A} = (A, +, -, ^\smile, ;, 0, 1, 1')$, where A is a non-empty set (the domain or universe of $\mathcal{A}$), $+$ and $;$ are binary functions, $^\smile$ and $-$ are unary functions, and $0, 1$, and $1'$ are constants. We require that $(A, +, -, 0, 1)$ is a boolean algebra (so we can use $\cdot$ and $\leq$ as abbreviations), and that the following equations hold, for all $x, y, z \in A$:

1. $x; 1' = 1'; x = x$

2. $(x;y);z = x;(y;z)$ (associativity)

3. $x^{\smile\smile} = x$

4. $(x+y)^{\smile} = x^{\smile} + y^{\smile}$

5. $(-x)^{\smile} = -(x^{\smile})$

6. $(x;y)^{\smile} = y^{\smile};x^{\smile}$

7. $(x;y)\cdot z^{\smile} = 0 \to (y;z)\cdot x^{\smile} = 0$ ('triangle axiom'). Admittedly, this is not an equation, but it can be shown that in the presence of the other axioms (1–6 above) it is equivalent to the equation $x^{\smile};-(x;y)\cdot y = 0$.

For a shorter axiomatisation, see [JT2 part II, definition 4.1] and [CT]. We write **RA** for the class of relation algebras.

A relation algebra is said to be *simple* if it satisfies $1;r;1 = 1$ for every non-zero r. Any relation algebra can be decomposed into simple relation algebras. So for simplicity, we will first be considering only simple relation algebras, postponing treatment of the general case until section 5.5.

A notational point: as is common in algebra, we will often use the same notation for a structure as for its domain. Thus, for example, we will write $a \in \mathcal{A}$ instead of $a \in A$, where $\mathcal{A}$ is as above.

3 Representations

A relation algebra $\mathcal{A}$ is intended to be a collection of binary relations, in which $+$ is union, $-$ is complement, ';' is relation composition, etc. (see example 4.1). So let us regard $\mathcal{A}$ as a binary relational signature (or similarity type). That is, each element of (the domain of) the algebra $\mathcal{A}$ will be regarded as a binary relation symbol. (This will not lead to ambiguity: for $r \in \mathcal{A}$, if we write $r(x,y)$, we are thinking of r as a relation symbol, but if we write simply r, we are thinking of r as an element of $\mathcal{A}$.) The following definition is now natural to make.

Definition 3.1 A *representation* of $\mathcal{A}$ is a model of the theory $T_{\mathcal{A}}$ consisting of:

$$\begin{array}{rl}
\forall x,y[1'(x,y) \leftrightarrow (x=y)] & \\
\forall x,y[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & \text{for each } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s+t \\
\forall x,y[1(x,y) \to (r(x,y) \leftrightarrow \neg s(x,y))] & \text{for each } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = -s \\
\forall x,y[r(x,y) \leftrightarrow s(y,x)] & \text{for each } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = s^{\smile} \\
\forall x,y[r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y))] & \text{for each } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s;t \\
\exists x,y\, r(x,y) & \text{for each } r \in \mathcal{A} \text{ with } \mathcal{A} \models r \neq 0.
\end{array}$$

Remark 3.2 If $\mathcal{A}$ is a simple relation algebra, then the last axiom follows from the preceding ones, and so can be dropped. Also, if $\mathcal{A}$ is simple, any representation of $\mathcal{A}$ is the disjoint union of representations satisfying $\forall xy\ 1(x,y)$. So we can (and will) add this axiom to $T_{\mathcal{A}}$ in this case.

Definition 3.3 Two representations M, N of a relation algebra $\mathcal{A}$ are said to be *isomorphic* if there is a model-theoretic isomorphism $\theta : M \to N$. That is, θ must be a bijection from the domain of M to that of N, and for all $x, y \in M$ and $r \in \mathcal{A}$ we must have $M \models r(x, y)$ iff $N \models r(\theta(x), \theta(y))$. (In algebraic logic, this notion is sometimes called 'base isomorphism'.) An *automorphism* of a representation M of $\mathcal{A}$ is simply an isomorphism $\theta : M \to M$ — i.e., an automorphism in the usual model-theoretic sense. These notions will be relevant in section 4.2 below.

Definition 3.4

1. A relation algebra $\mathcal{A}$ is said to be *representable* if it has a representation — i.e., if $T_{\mathcal{A}}$ is consistent.
2. We write **RRA** for the class of all representable relation algebras.

The axioms given in §2 are not enough to guarantee that a relation algebra is representable, and thus a 'real' collection of binary relations:

Theorem 3.5 (Jónsson & Tarski [JT1], Monk [Mo1]) *The class* **RRA** *is a variety (that is, it can be equationally axiomatised). However, it cannot be axiomatised by any finite set of first-order sentences.*

This is the chief negative result in the area. (Compare with the boolean algebra case, where *every* boolean algebra is representable as a set algebra.)

4 Examples

First, some basic examples of relation algebras and representations.

Examples 4.1

1. If $D \neq \emptyset$, the algebra
$$\mathcal{P} = (\wp(D^2), \cup, \sim, {}^\smile, ;, \emptyset, D \times D, \{(d, d) : d \in D\})$$
of all binary relations on D is a relation algebra, where $s^\smile$ is the converse relation of s (i.e., $s^\smile = \{(x, y) : (y, x) \in s\}$), and $s; t$ is the usual composition of relations: $s; t = \{(x, y) : \exists z \in D[(x, z) \in s \wedge (z, y) \in t]\}$. $\mathcal{P}$ is sometimes called the 'full relation algebra over D'. It is evidently representable, by the structure M with domain D in which each relation $r \in \mathcal{P}$ is interpreted as itself: $M \models r(d_1, d_2)$ iff $(d_1, d_2) \in r$.
2. Any subalgebra of $\mathcal{P}$ is also a representable relation algebra.
3. If L is a first-order signature and M is an L-structure, the L-formulas $\varphi(x, y)$ written with three variables, x, y, z, and x, y free, modulo equivalence in M, form a representable relation algebra. Composition, for example, is defined in the obvious way: $(\varphi; \psi)(x, y) = \exists z(\varphi(x, z) \wedge \psi(z, y))$.

Here, we wrote $\varphi(x, z)$ for the formula with free variables x, z obtained by swapping the variables y, z throughout $\varphi(x, y)$; the formula $\psi(z, y)$ is defined similarly.

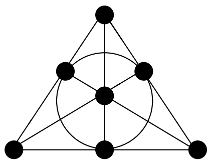
As a particular instance, suppose that $\mathcal{A}$ is a simple relation algebra, and that M^+ is a representation of $\mathcal{A}$. Let $L \subseteq \mathcal{A}$. (L is regarded equally as a subset and as a sub-signature of $\mathcal{A}$.) Let $M = M^+ \lceil L$, the reduct of M^+ to the signature L. Then it is an exercise to show that the relation algebra obtained from the three-variable L-formulas modulo M-equivalence, as above, is isomorphic to the subalgebra of $\mathcal{A}$ generated by L. When the boolean reduct of $\mathcal{A}$ is an atomic boolean algebra, and L is the set of atoms of $\mathcal{A}$, then this is the 'term algebra' over the atom structure of $\mathcal{A}$, in the terminology of section 9.

Lyndon [L1] was the first to give an example of a non-representable relation algebra. (Later, McKenzie [Mk, p286] found the smallest possible relation algebra that is not representable. It has 16 elements.) We will describe a family of algebras discovered by Lyndon [L3], which is a rich source of examples. First, some background information on projective planes; details can be found in the textbooks (for example, [HP]).

4.1 Projective planes

A *projective plane* is an incidence system of points and lines, such that any two distinct points lie on a unique line, and (dually) any two distinct lines meet in a unique point. Further, the plane must contain four points, no three of which lie on a single line.

Here is a diagram of the smallest projective plane. It has seven points and seven lines (one of the lines being drawn as a circle).



In a projective plane, all lines intersect the same number of points. The *order* of a projective plane is defined to be one less than the number of points on a line. The plane pictured above has order 2. For any prime p, and any $m \geq 1$, there is a projective plane of order p^m. One such plane may be obtained from the three-dimensional vector space over the finite field of order p^m, its points and lines being the one-dimensional and two-dimensional subspaces, respectively, and the point x lying on the line l iff $x \subseteq l$.

There are others (non-Desarguesian ones) not obtainable like this; but all known finite projective planes do have prime power order. By the Bruck–Ryser theorem (1949), if there is a projective plane of order n, and $n \equiv 1$ or 2 (mod 4), then n must be the sum of the squares of two non-negative whole numbers. So, for example, there is no projective plane of order 6 or 14. The cases $n = 10, 12, 15, 18, \ldots$ are not covered by this result. A long computation by Lam, Swiercz and Thiel (1989) showed that there is no projective plane of order 10; the other cases remain open.

4.2 The Lyndon algebras

We take n to be a whole number, at least 2. The Lyndon algebra $\mathcal{A}_n$ is finite, with $n+2$ atoms, say $1', a_0, \ldots, a_n$. It is defined by:

- $a_i; a_i = a_i + 1'$ if $n \geq 3$, and $a_i; a_i = 1'$ if $n = 2$.
- $a_i; a_j = \sum_{k \neq i,j} a_k$ if $i \neq j$,
- (necessarily) $a_i^{\smile} = a_i$,

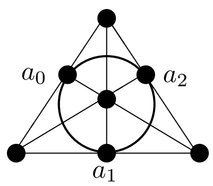
where $i, j, k \leq n$. On arbitrary elements, ';' can be calculated using distributivity over $\sum$. So can '$^{\smile}$': we have $r^{\smile} = r$ for all $r \in \mathcal{A}_n$.

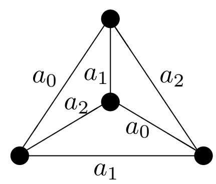
If there is a projective plane $\mathcal{P}_n$ of order n, choose a projective line l (called 'the line at infinity'), and let $\mathcal{S}_n$ be the 'affine' plane obtained from $\mathcal{P}_n$ by deleting l and all points on l. Then the set of points of $\mathcal{S}_n$ can be made into a representation of $\mathcal{A}_n$ by identifying the non-$1'$ atoms of $\mathcal{A}_n$ with the points of l (in any fashion), and defining, for any atom a_i, and any distinct points x, y of $\mathcal{S}_n$,

$$a_i(x, y) \text{ holds iff } \overline{xy} \cap l = \{a_i\}.$$

Here, $\overline{xy}$ is the unique projective line through x and y. (We can recover the interpretation of any $r \in \mathcal{A}_n$ from this.)

As an example, we obtain a representation of $\mathcal{A}_2$ from the projective plane pictured above, as follows:





We chose l to be the 'circular' line, and identified its points with the atoms a_0, a_1, a_2 of $\mathcal{A}_2$, as shown on the left. The affine plane on the right, obtained by deleting l and its points, gives a representation of $\mathcal{A}_2$, the atomic relations holding between its points being as illustrated. For example, the top two points are related by a_1, because, on the left, the (vertical) line joining them cuts the circular line in the point identified with the atom a_1 of $\mathcal{A}_2$.

It can be shown that any representation of any $\mathcal{A}_n$ must arise as an affine plane in the way described. So:

1. Any given Lyndon algebra $\mathcal{A}_n$ is representable iff there is a projective plane of order n. In particular, if there is no projective plane of order n (for example, $n = 6, 10, 14$) then $\mathcal{A}_n$ is not representable (Lyndon, [L3]; this is how some of the first non-representable relation algebras were found).

   It can be seen that the question of whether an arbitrary relation algebra is representable can be deep, and is in fact unsolved in the general case. In its starkest form, it is not known whether $\mathcal{A}_{12}$ is representable. As the different $\mathcal{A}_n$ appear superficially very similar, this illustrates the subtlety of the representability problem.

2. For $n \geq 4$, no representation of $\mathcal{A}_n$ is homogeneous (see definition 6.1 below for the meaning of this). The reason: an automorphism of the representation $\mathcal{S}_n$ must be induced by a collineation of the projective plane $\mathcal{P}_n$ that fixes l pointwise. (A *collineation* of a projective plane is a permutation of the plane that takes points to points, lines to lines, and preserves incidence between the two.) It follows that no automorphism can fix two points on an affine line and swap two others. Yet any two distinct points on such a line l' are related by the same atom — the one in $l \cap l'$.

3. For large n, e.g., $n = 29$, there are several non-isomorphic representations of $\mathcal{A}_n$. (Because the group of collineations of $\mathcal{P}_n$ fixing l does not induce the full symmetric group on the set of all points of l, the isomorphism type of $\mathcal{S}_n$ varies with the choice of association of atoms with the points of l.)

See [HH1] for details.

5 Axiomatising RRA

The standard reference work, [HMT] (part 1, page 461), identifies one of the two outstanding problems of the representation theory as '...the problem of providing a simple intrinsic characterization for all representable cylindric algebras ...' Characterisations do exist — for example, see [HMT], part 2, page 112, and for relation algebras, the paper [L2] — but the axioms are certainly not simple. In this section and the next, we will find axioms for the class **RRA** of representable relation algebras. Our methods seem very simple, and, as we will see, they are easily generalised to cylindric algebras.

By theorem 3.5, **RRA** is an elementary class — it is axiomatised by *some* set of first-order sentences. Hence, by the downward Löwenheim–Skolem theorem (see [Hodg2]), it suffices to find axioms that hold in a *countable* relation algebra if and only if it is representable. The same goes for the simple representable relation algebras, as these also form an elementary class. As we said, we will begin with the 'simple' case. The special advantage of this is that we will be able to obtain an *equational* axiomatisation.[1] Recall (theorem 3.5) that **RRA**

[1] Thanks to Yde Venema for pointing this out.

is a variety, so it must have such an axiomatisation. We would therefore like the method to provide one.

So fix a countable simple relation algebra, $\mathcal{A}$.

5.1 Networks

We will use 'forcing conditions' called $\mathcal{A}$-*networks.* As $\mathcal{A}$ is fixed, for now we will call them simply *'networks'.* A network is an approximation to a representation of $\mathcal{A}$.

Definition 5.1

1. A *pre-network* is simply a complete directed finite graph with edges labelled by elements of $\mathcal{A}$. Formally, it is a pair $N = (N_1, N_2)$, where N_1 is a finite non-empty set of nodes, and $N_2 : N_1 \times N_1 \to \mathcal{A}$ is a map assigning an element of $\mathcal{A}$ to each pair of nodes.

2. To free up some suffixes, we will abuse notation and write simply N for any of N, N_1, N_2 above, distinguishing them by context.

3. If N, N' are pre-networks, we write $N \subseteq N'$ if every node of N is a node of N', and, for all nodes x, y of N, we have $N'(x, y) \leq N(x, y)$.

4. A pre-network N is called a *network* if it satisfies:

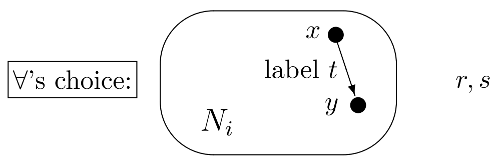
   (a) $N(x, x) \leq 1'A$ for all $x \in N$;

   (b) $(N(x, y); N(y, z)) \cdot N(x, z) \neq 0$ for all $x, y, z \in N$ ('triangle consistency')

5.2 The game G_n

Let N be any pre-network. We define a game, $G_n(N, \mathcal{A})$, between players $\forall$ and $\exists$, to build a chain $N = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$ of pre-networks if n is finite, and an infinite chain $N = N_0 \subseteq N_1 \subseteq \cdots$ if $n = \omega$. There are n rounds, numbered $0, 1, \ldots, i, \ldots$ for $i < n$. In each round, i, the players move as follows.

- $\forall$ chooses $x, y \in N_i$, and elements $r, s \in \mathcal{A}$:

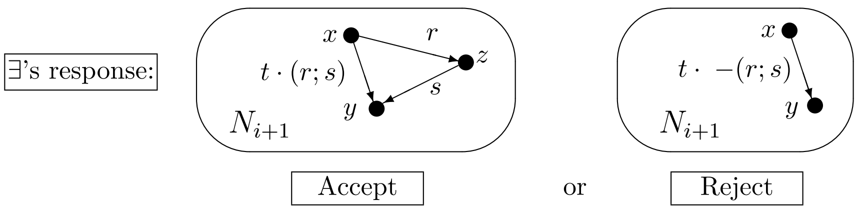


- $\exists$ responds with a pre-network $N_{i+1} \supseteq N_i$ such that one of the following holds:

(reject) N_{i+1} is the same as N_i, except that $N_{i+1}(x, y) = N_i(x, y) \cdot -(r; s)$

(accept) The nodes of N_{i+1} are those of N_i, plus a new one, z. The labels on edges of N_{i+1} are given as follows:

- $N_{i+1}(x, z) = r$
- $N_{i+1}(z, z) = 1'$
- $N_{i+1}(z, y) = s$
- $N_{i+1}(x, y) = N_i(x, y) \cdot (r; s)$
- If $x', y' \in N_i$ and $(x', y') \neq (x, y)$ then $N_{i+1}(x', y') = N_i(x', y')$.
- All other labels in N_{i+1} not yet mentioned are 1.



∃ wins if each pre-network $N_0, N_1, \ldots$ played during the game is actually a *network*. Otherwise, ∀ wins.

So in each round, ∀ challenges ∃ to add a certain triangle to an edge of the network, and ∃ must either do so, or claim instead that the relation on that edge is disjoint from the relation on the proposed side of the triangle. She will lose the game if, at some stage, both options lead to pre-networks violating the third, triangle-consistency condition of the definition of 'network'. She also loses $G_0(N, \mathcal{A})$ if N is not a network, but only a pre-network.

5.3 From representations to axioms, via games

Proposition 5.2 *$\mathcal{A} \in$ **RRA** iff ∃ has a winning strategy in $G_n(I, \mathcal{A})$ for all $n < \omega$, where I is the one-point network consisting of a single node, 0, with $I(0, 0) = 1'A$.*

Proof. $\Rightarrow$: If M is a representation of $\mathcal{A}$, ∃ can use M to help her decide whether to accept or reject in each round of $G_n(I, \mathcal{A})$. In more detail, she preserves the condition that for each $i < n$ there's a map $' : N_i \to M$ with $M \models [N_i(x, y)](x', y')$ for all nodes x, y of N_i. She starts by defining $0'$ to be any point of M (recall that 0 is the only node of the trivial network I). If ∀ plays x, y, r, s in round i:

1. If $M \models [r; s](x', y')$, she accepts, and extends the map $'$ by mapping the z of the game definition above to any $z' \in M$ with $M \models r(x', z') \wedge s(z', y')$.

2. If $M \not\models [r; s](x', y')$, she rejects. Recall from remark 3.2 that as $\mathcal{A}$ is simple, $M \models \forall uv\ 1(u, v)$. It follows that $M \models [-(r; s)](x', y')$, so the conditions on the map $'$ are kept into the next round.

$\Leftarrow$: Assume that $\exists$ has a winning strategy in $G_n(I, \mathcal{A})$ for infinitely many $n < \omega$. We claim that she also has a winning strategy in $G_\omega(I, \mathcal{A})$. This is because in any round, i, she can apply her winning strategies in the games G_n, for infinitely many $n > i$, to the current position. These strategies say whether to accept or reject $\forall$'s triangle. If infinitely many of them tell her to accept, then she accepts. If not, then infinitely many will advise rejection, and she rejects. In either case, she arrives at the next round in a position where infinitely many winning strategies are still running. So she can continue in the same way, forever. As she never loses in any round, she ends up winning $G_\omega(I, \mathcal{A})$.

Now, in a play of $G_\omega(I, \mathcal{A})$, let $\exists$ use her winning strategy, but also persuade $\forall$ to play at some stage x, y, r, s, for each pair x, y of nodes that arise during the game, and for every $r, s \in \mathcal{A}$. This is possible, as both $\mathcal{A}$ and the networks N_i in the game are countable — and she is on good terms with $\forall$. The outcome is (essentially) an $\mathcal{A}$-structure M, defined as follows. Let N^* be the set of nodes of all the networks played during the game. That is, $N^* = \bigcup_{i<\omega} N_i$. Define a binary relation $\sim$ on N^* by $x \sim y$ iff $\exists i < \omega\, (x, y \in N_i \wedge N_i(x, y) \leq 1')$. It can be checked that $\sim$ is an equivalence relation on N^*. If $x \in N^*$, we write $x/{\sim}$ for the $\sim$-class of x; we write $N^*/{\sim}$ for the set of all $\sim$-classes. The structure M is then defined as having domain $N^*/{\sim}$, and, for any $r \in \mathcal{A}$, we define $M \models r(x/{\sim}, y/{\sim})$ iff $\exists i < \omega\, (x, y \in N_i \wedge N_i(x, y) \leq r)$. As can be checked, this is well-defined.

One can now check that all axioms but the final one of definition 3.1 hold in M. The last axiom, '$M \models \exists xy\, r(x, y)$ for all non-zero $r \in \mathcal{A}$', follows from the others, since $\mathcal{A}$ is simple. Hence, M is a representation of $\mathcal{A}$. ∎

We need a little notation.

Definition 5.3

1. A *term network* is a complete directed finite graph N, each of whose edges is labelled with a term of the signature $\{+, -, 0, 1, 1', \smile, \ ;\}$ of relation algebras. If x, y are nodes of N, we write $N(x, y)$ for the term labelling the edge (x, y) of N.

2. Let N be any term network, and let ι be an assignment that maps the variables occurring in the labels of N to elements of $\mathcal{A}$. Then we obtain, in the obvious way, a pre-network, which we will denote by N^ι. Explicitly, N^ι has the same nodes as N, and for all such nodes, x, y, say, $N^\iota(x, y)$ is the value in $\mathcal{A}$ of the term $N(x, y)$ under the assignment ι.

3. Let N be a term network, and x, y nodes of N. Let u, v be any relation algebra terms. We define two term networks, $\mathsf{Acc}(N, x, y, u, v)$ and $\mathsf{Rej}(N_i, x, y, u, v)$, as follows.

   - $N' = \mathsf{Acc}(N, x, y, u, v)$ has, as nodes, the nodes of N plus a new one, z. We have $N'(x, z) = u$, $N'(z, z) = 1'$, $N'(z, y) = v$, $N'(x, y) =$

$N(x,y)\cdot(u;v)$, $N'(n,m) = N(n,m)$ for all other pairs of nodes n,m of N, and all other edges of N' are labelled by 1.

- $\mathsf{Rej}(N,x,y,u,v) = N''$ has the same nodes as N, and the same labels, except that $N''(x,y) = N(x,y)\cdot -(u;v)$.

It remains to prove:

Proposition 5.4 *For any n, there is an axiom ρ_n, whose definition is independent of $\mathcal{A}$, such that $\mathcal{A} \models \rho_n$ iff $\exists$ has a winning strategy in $G_n(I,\mathcal{A})$. Here, I is as in proposition 5.2.*

Proof. We will actually define formulas $\rho_n(N)$, where N is any term network. The free variables of $\rho_n(N)$ will be the variables occurring in the terms in the labels of N. We will require that for all assignments ι of these variables into $\mathcal{A}$, and for all $n < \omega$,

$$(*) \qquad \mathcal{A}, \iota \models \rho_n(N) \text{ iff } \exists \text{ has a winning strategy for } G_n(N^\iota, \mathcal{A}).$$

We let $\rho_0(N)$ be a quantifier-free formula saying that N^ι is a network:

$$\bigwedge_{x,y,z\in N} (N(x,x) \leq 1') \wedge ((N(x,y);N(y,z))\cdot N(x,z) \neq 0).$$

Then, inductively, we define $\rho_{n+1}(N)$ to be:

$$\bigwedge_{x,y\in N} \forall u,v\Big(\rho_n(\mathsf{Acc}(N,x,y,u,v)) \vee \rho_n(\mathsf{Rej}(N,x,y,u,v))\Big),$$

where u,v are new variables not occurring in the labels on N.

A simple induction on n now shows that $(*)$ holds. The inductive step uses the evident fact that for any pre-network N', $\exists$ has a winning strategy in $G_{n+1}(N',\mathcal{A})$ iff for all $x,y \in N$ and all $r,s \in \mathcal{A}$, she has a winning strategy in $G_n(N^+,\mathcal{A})$ or in $G_n(N^-,\mathcal{A})$, where N^+, N^- are, respectively, the pre-networks created from N' if she accepts or rejects when $\forall$ plays (x,y,r,s).

We now obtain the sentences ρ_n as $\rho_n(I_t)$, where I_t is the term network having only a single node, 0, and with $I_t(0,0) = 1'$. Note that $I_t^\iota = I$, for any ι. ∎

By propositions 5.2 and 5.4, we have proved:

Theorem 5.5 *The class of simple representable relation algebras is axiomatised by $\{\rho_n : n < \omega\}$, together with the axioms for simple relation algebras given in section 2.* ∎

5.4 Equational axioms

Note that the ρ_n of proposition 5.4 are universal sentences, because $\exists$ never has to choose any elements of $\mathcal{A}$. We can even transform them into equations. We first express ρ_n in prenex normal form, $\forall\bar{x}\psi_n(\bar{x})$, where ψ_n is quantifier free, so a boolean combination of equations. Now we use the following well-known lemma.

Lemma 5.6 *Let $\psi(\bar{x})$ be any boolean combination of equations. Then there is an equation, of the form $s = 0$ for some relation algebra term $s(\bar{x})$, that is equivalent in any simple relation algebra to ψ. That is, $\forall\bar{x}(\psi(\bar{x}) \leftrightarrow (s(\bar{x}) = 0))$ is true in any simple relation algebra.*

Proof. By induction on ψ. If ψ is the equation $t = u$, let $s = (t \cdot -u) + (u \cdot -t)$. Then ψ is equivalent in any relation algebra to $s = 0$.

Assume inductively that ψ is equivalent to $t = 0$, and χ to $u = 0$. Then:

- $\neg\psi$ is equivalent to $\neg(t = 0)$ and so (in simple relation algebras) to $1; t; 1 = 1$, and so to $(-(1; t; 1)) = 0$.
- $\psi \wedge \chi$ is clearly equivalent in any relation algebra to $(t + u) = 0$.

■

Using this lemma on ψ_n, we obtain an equation ε_n of the form $\forall\bar{x}(s(\bar{x}) = 0)$ for some relation algebra term $s(\bar{x})$, which is equivalent in any simple relation algebra to ρ_n. Doing this for each n, we obtain an equational axiomatisation of the simple representable relation algebras, in the sense that a simple relation algebra satisfies ε_n for all $n < \omega$ iff it is representable. (The *simple* relation algebras themselves do not form a variety and so are not equationally axiomatisable.)

5.5 The non-simple case

So far, we have only considered simple relation algebras. To extend our results to the general case, we may use games of the form $G_n(J, \mathcal{A})$ *for any one-point network J,* rather than simply the I of proposition 5.2. The proof of that proposition will now be a little more complicated, as to create a representation we will need to build structures 'M' in the games $G_\omega(J, \mathcal{A})$ for all possible J, and then take their 'disjoint union'. We will also need to adjust proposition 5.4. These complexities made us reluctant to take this approach in the exposition above. More importantly, we used the simplicity of the algebras in obtaining an equational axiomatisation.

There is a further reason to be reluctant: *the equations obtained for the simple case already characterise arbitrary representable relation algebras.*[2] To prove this, we use the fact [JT2, part II, theorem 4.15] that any relation algebra $\mathcal{A}$ is isomorphic to a subalgebra of a product relation algebra of the form

[2]We thank H. Andréka for pointing this out to us.

$\prod_{i\in I}\mathcal{A}_i$, where each $\mathcal{A}_i$ is simple and a homomorphic image of $\mathcal{A}$. Now by theorem 3.5, **RRA** is a variety, so is closed under taking subalgebras, products, and homomorphic images. Hence, $\mathcal{A}$ is representable iff each $\mathcal{A}_i$ is representable.

Let ε_n be an equation that is equivalent in simple relation algebras to ρ_n. Then $\mathcal{A}$ is representable iff each $\mathcal{A}_i$ is; and, by the foregoing work, this holds iff

$$(*) \qquad \mathcal{A}_i \models \varepsilon_n \text{ for all } i \in I, n < \omega.$$

But equations are preserved under subalgebras and products, so $(*)$ implies that $\mathcal{A} \models \varepsilon_n$ for all n. Conversely, if $\mathcal{A} \models \varepsilon_n$ for all n, then $(*)$ holds, as equations are preserved under homomorphic images. Hence $\mathcal{A}$ is representable iff $\mathcal{A} \models \varepsilon_n$ for all n.

6 Homogeneous representations

Homogeneity is a second-order property of a representation. It has been extensively studied in model theory and permutation group theory (see [C, Hodg2], for example).

Definition 6.1 *Let $\mathcal{A}$ be a relation algebra, and let M be a representation of $\mathcal{A}$.*

1. *A* local isomorphism *of M is a partial map $\theta : M \to M$ with finite domain, such that*

   $(*)$ *for all $x, y \in dom(\theta)$, and all $r \in \mathcal{A}$, we have $M \models r(x,y) \leftrightarrow r(\theta(x), \theta(y))$.*

2. *M is said to be* homogeneous *if any local isomorphism of M extends to an automorphism of M. (See definition 3.3.)*

3. *We write* **HRA** *for the class of all relation algebras that have a homogeneous representation.*

As the Lyndon algebras show, not every relation algebra has a homogeneous representation. However, we can use the techniques of the preceding section to axiomatise the finite relation algebras that do.

Theorem 6.2 *There are first-order sentences η_n $(n < \omega)$ which, with the basic relation algebra axioms, axiomatise the finite algebras of* **HRA**. *That is, a finite relation algebra has a homogeneous representation iff the axioms η_n are all true in it.*

Proof. [Sketch; see [HH1] for details.] We modify the old game G_n of section 6.1, by giving $\forall$ the option of moving differently. In any round, say i, he may move as before; but now, he may instead elect to provide $\exists$ with a partial 1–1 map $\theta : N_i \to N_i$. This represents a challenge to $\exists$ to amalgamate two copies of N_i over the stated common part given by θ. In such an eventuality,

1. $\exists$ may accept, by responding with a new network N_{i+1} and a pair of embeddings $\mu, \nu : N_i \to N_{i+1}$ such that $\nu \supseteq \mu \cdot \theta$, $N_{i+1}(\mu(x), \mu(y)) = N_{i+1}(\nu(x), \nu(y)) \leq N_i(x, y)$, for all $x, y \in N_i$, and $N_{i+1}(z, t) = 1$ for all other $z, t \in N_{i+1}$.

2. Or she may reject, by playing a network N_{i+1} with the same nodes as N_i, with $N_{i+1}(u, v) \leq N_i(u, v)$ for all nodes u, v, and such that $N_{i+1}(x, y) \cdot N_{i+1}(\theta(x), \theta(y)) = 0$ for some pair of nodes $x, y \in dom(\theta)$. (In effect, $\exists$ denies that θ is a local isomorphism at all.)

Call this game $H_n(N, \mathcal{A})$, where N is the starting network. It can be shown that a finite relation algebra $\mathcal{A}$ has a homogeneous representation iff $\exists$ can win $H_n(N, \mathcal{A})$ for all $n < \omega$ and all one-point networks N. This uses Fraïssé's well-known work on amalgamation and homogeneity (see [Hodg2, chapter 7], for example). As before, we can write down an axiom η_n (not universal!) saying that $\exists$ has a winning strategy in the game of length n. This completes the argument. ■

Interestingly, when $\exists$ rejects, she must choose values for $N_{i+1}(x, y)$ and $N_{i+1}(\theta(x), \theta(y))$ in $\mathcal{A}$. This means that the axioms η_n will not be universal sentences, though a little thought shows that equivalent $\forall\exists$ axioms exist, and, indeed, we can find universal axioms in a similarity type expanded by a unary predicate picking out the atoms of the algebra. We cannot eliminate $\exists$'s having to choose relations in $\mathcal{A}$, because **HRA** is not universally axiomatisable — it is not closed under subalgebras. For example, the Lyndon algebra $\mathcal{A}_4 \notin$ **HRA**, but it can be extended to a relation algebra with a homogeneous representation. To see this, let D be any representation of $\mathcal{A}_4$. (In fact, D must be an affine plane of order 4.) Form a relation algebra $\mathcal{P}_4$ with domain $\wp(D \times D)$, as in example 4.1. Then D is a homogeneous representation of $\mathcal{P}_4$ (in fact it is completely 'rigid' in that there are no non-trivial local isomorphisms), and $\mathcal{A}_4$ is isomorphic to a subalgebra of $\mathcal{P}_4$.

Remark. In this way, every representable relation algebra extends to a relation algebra in **HRA**.

Questions. What can be said about the infinite algebras in **HRA**? Is **HRA** elementary? Is it closed under elementary equivalence?

7 Cylindric algebras

These are the analogue of relation algebras for relations of larger arity than 2. Our techniques apply equally well to them.

Definition 7.1 • If U is a set and α an ordinal, ${}^{\alpha}U$ denotes the set of functions from α to U. A subset of ${}^{\alpha}U$ is called an α-ary relation on U.

- A *cylindric set algebra* of dimension α is a structure $(S, \cup, \sim, \emptyset, I, D_{ij}, C_i)_{i,j<\alpha}$ where S is a non-empty set of α-ary relations on some domain U, and, of the operations, $\cup$ is interpreted as union, $\sim$ as complement, I as $\bigcup_i {}^{\alpha}(U_i)$, where the U_i are pairwise disjoint non-empty subsets of U, the diagonal elements D_{ij} as $\{f \in I : f(i) = f(j)\}$ (for each $i, j < \alpha$), and the *cylindrification operators* C_i ($i < \alpha$) as follows:

$$\text{if } X \in S, \quad C_i X = \{f \in I : \exists g \in X\ \forall j < \alpha (j \neq i \Rightarrow g(j) = f(j))\}.$$

- A *cylindric algebra* of dimension α is defined to be a structure

$$\mathcal{C} = (C, +, -, 0, 1, d_{ij}, c_i)_{i,j<\alpha},$$

where $+$ is a binary function, $-, c_i$ ($i < \alpha$) are unary functions, and $0, 1, d_{ij}$ ($i, j < \alpha$) are constants. We require that $\mathcal{C}$ obeys the following axioms, for all $x, y \in C$ and $i, j, k < \alpha$:

    1. $(C, +, -, 0, 1)$ is a boolean algebra
    2. $c_i 0 = 0$
    3. $x \leq c_i x$
    4. $c_i(x \cdot c_i y) = c_i x \cdot c_i y$
    5. $c_i c_j x = c_j c_i x$
    6. $d_{ii} = 1$
    7. if $i \notin \{j, k\}$, then $d_{jk} = c_i(d_{ji} \cdot d_{ik})$
    8. if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

    These axioms are valid over cylindric set algebras.

- We write CA_α for the set of equations defining α-dimensional cylindric algebras.

- A cylindric algebra is said to be *representable* if it is isomorphic to a cylindric set algebra. The class of all representable cylindric set algebras of dimension α is denoted $\mathbf{RCA}_\alpha$. It is a variety.

- A *component* x of a cylindric algebra $\mathcal{C}$ is a non-zero element such that $c_i x = x$ for all $i < \alpha$. If the only component is 1, then $\mathcal{C}$ is said to be *simple*. A cylindric algebra is simple if and only if it has no non-trivial homomorphic images. By axioms 3–5, a cylindric algebra of finite dimension α is simple iff $c_0 c_1 \ldots c_{\alpha-1} r = 1$ for all non-zero r. Cylindric set algebras are simple if the universal element I is of the form ${}^{\alpha}U$. Any cylindric algebra can be decomposed into simple components.

7.1 Axiomatising $\mathbf{RCA}_n$ for $3 \leq n < \omega$

Using games, with a special argument for the non-simple case as for relation algebras, we may find equations ψ^n_m $(n, m < \omega)$ such that the following holds:

Theorem 7.2 *For any $n < \omega$, with $n \geq 3$, an n-dimensional cylindric algebra satisfies $\{\psi^n_m : m < \omega\}$ if and only if it is representable.*

The results on homogeneity carry through to cylindric algebras of finite dimension without change. See [HH1] for details.

7.2 Axiomatising $\mathbf{RCA}_\alpha$

The axiomatisation carries over to the α-dimensional representable cylindric algebras, for any $\alpha \geq \omega$. This can be easily read off from the following known algebraic results.

Definition 7.3 *For an ordinal α, let Ax_α consist of CA_α (i.e., the axioms defining α-dimensional cylindric algebras), together with all L_α-sentences of the following form:*

$$\forall v_1, \ldots, v_n (\exists v_0 \varphi(v_0, c_i v_1, \ldots, c_i v_n) \to \exists v_0 \varphi(c_i v_0, c_i v_1, \ldots, c_i v_n)),$$

where $\varphi(v_0, \ldots, v_n)$ is arbitrary, and $i < \alpha$ is such that c_i, d_{ij}, d_{ji} do not occur in φ for any $j < \alpha$.

Fact 7.4 (HMT, part 2, corollaries 4.1.15, 4.1.16) *Let α be an ordinal, and ε an equation of L_α. Then ε is valid in $\mathbf{RCA}_\alpha$ iff $Ax_{\max(\alpha,\omega)} \vdash \varepsilon$.*

To exploit this, we need some notation for renaming symbols of a signature. Write L_α for the signature of α-dimensional cylindric algebras. Let α, β be ordinals, and let $\mu : \alpha \to \beta$ be any 1–1 partial map.[3] For an L_α-formula φ, we define φ^μ to be the L_β-formula obtained by replacing every c_i in φ by $c_{\mu(i)}$, and replacing every d_{ij} by $d_{\mu(i),\mu(j)}$. So φ^μ is only defined if μ is defined on i, j, k for every c_i, d_{jk} occurring in φ.

For sets Φ of formulas on which μ is defined, we define $\Phi^\mu = \{\varphi^\mu : \varphi \in \Phi\}$.

Lemma 7.5 *Let α, β be ordinals, and $\mu : \alpha \to \beta$ be a partial 1–1 map.*

1. *Let Φ be any set of L_α-sentences, and suppose that μ is defined on them and also on some other L_α-sentence σ. Then $\Phi \vdash \sigma$ iff $\Phi^\mu \vdash \sigma^\mu$.*

2. *If μ is a total map, then $(CA_\alpha)^\mu \subseteq CA_\beta$, and $(Ax_\alpha)^\mu \subseteq Ax_\beta$.*

(Recall that CA_α is the set of axioms (equations) defining α-dimensional cylindric algebras.)

[3] Recall that an ordinal is the set of all smaller ordinals.

Proof. One can prove (1) by applying μ or its inverse to every sentence in a proof of σ from Φ, for example. (2) follows from the definitions. ■

We use this lemma implicitly in the following corollary to theorem 7.2.

Corollary 7.6 *Let $\alpha \geq \omega$ be an ordinal. Then $\mathbf{RCA}_\alpha$ is axiomatised by the set*

$$\Sigma = CA_\alpha \cup \{(\psi^n_m)^\mu : m, n < \omega, \ \ \mu : n \to \alpha \text{ is a 1–1 map}\}.$$

Proof. We begin with a claim.

Claim. For any L_α-equation ε, we have $Ax_\alpha \vdash \varepsilon$ iff $\Sigma \vdash \varepsilon$.
Proof of claim. For '$\Rightarrow$', fix an L_α-equation ε such that $Ax_\alpha \vdash \varepsilon$. There is a finite set $\Phi \subseteq Ax_\alpha$ such that $\Phi \vdash \varepsilon$. Choose a partial, surjective 1–1 map $\mu : \alpha \to \omega$ such that $\varepsilon^\mu, \varphi^\mu \, (\varphi \in \Phi)$ are all defined. Then $\Phi^\mu \vdash \varepsilon^\mu$. Examining the definition of Ax, we see that $\Phi^\mu \subseteq Ax_\omega$. Thus, $Ax_\omega \vdash \varepsilon^\mu$. By fact 7.4, ε^μ is valid in $\mathbf{RCA}_n$ for all large enough $n < \omega$. Fix such an n. By theorem 7.2, $CA_n \cup \{\psi^n_m : m < \omega\} \vdash \varepsilon^\mu$. So $(CA_n)^{\mu^{-1}} \cup \{(\psi^n_m)^{\mu^{-1}} : m < \omega\} \vdash \varepsilon$, whence $\Sigma \vdash \varepsilon$, as required.

The converse is easier. It is enough to show that any $(\psi^n_m)^\mu \in \Sigma$ is a consequence of Ax_α. Well, by theorem 7.2, ψ^n_m is an equation valid in $\mathbf{RCA}_n$, so, by fact 7.4, $Ax_\omega \vdash \psi^n_m$. Let $\nu : \omega \to \alpha$ be any 1–1 extension of μ to ω. Then $(Ax_\omega)^\nu \vdash (\psi^n_m)^\mu$. By definition of Ax, we have $(Ax_\omega)^\nu \subseteq Ax_\alpha$. So $Ax_\alpha \vdash (\psi^n_m)^\mu$, as required. This proves the claim.

As $\mathbf{RCA}_\alpha$ is a variety, any L_α-structure $\mathcal{C}$ is in $\mathbf{RCA}_\alpha$ iff it satisfies all equations of L_α that are valid in $\mathbf{RCA}_\alpha$. By fact 7.4, an L_α-equation is valid in $\mathbf{RCA}_\alpha$ iff it is a logical consequence of Ax_α. By the claim, this is iff it is a logical consequence of the set Σ given in the corollary. So $\mathcal{C}$ is in $\mathbf{RCA}_\alpha$ iff it satisfies all equational consequences of Σ. Since Σ itself consists of equations, this is iff $\mathcal{C} \models \Sigma$, as required. ■

8 Complete representations

Given an algebraic logic — boolean algebra, relation algebra or cylindric algebra, or indeed any structure with boolean operations — we may be interested in infinitary unions and intersections. In an algebra $\mathcal{A}$ with domain A, given any subset $S \subseteq A$, we can define the 'arbitrary union' $\sum S$ to be the least upper bound of S in $\mathcal{A}$, if it exists; we leave it undefined otherwise. Similarly, $\prod S$ is defined as the greatest lower bound of S, if it exists.

However, in a representation we have available a different notion of infinite union. A representation M of $\mathcal{A}$ is of course a structure in which every $r \in \mathcal{A}$ is interpreted as a *relation* r^M on M. The arity of these relations will depend on what kind of representation we have in mind — for example, if $\mathcal{A}$ is a boolean algebra, the r^M should be unary relations on M, if it is a relation algebra, they should be binary relations, and if it is an α-dimensional cylindric algebra, they should be α-ary relations. To be a representation, the interpretations of the elements of the algebra as relations on M should be connected to their algebraic properties in $\mathcal{A}$. Precisely what connections will be required depends on the particular case; but we would always expect the boolean operations on the algebra to be respected in the representation, so that in particular, if $\mathcal{A} \models r \leq s$ then $r^M \subseteq s^M$, for all algebra elements r, s. We would also expect that $r^M \neq \emptyset$ for each non-zero r in the algebra.

If now $S \subseteq A$, then we can associate with S the set-theoretic union of the relations s^M, for $s \in S$. Because the boolean operations are respected, the least upper bound of S in $\mathcal{A}$, if it exists, will be interpreted in M as a set containing this set-theoretic union; but the containment may be strict.

A complete representation (of boolean algebras, relation algebras, cylindric algebras, and so on) is one in which the two notions of 'union' agree:

Definition 8.1

1. A representation M is said to be *complete* if it respects arbitrary unions (hence also intersections) wherever they are defined. That is,

$$(\sum S)^M = \bigcup\{s^M : s \in S\}, \quad \text{whenever } \sum S \text{ is defined.}$$

2. An algebra is said to be *completely representable* if it has a complete representation.

This concept can be tackled in another way. Let M be a representation of an algebra $\mathcal{A}$; it is of no concern whether $\mathcal{A}$ is a boolean algebra, a relation algebra, or whatever. If $\bar{x} \in M$, we write $f_{\bar{x}}$ for the set $\{r \in \mathcal{A} : M \models r(\bar{x})\}$ of elements of $\mathcal{A}$. Evidently, if $M \models 1(\bar{x})$ then $f_{\bar{x}}$ is an ultrafilter on $\mathcal{A}$.

Definition 8.2 In this notation, M is said to be an *atomic representation* if, for all $\bar{x} \in M$ with $M \models 1(\bar{x})$, the ultrafilter $f_{\bar{x}}$ is principal — or, equivalently, there is some atom $\alpha \in \mathcal{A}$ with $M \models \alpha(\bar{x})$.

This is to say that 1^M is a union of interpretations of atoms. In [L2], relation algebras with such a representation are called *strongly representable,* after D. Scott.

Proposition 8.3 *A representation M of $\mathcal{A}$ is complete if and only if it is atomic. If $\mathcal{A}$ has a complete representation, then the boolean reduct of $\mathcal{A}$ is an atomic boolean algebra.*

Proof. An easy exercise. See [HH1,HH2] for details. ■

We may wish to know when an algebra has a complete representation. Our results can be summarised as follows:

- A boolean algebra has a complete representation iff it is atomic.

- The class of completely representable relation algebras is not an elementary class — it cannot be axiomatised by any set of first-order sentences.
- The class of completely representable cylindric algebras of any given finite dimension is not elementary, either. The proof is broadly similar to that for relation algebras.
- The same holds for infinite-dimensional cylindric algebras; the proof is a simple cardinality argument.

In what follows, we will outline some of the arguments used here.

8.1 Boolean Algebra

Proposition 8.4 *A boolean algebra $\mathcal{B}$ has a complete representation if and only if it is an atomic boolean algebra.*

Proof. For '$\Rightarrow$', see proposition 8.3. For '$\Leftarrow$', if $\mathcal{B}$ is atomic, then let X be the set of atoms of $\mathcal{B}$. Regard the elements of $\mathcal{B}$ as unary relation symbols, as usual. Define a structure M, with domain X, by

$$M \models b(x) \text{ iff } \mathcal{B} \models x \leq b, \text{ for all } b \in \mathcal{B},\, x \in X.$$

Then M is an atomic representation of $\mathcal{B}$, so, by proposition 8.3, a complete representation. ■

8.2 Relation Algebra

CRA denotes the class of completely representable relation algebras. By proposition 8.3, every $\mathcal{A} \in \mathbf{CRA}$ is atomic (as a boolean algebra). Of course, if $\mathcal{A}$ is finite, any representation of $\mathcal{A}$ is complete. But, unlike in boolean algebra theory, not every atomic relation algebra has a complete representation. (For example, any finite non-representable one.) Worse, even if $\mathcal{A}$ is atomic and has a representation, it still may not have a complete one. So 'atomicity' does not pick out the completely representable relation algebras from the representable ones. Indeed, no set of first-order properties does:

Theorem 8.5 [HH2] *The class* **CRA** *is not elementary.*

We indicate the main steps in the proof of theorem 8.5. The details, for both relation algebras and cylindric algebras, can be found in [HH2], and, for the relation algebra case alone, in [Hi]. Let $\mathcal{A}$ be an atomic relation algebra. An $\mathcal{A}$-network N is said to be *atomic* if every $N(x, y)$ is an atom of $\mathcal{A}$. We define a new game, $G^a_n(\mathcal{A})$, of length $n \leq \omega$. In it, the players $\forall, \exists$ build a chain $N_1 \subseteq N_2 \subseteq \cdots$ of atomic networks. In the first round, $\forall$ chooses for $\exists$ a one-point atomic network, N_1. In each later round, if N_i was the last network to be constructed,

- $\forall$ chooses $x, y \in N_i$ and atoms $a, b \in \mathcal{A}$ such that $a;b \geq N_i(x, y)$.

- $\exists$ responds with an atomic network N_{i+1} extending N_i (with more nodes), with $N_i(x,y) = N_{i+1}(x,y)$ for all nodes x, y of N_i, and such that there is $z \in N_{i+1}$ with $N_{i+1}(x,z) = a$ and $N_{i+1}(z,y) = b$.

So $\forall$ demands a certain triangle to be added to the network, and $\exists$ must comply. $\exists$ wins if she never gets stuck.

Lemma 8.6 *Let $\mathcal{B}$ be the ultrapower $\mathcal{A}^\omega/F$, where F is any non-principal ultrafilter over ω. Assume that $\exists$ has a winning strategy in $G^a_n(\mathcal{A})$ for all $n < \omega$. Then she has a winning strategy in $G^a_\omega(\mathcal{B})$.*

Proof. Construct a winning strategy along the ultrapower. The lemma actually holds where $\mathcal{B}$ is any ω-saturated relation algebra elementarily equivalent to $\mathcal{A}$. (Recall the standard fact that the ultrapower $\mathcal{A}^\omega/F$ is ω_1-saturated. See, for example, [Hodg2].) ∎

Lemma 8.7 *Let $\mathcal{B}$ be any relation algebra such that $\exists$ has a winning strategy for $G^a_\omega(\mathcal{B})$. Then there is a countable elementary subalgebra $\mathcal{C}$ of $\mathcal{B}$ such that $\exists$ has a winning strategy in $G^a_\omega(\mathcal{C})$.*

Proof. Let $\mathcal{C}_0$ be any countable, elementary subalgebra of $\mathcal{B}$. Make a chain of countable elementary subalgebras

$$\mathcal{C}_0 \preceq \mathcal{C}_1 \preceq \cdots$$

of $\mathcal{B}$ such that, in the game $G^a_\omega(\mathcal{B})$, if $\forall$'s moves are restricted to atoms in $\mathcal{C}_i$ then $\exists$'s winning strategy chooses only networks labelled by atoms in $\mathcal{C}_{i+1}$. The union of this chain, $\mathcal{C}$, say, is a countable elementary subalgebra of $\mathcal{B}$; and by construction, $\exists$ can win $G^a_\omega(\mathcal{C})$. ∎

Lemma 8.8 *Let $\mathcal{C}$ be any countable atomic relation algebra (or, more generally, one with countably many atoms). Then $\exists$ has a winning strategy in $G^a_\omega(\mathcal{C})$ iff $\mathcal{C}$ has a complete representation.*

Proof. This is much as in proposition 5.2. If M is a complete representation of $\mathcal{C}$, then $\exists$ can use M as a guide to her moves, and win $G^a_\omega(\mathcal{C})$. Conversely, if she has a winning strategy for $G^a_\omega(\mathcal{C})$, she can persuade $\forall$ to demand all 'atomic triangles' to be added everywhere possible, during the course of the game. This is possible because $\mathcal{C}$ has countably many atoms. Let $M_e = (\bigcup_{i<\omega} N_i)/\sim$ be the outcome of such a play, in which $\exists$ used her winning strategy, and in which $\forall$ began with a one-point network whose node is labelled by $e \in \mathcal{A}$ (necessarily, $0 < e \leq 1'$). Here, $\sim$ is as in proposition 5.2. Then the disjoint union $\bigcup_{0<e\leq 1'} M_e$ will be a representation of $\mathcal{B}$. But it will also be *complete,* because the N_i are always atomic. ∎

Combining lemmas 8.6, 8.7, and 8.8 gives

Theorem 8.9 *$\exists$ has a winning strategy in $G^a_n(\mathcal{A})$ (for all $n < \omega$) if and only if $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra.*

So, by lemma 8.8 and theorem 8.9, to show that **CRA** is not an elementary class, it is enough to construct a countable atomic relation algebra $\mathcal{A}$ such that $\exists$ can win $G^a_n(\mathcal{A})$ for all finite n, but not $G^a_\omega(\mathcal{A})$. This was done in [HH2], where it was also shown in a similar way that the completely representable cylindric algebras of any fixed finite dimension do not form an elementary class.

Remark 8.10 For each $n < \omega$, the statement '$\exists$ has a winning strategy in $G^a_n(\mathcal{A})$' can be written as a first-order condition on $\mathcal{A}$, in the manner of proposition 5.4. If we do so, we essentially obtain the so-called 'Lyndon conditions', given in [L1], and claimed[4] there to axiomatise **RRA**. By theorem 8.9, we see that an atomic relation algebra satisfies the Lyndon conditions if and only if it is elementarily equivalent to a completely representable relation algebra.

8.3 Infinite-dimensional cylindric algebras

A simple cardinality argument shows that the class of all cylindric algebras of any fixed infinite dimension that have a complete representation is not elementary.

Let α be a fixed infinite ordinal, the dimension. Below, $|\alpha|$ denotes the cardinality of α. Write $\mathbf{CCA}_\alpha$ for the class of all α-dimensional cylindric algebras $\mathcal{C}$ such that $\mathcal{C}$ has a complete representation. If $\mathcal{C} \in \mathbf{CCA}_\alpha$ then $\mathcal{C}$ must be atomic (cf. proposition 8.3). We write $At(\mathcal{C})$ for the set of all atoms of $\mathcal{C}$.

Lemma 8.11 *Let $\mathcal{C} \in \mathbf{CCA}_\alpha$ be such that $\mathcal{C} \models d_{01} < 1$. Then $|At(\mathcal{C})| \geq 2^{|\alpha|}$.*

Proof. Let h be an isomorphism from $\mathcal{C}$ onto a cylindric set algebra on the set ${}^\alpha M$ that respects all infs and sups where defined. Since $\mathcal{C} \models d_{01} < 1$, there is $\bar{a} \in {}^\alpha M$ with $\bar{a} \in h(1) \setminus h(d_{01})$. So if $x = \bar{a}_0, y = \bar{a}_1$, then $x \neq y$.

Let $S \subseteq \alpha$ be arbitrary such that $0 \in S$, and define $\bar{a}_S \in {}^\alpha M$ to be the sequence whose ith coordinate is x, if $i \in S$, and y, if $i \in \alpha \setminus S$. Then $\bar{a}_S \in h(1)$ by definition of cylindric set algebras, so by proposition 8.3, $\bar{a}_S$ is 'labelled' by an atom, in that $\bar{a}_S \in h(c)$ for some (unique) atom c of $\mathcal{C}$. Let $S, S' \subseteq \alpha$ be any distinct sets containing 0. Then, without loss of generality, there is $i < \alpha$ with $i \in S, i \notin S'$. Clearly, d_{0i} is above ($\geq$) the atom that labels $\bar{a}_S$, but not above the one labelling $\bar{a}_{S'}$. So these atoms must be different. Hence, the number of atoms in $\mathcal{C}$ is at least the number of subsets of α that contain 0 — i.e., at least $2^{|\alpha|}$. ∎

Corollary 8.12 *The class $\mathbf{CCA}_\alpha$ is not elementary.*

Proof. Take any $\mathcal{C} \in \mathbf{CCA}_\alpha$ such that $\mathcal{C} \models d_{01} < 1$. (There exist such $\mathcal{C}$ — for example, the full power set algebra $\wp({}^\alpha X)$, for any set X with at least two elements.) Since the cardinality of the language of α-dimensional cylindric

[4] The claim is correct for finite relation algebras but not for arbitrary ones; the correction is in [L2].

algebras is $|\alpha|$, we may use the downward Löwenheim–Skolem theorem to take $\mathcal{B} \preceq \mathcal{C}$ with $|\mathcal{B}| \leq |\alpha|$. Then $\mathcal{B} \models d_{01} < 1$, since $\mathcal{B}$ is an elementary substructure of $\mathcal{C}$. But $\mathcal{B}$ has at most $|\alpha|$ atoms. Hence, by the lemma, $\mathcal{B} \notin \mathbf{CCA}_\alpha$. So the class $\mathbf{CCA}_\alpha$ is not closed under elementary equivalence, and so cannot be elementary. ■

9 Atom structures

The relation algebra structure of an atomic relation algebra is determined by its boolean algebra structure and by the way the product operation behaves on its atoms. Hence we can extract the atom structure of an atomic relation algebra, and endow it with natural first-order relations from which the relation algebra structure of the given algebra may be recovered. Now the difficulties in finding representations for relation algebras mostly arise from the product structure — it is easy to find representations of a boolean algebra, and even complete representations. So the question naturally arises as to whether these problems can be pinned down to the atom structure, in the case of atomic relation algebras. That is, does representability of an atomic relation algebra depend only on its atom structure?

If the relation algebra is finite then the answer is clearly 'yes'. So we confine our attention to the infinite case. Here, there are many different relation algebras sharing the same atom structure $\mathcal{C}$. The biggest, $\mathcal{P}(\mathcal{C})$, is called the *complex algebra over* $\mathcal{C}$, and has as its universe the power set of $\mathcal{C}$. The smallest — call it the *term algebra* — is the subalgebra of $\mathcal{P}(\mathcal{C})$ generated by the atoms. Slightly more formally, any relation algebra $\mathcal{A}$ with atom structure $\mathcal{C}$ embeds into the complex algebra over $\mathcal{C}$, via $r \mapsto \{c \in \mathcal{C} : \mathcal{A} \models c \leq r\}$. Identifying $\mathcal{A}$ with its image under this embedding, the term algebra over $\mathcal{C}$ is a subalgebra of $\mathcal{A}$. Thus, up to isomorphism, any relation algebra with atom structure $\mathcal{C}$ lies between the term algebra and the complex algebra over $\mathcal{C}$.

Now, by theorem 3.5, if a relation algebra is representable then so is any subalgebra. So our question can be rephrased: if the term algebra of an atom structure $\mathcal{C}$ is representable, must the complex algebra over $\mathcal{C}$ be representable as well?

Our knowledge here is as follows. See [Hodk] for the proofs.

- There are two countable relation algebras $\mathcal{A}, \mathcal{B}$ with the same atom structure, $\mathcal{A}$ being representable, and $\mathcal{B}$ not representable. This is proved by the construction of [HH2] mentioned above. Hence the answer to the question above is 'no'.

- The class

  $$\{\mathcal{C} : \textit{some} \text{ relation algebra with atom structure } \mathcal{C} \text{ is representable}\}$$

  of atom structures of representable atomic relation algebras is elementary, and can be axiomatised explicitly using games. However, there is no axiomatisation using a finite number of first-order sentences.[5]

- The class of atom structures

  $$\mathcal{S} = \{\mathcal{C} : \textit{every} \text{ relation algebra with atom structure } \mathcal{C} \text{ is representable}\}$$

  is not finitely axiomatisable in first-order logic.[5]

- If an atom structure $\mathcal{C}$ satisfies the Lyndon conditions (see remark 8.10), then it is in $\mathcal{S}$. The converse fails: a counterexample can be given by modifying a construction of Maddux [Ma1], which in turn is related to one of Lyndon [L1].

- We do not know whether $\mathcal{S}$ is elementary; this question was asked in [Ma2]. We conjecture that it is not. Nor do we know whether it is closed under elementary equivalence, or whether membership of it is set-theoretically absolute.

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[5] Indeed, there is no axiomatisation by a sentence of the infinitary logic $L^{\omega}_{\infty\omega}$.

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