Connections between cylindric algebras and relation algebras

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Abstract

We investigate the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for $4 \leq n < \omega$ and survey some recent results. We see that RA_n — the subalgebras of relation algebras with relational bases — is too weak, and that the class of relation algebras whose canonical extension has an n-dimensional cylindric basis is too strong to define the class. We introduce the notion of an n-dimensional hyperbasis and show that for any relation algebra $\mathcal{A}$ the canonical extension $\mathcal{A}^+$ has such a hyperbasis if and only if $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We introduce techniques that can be used to show that the hierarchies $\mathsf{RA}_4 \supset \mathsf{RA}_5 \supset \ldots$ and $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supset \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supset \ldots$ are strict and each step is not finitely axiomatisable.

We outline a relativized semantics that characterises RA_n and another one for the class of subalgebras of relation algebras with n-dimensional cylindric bases.

1 Relation algebra reducts by bases

This abstract summarises our investigations into the connections between relation algebras and cylindric algebras. The detailed proofs have been omitted, but fuller accounts of the material can be found in [HH98, HHM98, HH99a, HH99b]. Our intention here is to provide a concise overview of a number of different but related results.

Algebraic logic is the study of algebraic counterparts to logical systems and historically an important part of algebraic logic concerns the algebraic treatment of relations. The simplest algebraic logic, *boolean algebra*, can be thought of as the algebra of unary relations. The correspondence between boolean algebras and fields of unary relations (or just fields of sets) is precise: every field of sets is a boolean algebra and every boolean algebra is isomorphic to a field of sets [Sto36].

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There are a number of algebras that are intended to correspond to relations of higher ranks, though the correspondence is less accurate. For binary relations, the most important kind of algebra is a *relation algebra*, based on the work of De Morgan, Peirce and Schröder and formalised by Tarski. But not every relation algebra is isomorphic to a field of binary relations (not every relation algebra is *representable*) and it requires infinitely many axioms to characterise the isomorphism class of the representable relation algebras [Lyn50, Mon64].

For higher-order relations there are alternative algebraizations: cylindric algebra, diagonal-free cylindric algebra, polyadic algebra and so on. The one which has received most attention is cylindric algebra. As with binary relations, not every n-dimensional cylindric algebra is representable as an n-dimensional generalised cylindric set algebra (which is a generalisation of a field of n-ary relations) and it requires infinitely many axioms to characterise this representation class.

There is a well-known technique for obtaining a relation algebra $\mathfrak{Ra}(\mathcal{C})$ from an n-dimensional cylindric algebra $\mathcal{C}$, for $n \geq 4$. $\mathfrak{Ra}(\mathcal{C})$ is constructed by taking the two-dimensional elements of $\mathcal{C}$ (i.e. the elements $a \in \mathcal{C}$ such that $c_i a = a$ for $2 \leq i < n$) and using the spare dimensions to define converse and composition [HMT85, definition 5.3.7]. $\mathfrak{Ra}(\mathcal{C})$ is called the relation algebra reduct of $\mathcal{C}$. The fact that $n \geq 4$ is enough to ensure that $\mathfrak{Ra}(\mathcal{C})$ is associative and hence a relation algebra [HMT85, theorem 5.3.8]. For $m < n$ there is a similar (and rather easier) method of constructing an m-dimensional cylindric algebra $\mathfrak{Nr}_m(\mathcal{C})$ from an n-dimensional one $\mathcal{C}$.

The focus of this paper is to investigate the connections between cylindric algebras of dimension $n \geq 4$ and relation algebras. In particular we will address the question: for which relation algebras $\mathcal{A}$ is there an n-dimensional cylindric algebra $\mathcal{C}$ such that $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$? Thus we wish to characterise the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}$ of subalgebras of relation algebra reducts of n-dimensional cylindric algebras.

A number (two) of candidates present themselves as characterisations of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Maddux defines n-dimensional cylindric bases and n-dimensional relational bases. Part of his motivation in defining these bases is the connection with reducts of relation algebras though, particularly for relational bases, Maddux was also interested in the connections with first-order proof theory using only n variables. Before we define these bases we need to define a *n-network* (sometimes referred to as a basic matrix or as an atomic network).

Definition 1 *An n-network N for an atomic relation algebra $\mathcal{A}$ is a function $N : n \times n \to At(\mathcal{A})$ such that for all $i, j, k < n$ we have $N(i,i) \leq 1'$ and $N(i,j); N(j,k) \geq N(i,k)$. From this we can also deduce that $N(i,j) = N(j,i)^\smile$.*

For two n-networks M, N and for $k < n$ we write $M \equiv_k N$ if M and *N agree off k, i.e. for $i, j \in n \setminus \{k\}$ we have $N(i,j) = M(i,j)$. Similarly, for $k, l < n$ we write $M \equiv_{kl} N$ if for all $i, j \in n \setminus \{k, l\}$ we have $N(i,j) = M(i,j)$.*

Definition 2 *[Mad92, section 4] An n-dimensional relational basis for an atomic relation algebra $\mathcal{A}$ is a set $\mathcal{R}$ of n-networks such that*

- *for all $a \in At(\mathcal{A})$ there is some $N \in \mathcal{R}$ with $N(0,1) = a$*

- *if $N \in \mathcal{R}$, $i, j < n$ and $a, b \in At(\mathcal{A})$ satisfy $N(i,j) \leq a;b$ and if $k \in n\setminus\{i,j\}$ then there is $M \in \mathcal{R}$ with $M \equiv_k N$ and $M(i,k) = a$, $M(k,j) = b$.*

Definition 3 *[Mad78, definition 11, page 138] An n-dimensional cylindric basis $\mathcal{S}$ for $\mathcal{A}$ is an n-dimensional relational basis such that if $M, N \in \mathcal{S}$, $i \neq j < n$ and $M \equiv_{ij} N$ then there is $L \in \mathcal{S}$ with $M \equiv_i L \equiv_j N$. We may refer to this condition as the* amalgamation condition. *CB_n denotes the class of all atomic relation algebras with n-dimensional cylindric bases.*

From a relational basis $\mathcal{R}$ for a relation algebra $\mathcal{A}$, you might hope to embed $\mathcal{A}$ in a cylindric algebra $\mathcal{C}$ whose domain is $\wp(\mathcal{R})$ (so that atoms are of the form $\{N\}$, which we identify with N, for $N \in \mathcal{R}$) where the diagonals and cylindrifications of $\mathcal{C}$ are defined by

- $N \leq d_{ij}$ if and only if $N(i,j) \leq 1'$ for $N \in \mathcal{R}$ and $i, j < n$.

- $c_i(N) = \{M \in \mathcal{R} : M \equiv_i N\}$.

This nearly gives a cylindric algebra but it may fail to satisfy the commutativity axiom

$$c_i c_j x = c_j c_i x$$

which holds in cylindric algebras. So the condition that $\mathcal{A}$ should extend to a relation algebra with an n-dimensional relational basis seems too weak to characterise the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Indeed, for $n \geq 5$, the algebra $\mathcal{A}_{n-1}$, defined below, has an n-dimensional relational basis but does not belong to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

So we consider atomic relation algebras with n-dimensional cylindric bases. From such a relation algebra we can use the elements of a cylindric basis as atoms in a cylindric algebra with the definitions given just above. This time, because of the amalgamation condition on cylindric bases, the commutativity axiom is satisfied. Thus, $\mathbf{S}\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ [Mad78, theorem 13, page 140]. Indeed, Maddux shows elsewhere that for any relation algebra $\mathcal{A}$, if its canonical embedding algebra $\mathcal{A}^+ \in \mathsf{CB}_n$ then $\mathcal{A} \subseteq \mathcal{A}^+ \in \mathfrak{Ra}\mathsf{CA}_n$.

One might hope for a converse to this. Starting from an arbitrary n-dimensional cylindric algebra $\mathcal{C}$, consider a relation algebra $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$. Note first that $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C}) \subseteq \mathfrak{Ra}(\mathcal{C}^+)$. Now we might try to show that $\mathcal{A}^+$ has an n-dimensional cylindric basis as follows. For each atom $x \in At(\mathcal{C}^+)$ define an n-network N_x for $\mathcal{A}^+$ by

$$N_x(i,j) = \{a \in \mathcal{A} : x \leq s_{ij} a\}$$

where s_{ij} is a certain string of substitutions mapping the 'free variables' 0 and 1 of a to i and j respectively, e.g. $s_{34} = s_3^0 s_4^1$. Now the set $N_x(i,j)$ is an ultrafilter of $\mathcal{A}$ and hence an atom of $\mathcal{A}^+$ and it can be shown (for $n \geq 4$), using elementary but technical properties of substitutions, that N_x is indeed an n-network. But the set $\{N_x : x \in At(\mathcal{C}^+)\}$ may fail to satisfy the amalgamation condition and hence it may fail to be a cylindric basis. The reason for this failure, roughly, is that from $N_x \equiv_{ij} N_y$ we cannot deduce that $x \leq c_i c_j y$. (If we could deduce that then of course we could find an atom z with $x \leq c_i z$ and $z \leq c_j y$. The amalgum would then be N_z.) So the condition that $\mathcal{A}^+$ has an n-dimensional cylindric basis is too strong a condition to characterise $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Indeed we can find a counter example: a finite relation algebra $\mathcal{A}$ in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ but such that $\mathcal{A}^+ \cong \mathcal{A}$ has no n-dimensional cylindric basis. The required example is a *Lyndon algebra* as in [Lyn61]. Lyndon showed that some of these algebras are representable and that a representation must be isomorphic to a certain affine geometry. It has also been shown that these representations are never *homogeneous*: there is a local isomorphism σ (i.e. a partial map on the representation X such that if $x, y \in dom(\sigma)$ then $X, x, y \models a \Leftrightarrow X, \sigma(x), \sigma(y) \models a$ for all $a \in \mathcal{A}$) such that σ does not extend to a full automorphism of X [HH97, section 6.2]. From the fact that $\mathcal{A}$ is representable, [HMT85, theorem 5.3.16] proves that $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for all $n < \omega$ but from the fact that it has no homogeneous representation it can be proved that there is an $n < \omega$ such that $\mathcal{A}$ has no n-dimensional cylindric basis, in fact it can be shown that $\mathcal{A}$ has no 5-dimensional cylindric basis.

Thus we seek a condition stronger than the existence of a relational basis but weaker than the existence of a cylindric basis. Let $\mathcal{A}$ be an atomic relation algebra, $n \geq 4$ and let Λ be any set (of 'labels').

Definition 4 *An* n-dimensional Λ-hypernetwork over $\mathcal{A}$, *$N = (N^-, N^+)$, is an n-network N^- extended with a labelling function $N^+ : {}^{<n-1}n \to \Lambda$ such that for every $\bar{a}, \bar{b} \in {}^{<n-1}n$ of equal length, if $N^-(a_i, b_i) \leq 1'$ for each $i < |\bar{a}|$, then $N^+(\bar{a}) = N^+(\bar{b})$.*

For two such hypernetworks N, M we write $N \equiv_i M$ to mean $N^-|_{n\setminus\{i\}} = M^-|_{n\setminus\{i\}}$ and $N^+|_{n\setminus\{i\}} = M^+|_{n\setminus\{i\}}$. $N \equiv_{ij} M$ is defined similarly.

Definition 5 *An* n-dimensional Λ-hyperbasis for $\mathcal{A}$ *is a set $\mathcal{H}$ of Λ-hypernetworks over $\mathcal{A}$ satisfying*

1. *If $a \in At(\mathcal{A})$ then there is $N \in \mathcal{H}$ with $N^-(0, 1) = a$*
2. *If $N \in \mathcal{H}$, $i, j < n$, $k \in n \setminus \{i, j\}$, and $a, b \in At(\mathcal{A})$ satisfy $N^-(i, j) \leq a; b$, then there is $M \in \mathcal{H}$ with $M \equiv_k N$, $M^-(i, k) = a$ and $M^-(k, j) = b$.*
3. *If $N, M \in \mathcal{H}$, $i, j < n$ are distinct, and $N \equiv_{ij} M$, then there is $P \in \mathcal{H}$ with $N \equiv_i P \equiv_j M$.*

This is the same as the definition of a cylindric basis, except the elements are hypernetworks instead of networks. This definition gives us the characterisation we require.

THEOREM 6 *Let $\mathcal{A}$ be a relation algebra and $4 \leq n < \omega$. Then $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if $\mathcal{A}^+$ has an n-dimensional Λ-hyperbasis, for some set Λ.*

The proof from right to left closely follows Maddux's proof for algebras with cylindric bases. The other direction involves technical arguments based on some results of Thompson on substitutions in cylindric algebras [Tho93].

2 A strict Hierarchy

Now it is quite easy to see that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$, because if $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ for some $\mathcal{C} \in \mathsf{CA}_{n+1}$, then it can be seen that $\mathfrak{Ra}(\mathcal{C}) = \mathfrak{Ra}(\mathfrak{Nr}_n(\mathcal{C}))$ and since $\mathfrak{Nr}_n(\mathcal{C}) \in \mathsf{CA}_n$, it follows that $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. A question raised in [AMN91, problem 17] is to show that each of these inclusions is strict, i.e. $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \neq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ for $3 \leq n < \omega$. For this, we define a finite relation algebra $\mathcal{A}_n$ and show that $\mathcal{A}_n \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. An alternative construction is in [HHM98]. The atoms of $\mathcal{A}_n$ are $1'$ and $a^r(p) : \ p < n-1, \ r < m_0$ where m_0 is some large integer, say $m_0 = n^n$. All atoms are self-converse. To define composition we let the set of *inconsistent triples of atoms* consist of permutations of $(1', x, y)$ $(x \neq y)$ and

$$(a^r(p), a^s(p), a^t(p)) \quad \text{any } p < n-1, \ r, s, t < m_0$$

so, if p is the 'colour' of $a^r(p)$ then monochromatic triples are not allowed. Composition is defined by this: for any $a, b \in \mathcal{A}_n$,

$$a; b = \sum\{\gamma \in At(\mathcal{A}_n) : \exists \alpha \leq a, \beta \leq b, \ (\alpha, \beta, \gamma) \text{ is not an inconsistent triple}\}$$

It is not too hard to show that the set of all n-networks forms an n-dimensional cylindric basis for $\mathcal{A}_n$ (exercise) so $\mathcal{A}_n \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, and a more complex argument proves that $\mathcal{A}_n$ does not have an $n+1$ dimensional hyperbasis and so cannot be in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. Maddux points out that this solves [HMT71, problem 2.12] negatively: $\mathbf{S}\mathfrak{Nr}_n\mathsf{CA}_{n+i} \neq \mathbf{S}\mathfrak{Nr}_n\mathsf{CA}_{n+i+1}$ for $2 < n < \omega, \ i < \omega$. A corollary (see [HHM98]) is that a certain $n+1$-variable proof system $\vdash_{n+1}$ defined in [TG87] using only one binary predicate symbol is strictly stronger than its n-variable counterpart $\vdash_n$.

We can go further and construct rather similar relation algebras $\mathcal{A}_{n,r} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for $4 \leq n, r < \omega$ and show that the second player can win the games $G^n_\omega(\mathcal{A}_{n,r})$ and $G^{n+1}_r(\mathcal{A}_{n,r})$ but not $G^{n+1}_\omega(\mathcal{A}_{n,r})$ (the game G^n_r tests membership of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ r times, see section 4 below). From this we can deduce that $\mathcal{A}_{n,r} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ (all r) but that player two has a winning strategy in $G^{n+1}_\omega(\mathcal{B})$ where $\mathcal{B}$ is a certain countable relation algebra, elementarily equivalent to a non-principal ultraproduct of the $\mathcal{A}_{n,r}$. From this, using Los' theorem, we deduce that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. A corollary of this is that for a certain standard n-variable first-order proof system $\vdash_{m,n}$ of m-variable formulas, there is no finite set of m-variable schemata whose m-variable instances, when added to $\vdash_{m,n}$ as axioms, yield $\vdash_{m,n+1}$. (Though our current version of this [HH99b] does not reduce the language to one with just a single binary predicate and equality, still we do not anticipate serious problems with this.)

A different algebraic construction was used in [HH98] to show that RA_{n+1} is not finitely axiomatisable over RA_n $(4 \leq n < \omega)$.

3 Representation Theory

Theorem 6 tells us that for any relation algebra $\mathcal{A}$ we have $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ iff $\mathcal{A}^+$ has an n-dimensional hyperbasis. But we would like to characterise when $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ purely in terms of $\mathcal{A}$, indeed we can obtain a recursive set of equations $\sigma_k : k < \omega$ such that $\mathcal{A} \models \{\sigma_k : k < \omega\} \Leftrightarrow \mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ and another recursive set of equations that characterise RA_n. We do this by first providing a relativized semantics for algebras in RA_n and $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Definition 7

- *A* relativized representation *of a relation algebra $\mathcal{A}$ is a map $h : \mathcal{A} \to \wp(X \times X)$ (for some domain X) such that, for all $a, b \in \mathcal{A}$,*

$$\begin{array}{rcl} h & is & 1-1, \quad h(a+b) = h(a) \cup h(b), \quad h(-a) = h(1) \setminus h(a) \\ h(1') & = & \{(x,x) \in h(1) : x \in X\}, \quad h(\breve{a}) = \{(y,x) \in h(1) : (x,y) \in h(a)\} \\ h(a;b) & = & \{(x,y) \in h(1) : \exists z \in X, (x,z) \in h(a) \wedge (z,y) \in h(b)\} \end{array}$$

  So this is like an ordinary representation except that all operations are relativized to some biggest binary relation $h(1) \subseteq X \times X$.

- *A* clique *C in a relativized representation h is a subset of the domain X such that for all $c, c' \in C$ we have $(c, c') \in h(1)$.*

- *An n-*square relativized representation *h has the property that for any clique C containing fewer than n points, any $c, c' \in C$ and any $r, s \in \mathcal{A}$ such that $(c, c') \in h(r;s)$, there is a point d in the domain of the relativized representation such that $C \cup \{d\}$ is a clique and $(c,d) \in h(r)$ and $(d, c') \in h(s)$.*

- *An n-*smooth relativized representation *h is an n-square representation with domain X such that if ρ is any local isomorphism on X with $|\rho| \leq n-2$ and if $\{x\} \cup dom(\rho)$ and $\{y\} \cup rng(\rho)$ are cliques ($x, y \in X$) then there is a point $x' \in X$ such that $\{x', y\} \cup rng(\rho)$ is a clique and $\rho \cup \{(x, x')\}$ is a local isomorphism.*

- *Let $\mathcal{L}_n(\mathcal{A})$ be the first-order language using only the variables $x_0, \ldots, x_{n-1}$ with atomic predicates $a(x_i, x_j)$ ($i, j < n$) for each $a \in \mathcal{A}$ and all $i, j < n$. (We use the same symbol a to stand for an element of $\mathcal{A}$ and a binary predicate symbol in this language.) Let h be a relativized representation of $\mathcal{A}$. We can evaluate a formula $\phi \in \mathcal{L}_n(\mathcal{A})$ over the cliques of h under any variable assignment $v : \{x_0, \ldots, x_{n-1}\} \to X$ such that $rng(v)$ is a clique, by*

  - $h, v \models a(x_i, x_j) \Leftrightarrow (v(x_i), v(x_j)) \in h(a)$
  - *boolean connectives behave as expected*
  - *$h, v \models \exists x_i \phi \Leftrightarrow h, v' \models \phi$ for some variable assignment v' (with $rng(v')$ a clique of h) that agrees with v except perhaps on x_i.*

*We say that h is n-*flat *if $h, v \models (\exists x_i \exists x_j \phi \leftrightarrow \exists x_j \exists x_i \phi)$ for any variable assignment v, such that $rng(v)$ is a clique, and any $\phi \in \mathcal{L}_n(\mathcal{A})$.*

These relativized representations exactly characterise the classes under consideration.

THEOREM 8 *Let $\mathcal{A}$ be any relation algebra (or more generally a non-associative algebra) and $n \geq 4$.*

1. *$\mathcal{A} \in \mathsf{RA}_n$ if and only if $\mathcal{A}$ has an n-square relativized representation.*
2. *For finite $\mathcal{A}$, $\mathcal{A} \in \mathsf{CB}_n$ if and only if $\mathcal{A}$ has an n-smooth relativized representation.*
3. *$\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if $\mathcal{A}$ has an n-flat relativized representation.*

4 Recursive axiomatisations

These relativized semantics can be used to obtain recursive axiomatisations of the classes under consideration. As an example consider n-flat relativized representations, which correspond to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. We define a game $G^n_r(\mathcal{A})$ played by two players $\forall$ and $\exists$ with r rounds in the game. Roughly, the two players build better and better approximations to an n-flat relativized representation, $\forall$ picks defects in the current approximation and $\exists$ repairs the defect if she can. We show that a winning strategy for $\exists$ in $G^n_\omega(\mathcal{A})$ is equivalent to the existence of an n-flat relativized representation. And using the fact that $\exists$ only has a finite number of choices for each of her moves we show that a winning strategy for her in each game $G^n_k(\mathcal{A})$ for $k < \omega$ is equivalent to a winning strategy for $\exists$ in $G^n_\omega(\mathcal{A})$. Finally we can find a first-order formula σ_k such that $\mathcal{A} \models \sigma_k$ iff $\exists$ has a winning strategy in $G^n_k(\mathcal{A})$ (any $k < \omega$). Thus $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \Leftrightarrow \mathcal{A} \models \{\sigma^n_k : k < \omega\}$. We can axiomatise the class RA_n in the same way.

5 Summary of relations between the classes

$$\mathsf{RRA} = \bigcap_{n<\omega} \mathsf{RA}_n = \bigcap_{n<\omega} \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \bigcap_{n<\omega} \mathbf{S}\mathsf{CB}_n$$

$$\mathsf{RA}_4 \supset \mathsf{RA}_5 \supset \ldots, \qquad \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supset \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supset \ldots$$

[HMT85, Mad92, HH98] and each of these inclusions is strict and not finitely axiomatisable.

$$\mathsf{RA}_n \supset \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \supseteq \mathbf{S}\mathsf{CB}_n, \quad \text{for each } n \text{ with } 5 \leq n < \omega$$

(whether the second inclusion is strict or not remains open.) For a finite relation algebra $\mathcal{A}$ we have

$$\mathcal{A} \in \bigcap_{n<\omega} \mathsf{CB}_n \Leftrightarrow \mathcal{A} \text{ has a homogeneous representation}$$

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