THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

RELATION ALGEBRA REDUCTS OF CYLINDRIC ALGEBRAS AND COMPLETE REPRESENTATIONS

ROBIN HIRSCH

Abstract. We show, for any ordinal $\gamma \geq 3$, that the class $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary and has a recursively enumerable elementary theory. $\mathbf{S_c}K$ denotes the class of strong subalgebras of members of the class K. We devise games, F^n $(3 \leq n \leq \omega)$, G, H, and show, for an atomic relation algebra $\mathcal{A}$ with countably many atoms, that

$$\begin{array}{rcl} \exists \text{ has a winning strategy in } F^\omega(\mathsf{At}(\mathcal{A})) & \Leftrightarrow & \mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega \\ \exists \text{ has a winning strategy in } F^n(\mathsf{At}(\mathcal{A})) & \Leftarrow & \mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n \\ \exists \text{ has a winning strategy in } G(\mathsf{At}(\mathcal{A})) & \Leftarrow & \mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\omega \\ \exists \text{ has a winning strategy in } H(\mathsf{At}(\mathcal{A})) & \Rightarrow & \mathcal{A} \in \mathfrak{Ra}\mathbf{RCA}_\omega \end{array}$$

for $3 \leq n < \omega$. We use these games to show, for $\gamma \geq 5$ and any class K of relation algebras satisfying

$$\mathfrak{Ra}\mathbf{RCA}_\gamma \subseteq K \subseteq \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5,$$

that K is not closed under subalgebras and is not elementary. For infinite γ, the inclusion $\mathfrak{Ra}\mathbf{CA}_\gamma \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ is strict.

For infinite γ and for a countable relation algebra $\mathcal{A}$ we show that $\mathcal{A}$ has a complete representation if and only if $\mathcal{A}$ is atomic and $\exists$ has a winning strategy in $F(\mathsf{At}(\mathcal{A}))$ if and only if $\mathcal{A}$ is atomic and $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$.

§1. Introduction. There are two kinds of algebras of relations largely due to Alfred Tarski: *relation algebra* (although the history of relation algebra goes much further back [10, 3]) and n-dimensional *cylindric algebra*, for various n. Relation algebras are closely related to fields of binary relations and n-dimensional cylindric algebras are based on fields of n-ary relations. Both types of algebra have been studied intensively and are widely used.

For any n-dimensional cylindric algebra $\mathcal{C}$ $(n \geq 3)$ the *relation algebra reduct* $\mathfrak{Ra}(\mathcal{C})$ can be defined by taking the two dimensional elements of $\mathcal{C}$ and using the third dimension to define converse and composition. The relation algebra reduct is the key tool for connecting cylindric algebras to relation algebras. Quite a lot is

---

1991 *Mathematics Subject Classification.* 03G15.

Key words and phrases. Relation Algebra, Cylindric Algebra, Complete Representation.

Thanks to Tarek Sayed Ahmed for useful discussions and for suggesting this line of research, to the anonymous referee for suggesting improvements, to Ian Hodkinson for pointing out serious errors in a draft of this paper and to Hajnal Andréka and Istvan Németi for a significant contribution to these results. Research partially supported by the London Mathematical Society, ref. 5506.

0022-4812/00/0000-0000/$00.00

known about the class of subalgebras of relation algebra reducts of n-dimensional cylindric algebras: the class is a canonical variety [6, proposition 5.48], for example. The neat embedding theorem [5, 5.3.13, 5.3.16] says that a relation algebra is representable if and only if it is a subalgebra of a relation algebra reduct of an ω-dimensional cylindric algebra.

Much less is known about the class of relation algebra reducts of n-dimensional cylindric algebras (here we do not take subalgebras). A relation algebra in this class is more directly connected with an n-dimensional cylindric algebra. We can at least show that the class is pseudo-elementary, even for infinite n, and that the elementary theory of the class is recursively enumerable provided n is not uncountably infinite. But Monk proved that $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under subalgebras for any $n \geq 5$ (including infinite n) [12], Maddux and Németi independently proved that $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under subalgebras for $n \geq 4$ [8, 14] and later the same was proved for $n = 3$ [19, 15]. It is also known that the related classes $\mathfrak{Nr}_m\mathbf{CA}_n$ of neat reducts of cylindric algebras, for $2 \leq m < n$, are not closed under subalgebras [13, 18]. In this article we will show, for $n \geq 5$, that $\mathfrak{Ra}\mathbf{CA}_n$ is not closed under elementary equivalence. [A corresponding result was already established for neat reducts of cylindric algebras [17], but the construction used there does not seem to work for relation algebra reducts.]

A complete representation of a relation algebra is a representation where arbitrary suprema are preserved in the representation, wherever they exist in the algebra. There are connections between relation algebra reducts and **CRA**, the class of *completely representable relation algebras*, although the relation algebra reduct is algebraically defined whereas **CRA** has a semantic definition. **CRA** is known to be pseudo-elementary but not closed under elementary equivalence [6, theorem 17.6], so it cannot be defined by any first-order theory. The same negative properties hold for $\mathfrak{Ra}\mathbf{CA}_n$ ($n \geq 5$).

It is known that a completely representable relation algebra must be atomic and representable (of course), but representability and atomicity do not suffice to prove that a relation algebra is completely representable (see [7, examples 23, p. 154 ff.] for an atomic, representable relation algebra with no complete representation). This makes you wonder what else is needed to ensure that a relation algebra is completely representable. A transfinite game can be defined that does characterise **CRA**, but not everyone likes transfinite games. In this paper we will demonstrate another connection between the class $\mathfrak{Ra}\mathbf{CA}_\omega$ and the class **CRA**, at least for countable algebras, by proving a complete version of the neat embedding theorem: a countable relation algebra has a complete representation if and only if it is atomic and it is a strong subalgebra of the relation algebra reduct of an ω-dimensional cylindric algebra. (An algebra is a strong subalgebra of another if it embeds into it in such a way that arbitrary suprema are preserved by the embedding, whenever they exist in the algebra.) This can be thought of as an algebraic characterisation of **CRA**, at least for the countable case. Whether this characterisation works for uncountable algebras remains unknown.

In the next section we run through the necessary prerequisites from algebraic logic, including the definition of the relation algebra reduct, the relation algebra atom structure, the complete representation, the strong subalgebra etc. and some basic lemmas. In section 3 we prove that $\mathfrak{Ra}\mathbf{CA}_n$ is pseudo-elementary. In section 4 we define three games played over a relation algebra atom structure and use these games to determine if a relation algebra is a strong subalgebra of a relation algebra reduct, as well as some other results concerning other classes. We use one of these games to re-prove a result of Sayed-Ahmed: a countable relation algebra is completely representable iff it is atomic and a strong subalgebra of an ω-dimensional cylindric algebra. In section 5 we construct a particular atom structure (sometimes called a rainbow algebra atom structure) and use this in section 6 to demonstrate that a range of classes, including $\mathfrak{Ra}\mathbf{CA}_n$ for $n \geq 5$, are not closed under elementary subalgebras. The paper includes a series of open problems.

§2. Preliminaries.

2.1. General. $\wp(X)$ is the *power set* of X. ${}^{\gamma}X$ denotes the set of all functions from the ordinal γ to the set X. Equivalently, we may consider $\bar{x} \in {}^{\gamma}X$ as a sequence $(x_0, x_1, \ldots) = (x_i : i < \gamma)$, it will be implicit that the i'th element of $\bar{x}$ (or equivalently $\bar{x}(i)$) is x_i. We write ${}^{<\omega}X$ for $\bigcup_{n<\omega} {}^{n}X$. For $f \in {}^{\gamma}X$, $i < \gamma$ and $x \in X$ we write $f[i/x]$ for the function which is identical to f except that $f[i/x](i) = x$. If $\theta : X \to Y$ is any function and $\bar{x} \in {}^{\gamma}X$ then $\theta(\bar{x}) \in {}^{\gamma}Y$ is defined by $(\theta(\bar{x}))(i) = \theta(\bar{x}(i))$. Fix some $n \in \omega$. For $i, j < n$ we write $[i/j], Id_{-i}$ for the functions $n \to n$ defined by

$$[i/j](k) = \begin{cases} k & \text{if } k \neq i,\ k < n \\ j & \text{if } k = i \end{cases}$$

$$Id_{-i} = \{(k, k) : k \in n \setminus \{i\}\}$$

Note that all these definitions depend implicitly on n, the first one is a total function and the second one is a partial function.

2.2. Boolean algebra with operators. We assume some knowledge about cylindric algebras and relation algebras [9, 4, 5, 6]. For any ordinal γ, $\mathbf{CA}_\gamma$ denotes the class of γ-dimensional cylindric algebras [4, definition 1.1.1]. If Γ is a finite subset of γ we write $\mathsf{c}_{(\Gamma)}x$ for $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} x$, where $i_0, \ldots, i_k$ is an arbitrary enumeration of Γ. Since $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$ is one of the cylindric algebra axioms, the order of the enumeration makes no difference.

RA is the class of all relation algebras. One of the defining axioms for relation algebras is the *Peircean law* which states

$$a;b \,.\, c^{\smile} = 0 \iff b;c \,.\, a^{\smile} = 0$$

$\mathbf{CA}_\gamma, \mathbf{RA}$ are examples of classes of boolean algebras with (completely additive) operators. A boolean algebra with operators is *simple* if any homomorphism defined on the algebra is an isomorphism or has a degenerate image. For similar boolean algebras with operators $\mathcal{A}, \mathcal{B}$ we write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{B}$ (often we will identify $\mathcal{A}$ with its isomorphic image). $\prod_{i \in I} \mathcal{A}_i$ denotes the direct product of the similar boolean algebras with operators $(\mathcal{A}_i : i \in I)$. If K is a class of similar boolean algebras with operators then $\mathbf{H}K, \mathbf{S}K, \mathbf{P}K$ denote the classes of homomorphic images, subalgebras, direct products of members of K, respectively.

2.3. Representations. A relation algebra $\mathcal{A}$ is *representable* if there is a structure $\mathcal{M}$ in which each element $a \in \mathcal{A}$ is interpretted as a binary relation $a^{\mathcal{M}}$ over the domain of $\mathcal{M}$ faithfully (i.e. $a \neq b \in \mathcal{A} \to a^{\mathcal{M}} \neq b^{\mathcal{M}}$) so as to preserve all the relation algebra operators, i.e. $0^{\mathcal{M}} = \emptyset$, $(a+b)^{\mathcal{M}} = a^{\mathcal{M}} \cup b^{\mathcal{M}}$, $(-a)^{\mathcal{M}} = 1^{\mathcal{M}} \setminus a^{\mathcal{M}}$, $1'^{\mathcal{M}} = \{(m,m) : m \in \mathcal{M}\}$, $a^{\smile\mathcal{M}} = \{(n,m) : (m,n) \in a^{\mathcal{M}}\}$ and $(a;b)^{\mathcal{M}} = a^{\mathcal{M}}|b^{\mathcal{M}}$. It follows, using $1^{\smile} = 1,\ 1;1 = 1$, that $1^{\mathcal{M}}$ is an equivalence relation over the domain of $\mathcal{M}$, if $\mathcal{M}$ is a representation. **RRA** is the class of all *representable relation algebras*. A *partial isomorphism* θ of the representation $\mathcal{M}$ of $\mathcal{A}$ is a partial map on the base of $\mathcal{M}$ such that for all $i,j \in \mathsf{dom}(\theta)$ and all $a \in \mathcal{A}$ we have $(i,j) \in a^{\mathcal{M}} \Leftrightarrow (\theta(i), \theta(j)) \in a^{\mathcal{M}}$.

The cylindric set algebra of dimension γ on a base D is $(\wp({}^{\gamma}D), \emptyset, {}^{\gamma}D, \cup, \setminus, D_{ij}, C_i : i,j < \gamma)$, where $D_{ij} = \{f \in {}^{\gamma}D : f(i) = f(j)\}$ and for $S \subseteq {}^{\gamma}D$, $C_i(S) = \{f \in {}^{\gamma}D : \exists d \in D, f[i/d] \in S\}$. A cylindric algebra $\mathcal{C} \in \mathbf{CA}_\gamma$ is *representable* if it is isomorphic to a subalgebra of a product of cylindric set algebras of dimension γ (see [4, page 171] and [5, definition 3.1.1], but note that in the latter definition instead of closing under direct products the equivalent notion of a *generalised cylindric field of sets* is used). We write $\mathbf{RCA}_\gamma$ for the class of representable γ-dimensional cylindric algebras.

PROPOSITION 1 ([3.1.108]HMT2). *$\mathbf{RCA}_\gamma$ is a variety, hence closed under subalgebras, direct products and homomorphic images.*

2.4. Locally finite cylindric algebras and weak set algebras. Define the *dimension set* of x, for $x \in \mathcal{C} \in \mathbf{CA}_\gamma$, by $\Delta(x) = \{i < \gamma : \mathsf{c}_i x \neq x\}$. $\mathcal{C}$ is said to be *locally finite* if $|\Delta(x)|$ is finite, for all $x \in \mathcal{C}$. If γ is finite then every γ-dimensional cylindric algebra is locally finite.

Let $\gamma \geq \omega$ and let D be a set. Fix $f_0 \in {}^{\gamma}D$ and let $W = \{f \in {}^{\gamma}D : \{i : f(i) \neq f_0(i)\}$ is finite$\}$. The γ-dimensional cylindric algebra $(\wp(W), \emptyset, W, D_{ij}, C_i : i,j < \omega)$ where D_{ij}, C_i are defined as for cylindric set algebras, but relativized to W, is called a *weak cylindric set algebra* of dimension γ [5, definition 3.1.2].

PROPOSITION 2 ([5, 3.1.102]). *Every weak cylindric set algebra is representable.*

2.5. Atom structures. An atom of a boolean algebra with operators is a minimal non-zero element. A boolean algebra with operators is atomic if every non-zero element is above some atom. If $\mathcal{A}$ is an atomic relation algebra, the *relation algebra atom structure* $\mathsf{At}(\mathcal{A}) = (A, Id, ^{\smile}, C)$ consists of the set A of atoms of $\mathcal{A}$, the set Id of atoms below the identity of $\mathcal{A}$, the function $^{\smile}$ that takes an atom to its converse, and the list C of consistent triples of atoms (a,b,c) — those where $a;b \geq c$. Since the relation algebra operators are completely additive, the atom structure suffices to define the operators over arbitrary elements of $\mathcal{A}$. The following properties always hold in a relation algebra atom structure [6, lemma 3.24]. For all $x,y,z,t \in A$,

- $x = y$ iff there is $e \in Id$ such that $(x,e,y) \in C$.
- If $(x,y,z) \in C$ then $(\breve{x}, z, y), (\breve{y}, \breve{x}, \breve{z}) \in C$.
- $(\exists u \in A\ ((x,y,u),(u,z,t) \in C)) \Leftrightarrow (\exists v \in A\ ((y,z,v),(x,v,t) \in C))$.

Conversely, if $\alpha = (A, Id, ^{\smile}, C)$ has the type of a relation algebra atom structure we can define the *complex algebra* of α, which has the type of a relation algebra, by $\mathfrak{Cm}(\alpha) = (\wp(A), \emptyset, A, \cup, \setminus, Id, ^\smile, ;)$, where the converse operator $^\smile$ is extended from atoms to sets of atoms by $S^\smile = \{s^\smile : s \in S\}$, and composition of sets of atoms is defined by $S;T = \{a \in \alpha : \exists s \in S,\ \exists t \in T,\ (s,t,a) \in C\}$, where $S, T \subseteq A$. It turns out that the three conditions above are not only necessary for an atom structure to arise from the atoms of a relation algebra, but they are also sufficient — the complex algebra of such an structure will be a relation algebra.

An atomic relation algebra $\mathcal{A}$ is simple if and only if $\mathsf{At}(\mathcal{A}) = (A, Id, ^\smile, C) \models \forall a, b \in A, \exists c, d, f \in A,\ (a,c,d), (f,d,b) \in C$. (Rewriting this with the composition operator instead of the list of consistent triples we get $\forall a, b \in A, \exists c, f \in A,\ f;a;c \geq b$ and this is equivalent to the more familiar statement $x \neq 0 \to 1;x;1 = 1$.)

2.6. Substitutions. Let γ be an ordinal and $\mathcal{C} \in \mathbf{CA}_\gamma$, $i, j < \gamma$ and $x \in \mathcal{C}$. Define

$$s^i_j x = \begin{cases} x & \text{if } i = j \\ \mathsf{c}_i(d_{ij} \cdot x) & \text{otherwise} \end{cases}$$

FACT 3. *Let $\mathcal{C} \in \mathbf{CA}_\gamma$ (some $\gamma \geq 3$), $x, y \in \mathcal{C}$, $i, j, k, l < \gamma$.*

1. *s^i_j is a completely additive endomorphism of $\mathcal{C}$ [4, 1.5.3].*
2. *If $i \neq j$ then $\mathsf{c}_i s^i_j x = s^i_j x$ (from the definition of s^i_j and the cylindric algebra axiom $\mathsf{c}_i \mathsf{c}_i y = \mathsf{c}_i y$).*
3. *If $x \cdot \mathsf{c}_i y = 0$ then $y \cdot \mathsf{c}_i x = 0$ [4, 1.2.5].*
4. *If $k \notin \{i, j\}$ then $\mathsf{c}_k s^i_j x = s^i_j \mathsf{c}_k x$ [4, 1.5.8(ii)].*
5. *$s^i_j \mathsf{c}_i x = \mathsf{c}_i x$ [4, 1.5.8(i)]*
6. *If $i \neq j$ then $\mathsf{c}_i s^i_j x = s^i_j x$ [4, 1.5.9(ii)]*
7. *$\mathsf{c}_j s^i_j x = \mathsf{c}_i s^j_i x$ [4, 1.5.9(i)]*
8. *If $i \neq k$ then $s^i_j s^i_k x = s^i_k x$ [4, 1.5.10(i)]*
9. *$s^i_j s^j_i x = s^i_j x$ [4, 1.5.10(v)]*

DEFINITION 4. *Let $n \geq 3$ be an ordinal and $i, j < n$. We define a string of substitutions s_{ij} that 'move dimensions $0, 1$ to i, j' as follows.*

$$s_{ij} = \begin{cases} s^0_i s^1_j & \textit{if } j \neq 0 \\ s^1_0 s^0_i & \textit{if } j = 0,\ i \neq 1 \\ s^2_0 s^0_1 s^1_2 & \textit{if } j = 0,\ i = 1 \end{cases}$$

[In the notation of [6, definition 5.23, lemma 13.29], $\widehat{s_{ij}}$ is the function $n \to n$ taking $0, 1$ to i, j, respectively, and fixing all $k \in n \setminus \{i, j\}$.]

2.7. Neat reducts and relation algebra reducts.

DEFINITION 5. *Let $\lambda \leq \mu$ be ordinals and let $\mathcal{C} \in \mathbf{CA}_\mu$. The* neat λ-reduct *$\mathfrak{Nr}_\lambda(\mathcal{C}) \in \mathbf{CA}_\lambda$ has as its domain $\{x \in \mathcal{C} : \lambda \leq i < \beta \to \mathsf{c}_i x = x\}$ and all the operators are inherited from $\mathcal{C}$. $\mathfrak{Nr}_\lambda \mathbf{CA}_\mu$ denotes the class $\{\mathfrak{Nr}_\lambda(\mathcal{C}) : \mathcal{C} \in \mathbf{CA}_\mu\}$.*

Let $\lambda \geq 3$ and let $\mathcal{C} \in \mathbf{CA}_\lambda$. The relation algebra reduct $\mathfrak{Ra}(\mathcal{C})$ *is the algebra of the type of relation algebras whose domain is the same as that of $\mathfrak{Nr}_2(\mathcal{C})$, with boolean operators inherited from $\mathcal{C}$ and with the relation algebra operators defined* *by*

$$\begin{aligned} 1' &= d_{01} \\ a^{\smile} &= s_0^2 s_1^0 s_2^1 a \\ a;b &= \mathsf{c}_2(s_2^1 a \cdot s_2^0 b) \end{aligned}$$

for $a, b \in \mathfrak{Nr}_2(\mathcal{C})$*. Observe, in the notation of definition 4, that* $a^{\smile} = s_{10}a$ *and* $a;b = \mathsf{c}_2(s_{02}a \cdot s_{21}b)$*. For* $\lambda \geq 4$, $\mathfrak{Ra}(\mathcal{C})$ *is a relation algebra [5, 5.3.8].* $\mathfrak{Ra}\mathbf{CA}_\lambda$ *denotes the class* $\{\mathfrak{Ra}(\mathcal{C}) : \mathcal{C} \in \mathbf{CA}_\lambda\}$*.*

LEMMA 6. *Let* $2 \leq \lambda \leq \mu \leq \gamma$ *and* $3 \leq \mu$*.*

- $\mathfrak{Nr}_\lambda(\mathfrak{Nr}_\mu \mathbf{CA}_\gamma) = \mathfrak{Nr}_\lambda(\mathbf{CA}_\gamma)$ *and* $\mathfrak{Ra}(\mathfrak{Nr}_\mu \mathbf{CA}_\gamma) = \mathfrak{Ra}(\mathbf{CA}_\gamma)$*.*
- $\mathfrak{Nr}_\lambda(\mathbf{CA}_\gamma) \subseteq \mathfrak{Nr}_\lambda(\mathbf{CA}_\mu)$ *and* $\mathfrak{Ra}(\mathbf{CA}_\gamma) \subseteq \mathfrak{Ra}(\mathbf{CA}_\mu)$*.*

The neat embedding theorem was first proved in the closely related setting of neat reducts of cylindric algebras [12, theorems 4.1,9.11,9.12], see [11, p.112] for a similar result with relational bases.

THEOREM 7 (Neat embedding theorem, Henkin, Maddux, Monk). *Let* $\gamma \geq \omega$*.*

$$\mathbf{RRA} = \bigcap_{n<\omega} \mathbf{S}\mathfrak{Ra}\mathbf{CA}_n = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_\gamma$$

The theorem and a proof can be found in [6, proposition 13.48]. Thus $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_\gamma$ is constant, for $\gamma \geq \omega$. By contrast, it is strictly decreasing for $3 \leq \gamma < \omega$ [6, theorem 15.1] and therefore $\mathfrak{Ra}\mathbf{CA}_\gamma$ strictly decreases as γ increases, for finite γ. We might ask what happens to $\mathfrak{Ra}\mathbf{CA}_\gamma$ as $\gamma \geq \omega$ increases. We thank Andréka and Németi for this result[1].

THEOREM 8 (Andréka and Németi). *For* $\gamma \geq \omega$ *we have* $\mathfrak{Ra}\mathbf{CA}_\gamma = \mathfrak{Ra}\mathbf{CA}_\omega$*.*

PROOF. The inclusion $\mathfrak{Ra}\mathbf{CA}_\gamma \subseteq \mathfrak{Ra}\mathbf{CA}_\omega$ is lemma 6. Conversely, let $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\omega$, say $\mathcal{A} = \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. We have to show that $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\gamma$. Let $\mathcal{C}'$ be the subalgebra of $\mathcal{C}$ generated (using the cylindric algebra operators) by $\mathcal{A}$. Then $\mathcal{A} = \mathfrak{Ra}\mathcal{C}'$ and $\mathcal{C}'$ is a locally finite, ω-dimensional cylindric algebra. By [4, 2.6.74(ii)], every locally finite ω-dimensional cylindric algebra is the neat reduct of a locally finite γ-dimensional cylindric algebra, so $\mathcal{C}' = \mathfrak{Nr}_\omega\mathcal{D}$ for some locally finite $\mathcal{D} \in \mathbf{CA}_\gamma$. Hence $\mathcal{A} = \mathfrak{Ra}(\mathcal{C}') = \mathfrak{Ra}(\mathfrak{Nr}_\omega\mathcal{D}) = \mathfrak{Ra}(\mathcal{D})$ (by lemma 6), so $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\gamma$, as required. ⊣

PROBLEM 9. *That still leaves one case: is* $\mathfrak{Ra}\mathbf{CA}_\omega = \bigcap_{n<\omega} \mathfrak{Ra}\mathbf{CA}_n$*? Andréka and Németi have proved that every relation algebra in* $\bigcap_{n<\omega} \mathfrak{Ra}\mathbf{CA}_n$ *has an elementary subalgebra in* $\mathfrak{Ra}\mathbf{CA}_\omega$*, but the question as stated remains open.*

THEOREM 10. $\mathfrak{Ra}\mathbf{CA}_\omega = \mathfrak{Ra}\mathbf{RCA}_\omega$*.*

PROOF. The inclusion $\mathfrak{Ra}\mathbf{CA}_\omega \supseteq \mathfrak{Ra}\mathbf{RCA}_\omega$ is trivial. To prove the other inclusion, let $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\omega$, say $\mathcal{A} = \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. Let $\mathcal{C}'$ be the subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$. Then $\mathcal{C}'$ is locally finite and $\mathcal{A} = \mathfrak{Ra}(\mathcal{C}')$. By proposition 2, $\mathcal{C}' \in \mathbf{RCA}_\omega$ so $\mathcal{A} \in \mathfrak{Ra}\mathbf{RCA}_\omega$. ⊣

[1] Personal communication to the author.

PROPOSITION 11 ([6, 13.31]). *Let* $4 \leq \gamma$, $\mathcal{C} \in \mathbf{CA}_\gamma$, $i, j, k < \gamma$, $k \notin \{i, j\}$ *and* $\alpha, \beta, \gamma \in \mathfrak{Ra}(\mathcal{C})$.

$$s_{ij}(\alpha;\beta) = \mathsf{c}_k(s_{ik}\alpha \,.\, s_{kj}\beta)$$

For $\gamma \geq 3$ it is known that $\mathfrak{Ra}\mathbf{CA}_\gamma$ is not closed under subalgebra. It is easy to check that it is closed under direct products: $\prod_{i\in I} \mathfrak{Ra}\mathcal{C}_i \cong \mathfrak{Ra}\prod_{i\in I}\mathcal{C}_i$, where $\mathcal{C}_i \in \mathbf{CA}_\gamma$. Andréka and Németi proved[2] that the class is also closed under homomorphic images.

THEOREM 12 (Andréka and Németi). *Let* $\gamma \geq 3$. $\mathbf{H}\mathfrak{Ra}\mathbf{CA}_\gamma = \mathfrak{Ra}\mathbf{CA}_\gamma$.

PROOF. See [13, theorem 1(i)]. Let $\mathcal{A} = \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\gamma$ and let $h : \mathcal{A} \to \mathcal{B}$ be a relation algebra homomorphism. We have to show that $\mathcal{B} \in \mathfrak{Ra}\mathbf{CA}_\gamma$. We can assume that the algebra generated by $\mathcal{A}$ in $\mathcal{C}$ using the cylindric algebra operations is the whole of $\mathcal{C}$ (else replace $\mathcal{C}$ by the subalgebra generated by $\mathcal{A}$). Then $\mathcal{C}$ is locally finite. Let I be the kernel of h, an ideal of $\mathcal{A}$. The domain of $\mathcal{A}$ is the same as the domain of $\mathfrak{Nr}_2(\mathcal{C})$ and I is also an ideal of $\mathfrak{Nr}_2(\mathcal{C})$. Hence [4, 2.3.8] I extends to an ideal I' of $\mathcal{C}$ and $I = I' \cap \mathcal{A}$. Now $\mathcal{B} \cong \mathcal{A}/I \cong \mathcal{A}/I' \subseteq \mathfrak{Ra}(\mathcal{C}/I')$. We have to show that the inclusion is not proper.

Let $x \in \mathfrak{Ra}(\mathcal{C}/I')$ be arbitrary, say $x = c/I'$ for some $c \in \mathcal{C}$, we will show that $x = a/I'$ for some $a \in \mathcal{A}$. Since $\mathcal{C}$ is locally finite, the dimension set $\Delta(c)$ is finite. Let $\Gamma = \Delta(c) \setminus \{0, 1\}$ (a finite subset of γ) and let $c' = \mathsf{c}_{(\Gamma)}c$. Now $c' \in \mathcal{A}$ and $x = c/I' = c'/I'$, since $x \in \mathfrak{Nr}_2(\mathcal{C}/I')$. Thus $\mathcal{B} \cong \mathfrak{Ra}(\mathcal{C}/I')$ as required. ⊣

2.8. Complete representations and strong embeddings.

DEFINITION 13. *For any boolean algebras with operators* $\mathcal{A} \subseteq \mathcal{B}$ *we say "*$\mathcal{A}$ *is a strong subalgebra of* $\mathcal{B}$*" and we write* $\mathcal{A} \subseteq_c \mathcal{B}$ *if whenever the supremum* $\sum^{\mathcal{A}} X$ *exists in* $\mathcal{A}$ *then the supremum exists in* $\mathcal{B}$ *and* $\sum^{\mathcal{A}} X = \sum^{\mathcal{B}} X$. *Examples of cases where* $\mathcal{A} \subseteq_c \mathcal{B}$ *include the case where* $\mathcal{A}$ *is a finite subalgebra of* $\mathcal{B}$ *and the case where* $\mathcal{B}$ *is the MacNeille completion of* $\mathcal{A}$. *We write* $\mathbf{S_c}K$ *for* $\{\mathcal{A} : \exists \mathcal{B} \in K,\ \mathcal{A} \subseteq_c \mathcal{B}\}$.

A representation $\mathcal{M}$ *with base* D *of a boolean algebra* $\mathcal{B}$ *interprets each element* $b \in \mathcal{B}$ *as a distinct subset of* D *such that* $0^{\mathcal{M}} = \emptyset$, $1^{\mathcal{M}} = D$, $(-b)^{\mathcal{M}} = D \setminus b^{\mathcal{M}}$ *and* $(b + b')^{\mathcal{M}} = b^{\mathcal{M}} \cup b'^{\mathcal{M}}$, *for all* $b, b' \in \mathcal{B}$. *So* $\mathcal{M}$ *is a representation of* $\mathcal{B}$ *with base* D *iff the map* $b \mapsto b^{\mathcal{M}}$ *is an embedding:* $\mathcal{B} \subseteq \mathfrak{P}(\mathcal{M}) =_{def.} (\wp(D), D, \emptyset, \cup, \setminus)$. *If* $\mathcal{B} \subseteq_c \mathfrak{P}(\mathcal{M})$ *then we say that* $\mathcal{M}$ *is a* complete boolean representation $\mathcal{B}$. *Equivalently, for any subset* X *of the universe of* $\mathcal{B}$*, if the supremum* $\sum^{\mathcal{B}} X$ *exists in* $\mathcal{B}$ *then* $(\sum^{\mathcal{B}} X)^{\mathcal{M}} = \bigcup_{b\in X} b^{\mathcal{M}}$.

A relation algebra $\mathcal{A}$ *is completely representable if there is a representation* $\mathcal{M}$ *of* $\mathcal{A}$ *such that the reduct of* $\mathcal{M}$ *to the boolean part of the signature is a complete boolean representation of the boolean part of* $\mathcal{A}$. **CRA** *denotes the class of all completely representable relation algebras.*

It is easy to show, using the De Morgan laws, that infima are also preserved by strong subalgebras: if $\mathcal{A} \subseteq_c \mathcal{B}$ then whenever $\prod^{\mathcal{A}} X$ exists then $\prod^{\mathcal{B}} X = \prod^{\mathcal{A}} X$

[2] Personal communication to the author.

also exists. Similarly, if $\mathcal{M}$ is a complete representation of $\mathcal{B}$ then whenever $\prod^{\mathcal{B}} X$ exists $(\prod^{\mathcal{B}} X)^{\mathcal{M}} = \bigcap_{b\in X} b^{\mathcal{M}}$.

The next few lemmas are are about boolean algebras but apply equally to any boolean algebras with operators.

LEMMA 14 ([6, lemma 2.16]). *If $\mathcal{B}$ is an atomic boolean algebra and $\mathcal{A} \subseteq_c \mathcal{B}$ then $\mathcal{A}$ is atomic too.*

PROOF. Suppose that $\mathcal{B}$ is atomic but $\mathcal{A}$ is not. Then there is $a \in \mathcal{A}$ with $a \neq 0$ with no atom of $\mathcal{A}$ below a. But $\mathcal{B} \supseteq \mathcal{A}$ is atomic so there is $\beta \in \mathsf{At}(\mathcal{B})$ with $\beta \leq a$. Let $F = \{r \in \mathcal{A} : \beta \leq r\}$. We have $a \in F$. Then $\prod^{\mathcal{A}} F = 0$ but $\beta \leq \prod^{\mathcal{B}} F$. Hence $\mathcal{A} \not\subseteq_c \mathcal{B}$. ⊣

LEMMA 15. *Let $\mathcal{A} \subseteq \mathcal{B}$ be boolean algebras and let $\mathcal{A}$ be atomic. $\mathcal{A} \subseteq_c \mathcal{B}$ if and only if for all $b \in \mathcal{B} \setminus \{0\}$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $a.b \neq 0$.*

PROOF. If $b \in \mathcal{B} \setminus \{0\}$ and for all $a \in \mathsf{At}(\mathcal{A})$ $a.b = 0$ then $\sum^{\mathcal{A}} \mathsf{At}(\mathcal{A}) = 1$ but $\sum^{\mathcal{B}} \mathsf{At}\mathcal{A} \leq 1 - b$ if it exists, so $\mathcal{A} \not\subseteq_c \mathcal{B}$.

Conversely, if $\mathcal{A} \not\subseteq_c \mathcal{B}$ then there is a set $S \subseteq \mathcal{A}$ such that $\sum^{\mathcal{A}} S$ exists but there is $b \in \mathcal{B}$ with $b \not\geq \sum^{\mathcal{A}} S$ and b is an upper bound for S. But then, $b' = \sum^{\mathcal{A}} S - b \neq 0$ must be disjoint from all atoms of $\mathcal{A}$. ⊣

LEMMA 16 ([6, theorem 2.21]). *Let $\mathcal{A}$ be a boolean algebra and let $\mathcal{M}$ be a representation of $\mathcal{A}$. The following are equivalent.*

- *$\mathcal{M}$ is a* complete *representation of $\mathcal{A}$.*
- *$\mathcal{M}$ is an* atomic *representation of $\mathcal{A}$ i.e. $1^{\mathcal{M}} = \bigcup\{\beta^{\mathcal{M}} : \beta \in \mathsf{At}(\mathcal{A})\}$.*

PROOF. Let $\mathcal{M}$ be any representation of $\mathcal{A}$, i.e. $\mathcal{A} \subseteq \mathfrak{P}(\mathcal{M}) = (\wp(1^{\mathcal{M}}), 1^{\mathcal{M}}, \emptyset, \cup, \setminus)$. By definition 13,

$$\mathcal{M} \text{ is a complete representation of } \mathcal{A} \iff \mathcal{A} \subseteq_c \mathfrak{P}(\mathcal{M}) \tag{1}$$

If $\mathcal{M}$ is a complete representation of $\mathcal{A}$ then, since $\mathfrak{P}(\mathcal{M})$ is atomic and by lemma 14, $\mathcal{A}$ is also atomic. By lemma 15, for all $b \in \mathfrak{P}(\mathcal{M}) \setminus \{0\}$ there is $a \in \mathsf{At}(\mathcal{A})$ with $a.b \neq 0$. It follows that $1^{\mathcal{M}} = \bigcup_{a\in\mathsf{At}(\mathcal{A})} a^{\mathcal{M}}$, so $\mathcal{M}$ is an atomic representation.

Conversely, if $\mathcal{M}$ is an atomic representation of $\mathcal{A}$, i.e. $1^{\mathcal{M}} = \bigcup_{a\in\mathsf{At}(\mathcal{A})} a^{\mathcal{M}}$, then $\mathcal{A}$ must be atomic. By lemma 15, $\mathcal{A} \subseteq_c \mathfrak{P}(\mathcal{M})$, hence $\mathcal{M}$ is a complete representation of $\mathcal{A}$, by (1).

⊣

LEMMA 17. *If $\mathcal{M}$ is a complete representation of $\mathcal{B}$ and $\mathcal{A} \subseteq_c \mathcal{B}$ then $\mathcal{M}$ induces a complete representation of $\mathcal{A}$.*

PROOF. Suppose for contradiction that $\mathcal{M}$ is a complete representation of $\mathcal{B}$ but it does not induce a complete representation of $\mathcal{A}$. By lemma 16 there is $m \in \mathcal{M}$ with $m \in 1^{\mathcal{M}}$ but $m \notin \alpha^{\mathcal{M}}$ for all $\alpha \in \mathsf{At}(\mathcal{A})$. By the same lemma, $m \in \beta^{\mathcal{M}}$ for some $\beta \in \mathsf{At}(\mathcal{B})$. Let $F = \{a \in \mathcal{A} : \beta \leq a\}$. Then $\prod^{\mathcal{A}} F = 0$ but $\prod^{\mathcal{B}} F \geq \beta$. This contradicts $\mathcal{A} \subseteq_c \mathcal{B}$.

⊣

Now we apply this to relation algebra. **FRA** denotes the class of full relation algebras — the closure under isomorphism of the class of relation algebras of the form $\mathfrak{Re}(D) =_{\text{def.}} (\wp(D \times D), D \times D, \emptyset, \cup, \setminus, Id_D, \smile, ;)$ for some domain D.

THEOREM 18. $\mathbf{CRA} = \mathbf{S_cP}(\mathbf{FRA})$.

PROOF. Let $\mathcal{A} \in \mathbf{CRA}$ and let $\mathcal{M}$ be a complete representation of $\mathcal{A}$. From definition 13, we have $\mathcal{A} \subseteq_c \mathfrak{P}(\mathcal{M})$. $1^{\mathcal{M}}$ is an equivalence relation over the base of $\mathcal{M}$, as we saw earlier, and $\mathfrak{P}(\mathcal{M}) \cong \prod_{\text{equiv. classes } D} \mathfrak{Re}(D) \in \mathbf{P}(\mathbf{FRA})$, so $\mathcal{A} \in \mathbf{S_cP}(\mathbf{FRA})$.

Conversely, let $\mathcal{A} \in \mathbf{S_cP}(\mathbf{FRA})$, say $\mathcal{A} \subseteq_c \prod_{D \in \Delta} \mathfrak{Re}(D)$, for some Δ. Now $\prod_{D \in \Delta} \mathfrak{Re}(D)$ is completely representable — just interpret each element as itself. **CRA** is closed under strong subalgebras (lemma 17) so $\mathcal{A} \in \mathbf{CRA}$. Hence $\mathbf{S_cP}(\mathbf{FRA}) \subseteq \mathbf{CRA}$.

⊣

LEMMA 19. *Let $n \geq 3$ and let $\mathcal{A}$ be an atomic relation algebra, $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$ for some $\mathcal{C} \in \mathbf{CA}_n$. For all $x \in \mathcal{C} \setminus \{0\}$ and all $i, j < n$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $s_{ij}a \cdot x \neq 0$.*

PROOF. Recall from fact 3.1, that s^i_j is a completely additive operator (any i, j), hence s_{ij} is too (see definition 4). So $\sum\{s_{ij}a : a \in \mathsf{At}(\mathcal{A})\} = s_{ij} \sum \mathsf{At}(\mathcal{A}) = s_{ij}1 = 1$, for any $i, j < n$. Let $x \in \mathcal{C} \setminus \{0\}$. It is impossible that $s_{ij}a \cdot x = 0$ for all $a \in \mathsf{At}(\mathcal{A})$ because this would imply that $1 - x$ was an upper bound for $\{s_{ij}a : a \in \mathsf{At}(\mathcal{A})\}$, contradicting $\sum\{s_{ij}a : a \in \mathsf{At}(\mathcal{A})\} = 1$. ⊣

§3. $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary.

DEFINITION 20. *Let K be a class of structures in a signature L. We say that K is* pseudo-elementary *if there is a many-sorted signature L^s, where the signature L_1 of the first sort contains L, and some L^s-theory U such that $K = \{M^1\restriction_L : M \models U\}$. Here $M^1\restriction_L$ is the L-structure obtained from M by (a) restricting the domain to the first-sorted elements only and (b) restricting the language to L.*

THEOREM 21. *For any ordinal $\gamma \geq 3$ the class $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary.*

PROOF. For finite γ it is quite easy to define $\mathfrak{Ra}\mathbf{CA}_\gamma$ in a two-sorted language. The first sort is for relation algebra elements and the second sort is for cylindric algebra elements. The defining theory includes sentences requiring the second-sorted elements to form an γ-dimensional cylindric algebra. The signature of the defining theory also includes a function I from sort one to sort two and the defining theory includes a sentence requiring that I respects the operators (e.g. $I(1') = d_{01}$) and is injective. Finally, there is a sentence saying, for any cylindric algebra element y, that $\bigwedge_{2 \leq i < \gamma} \mathsf{c}_i y = y$ if and only if that there is a relation algebra element x such that $y = I(x)$. This ensures that I is a surjection onto the relation algebra reduct of the cylindric algebra.

For infinite γ this method won't work because the conjunction $\bigwedge_{2 \leq i \leq \gamma} \mathsf{c}_i y = y$ is infinitary. Instead, we use a three sorted defining theory, with one sort for a relation algebra (r), the second sort for the boolean part of a cylindric algebra (b)

and the third sort for a set of dimensions (δ). We will use superscripts r, b, δ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the relation algebra, the boolean part of the cylindric algebra or the dimension set, respectively. Our signature includes dimension sort constants i^δ, for each $i < \gamma$ to represent the dimensions. It also includes the relation algebra operators for the first sort, a function d^b taking two dimension sort arguments and returning a boolean sort element, and a function c^b taking one argument of sort δ and a second argument of sort b and returning an element of sort b. The defining theory for $\mathfrak{Ra}\mathbf{CA}_\gamma$ includes sentences demanding that the constants i^δ for $i < \gamma$ are distinct, and that the last two sorts define a cylindric algebra of dimension at least γ. For example, in place of the cylindric algebra axiom $d_{ij} = \mathsf{c}_k(d_{ik}.d_{kj})$ (all $i, j, k < \gamma$) we have the sentence

$$\forall x^\delta, y^\delta, z^\delta \; (d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta).^b d^b(z^\delta, y^\delta)))$$

(here $x^\delta, y^\delta, z^\delta$ are variables of sort δ, $.^b$ is the boolean intersection operator for cylindric algebras, henceforth we drop sort superscripts for boolean operators) with similar translations of the other cylindric algebra axioms. We also have a function I^b from sort r to sort b and sentences requiring I^b to be injective and to respect the relation algebra operations as follows: for all x^r, y^r,

$$\begin{aligned} I^b(1') &= d^b(0^\delta, 1^\delta) \\ I^b(x^r) &= s_0^2 s_1^0 s_2^1 I^b(x^{\smile r}) \\ I^b(x^r; y^r) &= c_2^b(s_2^1 I^b(x) \; . \; s_2^0 I^b(y)) \end{aligned}$$

where s^i_j the substitution operator from sort b to sort b. More precisely, an equation $x^b = s^i_j y^b$ abbreviates the formula

$$[(i^\delta = j^\delta) \to (x^b = y^b)] \wedge [(i^\delta \neq j^\delta) \to (x^b = c^b(i^\delta, (d^b(i^\delta, j^\delta).y^b)))]$$

Finally, we require that I^b maps *onto* the set of two dimensional elements:

$$\forall y^b \; ((\forall z^\delta (z^\delta \neq 0^\delta, 1^\delta \to c^b(z^\delta, y^b) = y^b)) \leftrightarrow \exists x^r \; (y^b = I^b(x^r)))$$

Clearly, any algebra of the type of a relation algebra $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\gamma$ is the first sort of a model of this theory. Conversely, a model of this theory will consist of a relation type algebra (sort r) and a cylindric algebra whose dimension is the cardinality of the set of δ-sorted elements. This cardinality is at least $|\gamma|$ since we required that all the constants $\{i^\delta : i < \gamma\}$ are distinct. So the first sort of a model will be the relation algebra reduct of a cylindric algebra of dimension $\gamma' \geq \gamma$. By lemma 6 this implies that the first sort of a model must belong to $\mathfrak{Ra}\mathbf{CA}_\gamma$. Hence this three sorted theory does define $\mathfrak{Ra}\mathbf{CA}_\gamma$. ⊣

COROLLARY 22. *For countable $\gamma \geq 3$ the elementary theory of $\mathfrak{Ra}\mathbf{CA}_\gamma$ is recursively enumerable.*

PROOF. The defining three-sorted theory in the proof of the previous theorem is recursive. Use [6, theorem 9.37]. ⊣

§4. **Games.** Since $\mathfrak{Ra}\mathbf{CA}_\gamma$ is pseudo-elementary and the defining theory is recursive for countable γ, it is possible to devise a two-player game $\Gamma(\mathcal{A})$ to test if a relation algebra $\mathcal{A}$ belongs to this class [6, definition 9.32, proposition 9.33]. The number of rounds in a play of $\Gamma(\mathcal{A})$ is the cardinal $|\mathcal{A}| + |\gamma| + \omega$. In each of these rounds the first player, $\forall$, makes a move and the second player, $\exists$, has to respond. There are rules which stipulate which responses by $\exists$ are legal and which are not. If $\exists$ makes an illegal response in any round then $\forall$ wins the play, otherwise $\exists$ makes a legal response in every round and $\exists$ wins the play. $\exists$ has a winning strategy in $\Gamma(\mathcal{A})$ if and only if $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\gamma$.

For $n < \omega$, a shortened version of this game, $\Gamma_n(\mathcal{A})$, can be defined. This is very similar, but play stops after n rounds. If $\exists$ responds legally in each of the n rounds she wins the play, otherwise $\forall$ wins. [6, Propositions 9.34, 9.36] state (in the more general setting of arbitrary pseudo-elementary classes) that for each $n < \omega$ there is a first-order formula η_n in the signature of relation algebras such that $\exists$ has a winning strategy (w.s.) in $\Gamma_n(\mathcal{A})$ if and only if $\mathcal{A} \models \eta_n$, and that if $\exists$ has a winning strategy in $\Gamma_n(\mathcal{A})$ for all $n < \omega$ then $\mathcal{A}$ is elementarily equivalent to a member of $\mathfrak{Ra}\mathbf{CA}_\gamma$. Thus $\{\eta_n : n < \omega\}$ axiomatises the elementary theory of $\mathfrak{Ra}\mathbf{CA}_\gamma$.

However, the game $\Gamma(\mathcal{A})$ is not very easy to use in practice — it seems that games that use the atoms of an atomic boolean algebra with operators are easier to use then these more general games. Furthermore, we want to prove not only that $\mathfrak{Ra}\mathbf{CA}_\gamma$ is not elementary, but various other classes also fail to be elementary (see theorem 45). We also want to draw out the connection between relation algebra reducts and complete representations. For these reasons, we omit details of the game $\Gamma(\mathcal{A})$ and define three other games $F^n(\alpha), G(\alpha), H(\alpha)$ played on the *atom structure* of an atomic relation algebra. The games are increasingly difficult for $\exists$ to win (and increasingly easy for $\forall$ to win), so

$$\text{w.s. for } \exists \text{ in } H(\alpha) \Rightarrow \text{ w.s. for } \exists \text{ in } G(\alpha) \Rightarrow \text{w.s. for } \exists \text{ in } F^\omega(\alpha)$$

For countable α, we will prove

$$\text{w.s. for } \exists \text{ in } F^\omega(\alpha) \Leftrightarrow \alpha \in \mathsf{At}(\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega) \qquad \text{(thm. 29)}$$
$$\text{w.s. for } \exists \text{ in } H(\alpha) \Rightarrow \alpha \in \mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega) \Rightarrow \text{w.s. for } \exists \text{ in } G(\alpha) \qquad \text{(thms. 39, 34)}$$

We are not sure about the converses of the last two implications. We will also prove that there is a relation algebra atom structure $\alpha \in \mathsf{At}\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega \setminus \mathsf{At}\mathfrak{Ra}\mathbf{CA}_\omega$ (theorem 36). It follows that it is strictly harder for $\exists$ to win $H(\alpha)$ than $F^\omega(\alpha)$. The game $G(\alpha)$ is in between, but we do not know if it is equivalent to $F^\omega(\alpha)$ or $H(\alpha)$ or neither.

DEFINITION 23 (Networks and Hypernetworks). *Let α be a relation algebra atom structure. A* network over α *(sometimes called an atomic network, also known as a basic matrix) is a complete labelled graph N whose nodes* $\mathsf{nodes}(N)$ *form a set of natural numbers, with each edge labelled by an atom from α such that*

I. $N(i,i) \leq 1'$,
II. $N(j,i) = N(i,j)^\smile$,
III. $N(i,j); N(j,k) \geq N(i,k)$,

for all nodes $i, j, k \in \mathsf{nodes}(N)$*. In fact if* N *satisfies conditions I and III then, by the relation algebra axioms, it must also satisfy condition II. A network* N *is* strict *if* $N(i,j) \leq 1' \iff i = j$.

Define an equivalence relation $\sim$ *over the set of all finite sequences over* $\mathsf{nodes}(N)$ *by* $\bar{x} \sim \bar{y}$ *iff* $|\bar{x}| = |\bar{y}|$ *and* $N(x_i, y_i) \leq 1'$ *for all* $i < |\bar{x}|$.

A hypernetwork $N = (N^a, N^h)$ *consists of a network* N^a *together with a labelling function for hyperlabels* $N^h : {}^{<\omega}\mathsf{nodes}(N) \to \Lambda$ *(some arbitrary set of hyperlabels* Λ*) such that for* $\bar{x}, \bar{y} \in {}^{<\omega}\mathsf{nodes}(N)$

IV. $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$.

If $|\bar{x}| = k \in \mathbb{N}$ *and* $N^h(\bar{x}) = \lambda$ *then we say that* λ *is a* k*-ary hyperlabel. When there is no risk of ambiguity we may drop the superscripts* a, h.

The following notation is defined for hypernetworks, but applies equally to networks. If N *is a hypernetwork and* S *is any set then* $N{\restriction_S}$ *is the* n*-dimensional hypernetwork defined by restricting* N *to the set of nodes* $S \cap \mathsf{nodes}(N)$*. For hypernetworks* M, N *if there is a set* S *such that* $M = N{\restriction_S}$ *then we write* $M \subseteq N$*. If* $N_0 \subseteq N_1 \subseteq \ldots$ *is a nested sequence of hypernetworks then we let the* limit $N = \bigcup_{i<\omega} N_i$ *be the hypernetwork defined by* $\mathsf{nodes}(N) = \bigcup_{i<\omega} \mathsf{nodes}(N_i)$, $N^a(x,y) = N_i^a(x,y)$ *if* $x, y \in \mathsf{nodes}(N_i)$*, and* $N^h(\bar{x}) = N_i^h(\bar{x})$ *if* $\mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(N_i)$*. This is well-defined since the hypernetworks are nested and since hyperedges* $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ *are only finitely long.*

For hypernetworks M, N *and any set* S*, we write* $M \equiv^S N$ *if* $N{\restriction_S} = M{\restriction_S}$*. For hypernetworks* M, N*, and any set* S*, we write* $M \equiv_S N$ *if the symmetric difference* $\Delta(\mathsf{nodes}(M), \mathsf{nodes}(N)) \subseteq S$ *and* $M \equiv^{(\mathsf{nodes}(M) \cup \mathsf{nodes}(N)) \setminus S} N$*. We write* $M \equiv_k N$ *for* $M \equiv_{\{k\}} N$.

Let N *be a network and let* θ *be any function. The network* $N\theta$ *is a complete labelled graph with nodes* $\theta^{-1}(\mathsf{nodes}(N)) = \{x \in \mathsf{dom}(\theta) : \theta(x) \in \mathsf{nodes}(N)\}$*, and labelling defined by* $(N\theta)(i,j) = N(\theta(i), \theta(j))$*, for* $i, j \in \theta^{-1}(\mathsf{nodes}(N))$*. Similarly, for a hypernetwork* $N = (N^a, N^h)$*, we define* $N\theta$ *to be the hypernetwork* $(N^a\theta, N^h\theta)$ *with hyperlabelling defined by* $N^h\theta(x_0, x_1, \ldots) = N^h(\theta(x_0), \theta(x_1), \ldots)$ *for* $(x_0, x_1, \ldots) \in {}^{<\omega}\theta^{-1}(\mathsf{nodes}(N))$.

Let M, N *be hypernetworks. A* partial isomorphism $\theta : M \to N$ *is a partial map* $\theta : \mathsf{nodes}(M) \to \mathsf{nodes}(N)$ *such that for any* $i, j \in \mathsf{dom}(\theta) \subseteq \mathsf{nodes}(M)$ *we have* $M^a(i,j) = N^a(\theta(i), \theta(j))$ *and for any finite sequence* $\bar{x} \in {}^{<\omega}\mathsf{dom}(\theta)$ *we have* $M^h(\bar{x}) = N^h\theta(\bar{x})$*. If* $M = N$ *we may call* θ *a partial isomorphism of* N.

A hyperedge $\bar{x} \in {}^{<\omega}\mathsf{nodes}(N)$ *of* N *is called* short *if there are* $y_0, y_1 \in \mathsf{nodes}(N)$ *and for all* $i < |\bar{x}|$ *either* $N(x_i, y_0) \leq 1'$ *or* $N(x_i, y_1) \leq 1'$*. Other hyperedges are called* long*. A hypernetwork* N *is called* λ-neat *if* $N(\bar{x}) = \lambda$*, for all short hyperedges* $\bar{x}$ *of* N*. If* N *is a* λ*-neat hypernetwork then* $N\theta$ *is a* λ*-neat hypernetwork.*

REMARK 24. *We will fix some hyperlabel* λ_0 *and use* λ_0*-neat hypernetworks extensively in what follows. The idea is to keep a constant label* (λ_0) *on short hyperedges of the hypernetworks we use. These hypernetworks can be used to form the atoms of a cylindric algebra (at least in the finite dimensional case). The fact that short hyperlabels are constant means that the atoms of the relation algebra reduct of this cylindric algebra should be no smaller than the atoms of the original relation algebra. This will help us prove that the relation algebra is a relation algebra reduct of a cylindric algebra.*

DEFINITION 25. *For $n \geq 3$ and $\mathcal{C} \in \mathbf{CA}_n$, if $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ is an atomic relation algebra and N is an $\mathcal{A}$-network then we define $\widehat{N} \in \mathcal{C}$ by*

$$\widehat{N} = \prod_{i,j \in \mathsf{nodes}(N)} s_{ij} N(i,j)$$

$\widehat{N} \in \mathcal{C}$ depends implicitly on $\mathcal{C}$.

LEMMA 26. *Let $3 \leq n$, $\mathcal{C} \in \mathbf{CA}_n$ and let $\mathcal{A} \subseteq_c \mathfrak{Ra}\mathcal{C}$ be an atomic relation algebra.*

1. *For any $x \in \mathcal{C} \setminus \{0\}$ and any finite set $I \subseteq n$ there is a network N such that $\mathsf{nodes}(N) = I$ and $x \,.\, \widehat{N} \neq 0$.*
2. *For any networks M, N if $\widehat{M} \,.\, \widehat{N} \neq 0$ then $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$.*

PROOF. The proof of the first part is based on repeated use of lemma 19. We define the edge labelling of N one edge at a time. Initially no edges are labelled. Suppose $E \subseteq \mathsf{nodes}(N) \times \mathsf{nodes}(N)$ is the set of labelled edges of N (initially $E = \emptyset$) and $x \,.\, \prod_{(i,j) \in E} s_{ij} N(i,j) \neq 0$. Pick $k, l \in I$ such that $(k,l) \notin E$. By lemma 19 there is $a \in \mathsf{At}(\mathcal{A})$ such that $x \,.\, \prod_{(i,j) \in E} s_{ij} N(i,j) \,.\, s_{kl} a \neq 0$. If $k = l$ then we can find such an a with $a \leq 1'$ (note that $s_{ii} d_{01} = 1$). Extend the labelling of N so that $N(k,l) = a$ and include the edge (k,l) in E. Eventually, all edges will be labelled, so we obtain a completely labelled graph N with $\widehat{N} \neq 0$. Network condition I in definition 23 is true by the way we selected the label of reflexive edges. For condition III, let $i, j, k < n$. We have $s_{ij} N(i,j) \,.\, s_{jk} N(j,k) \,.\, s_{ik} N(i,k) \geq \widehat{N} \neq 0$ so by proposition 11, $0 < \mathsf{c}_j(s_{ij} N(i,j) \,.\, s_{jk} N(j,k)) \,.\, s_{ik} N(i,k) = s_{ik}(N(i,j); N(j,k)) \,.\, s_{ik} N(i,k)$, hence $N(i,j); N(j,k) \,.\, N(i,k) \neq 0$, by fact 3.1, so N satisfies network condition III. Network condition II follows from I and III, hence N is a network.

For the second part, if it is not true that $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$ then there are $i, j \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ such that $M(i,j) \neq N(i,j)$. Since edges are labelled by atoms we have $M(i,j).N(i,j) = 0$ so $0 = s_{ij} 0 = s_{ij} M(i,j) \,.\, s_{ij} N(i,j) \geq \widehat{M} \,.\, \widehat{N}$.

$\dashv$

LEMMA 27. *Let $n \geq 3$, $\mathcal{C} \in \mathbf{CA}_n$ and let $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ be atomic. Let N be a network over $\mathcal{A}$ and $i, j < n$.*

1. *If $i \notin \mathsf{nodes}(N)$ then $\mathsf{c}_i \widehat{N} = \widehat{N}$.*
2. *$\widehat{N Id_{-j}} \geq \widehat{N}$.*
3. *If $i \notin \mathsf{nodes}(N)$ and $j \in \mathsf{nodes}(N)$ then $\widehat{N} \neq 0 \rightarrow \widehat{N[i/j]} \neq 0$.*
4. *If θ is any partial, finite map $n \rightarrow n$ and if $\mathsf{nodes}(N)$ is a proper subset of n, then $\widehat{N} \neq 0 \rightarrow \widehat{N\theta} \neq 0$.*

PROOF. The first part is by facts 3.1, 3.2 and 3.4. The second part is by definition of $\widehat{\ }$. For the third part suppose $\widehat{N} \neq 0$. Since $i \notin \mathsf{nodes}(N)$, by part 1, we have $\mathsf{c}_i \widehat{N} = \widehat{N}$. By cylindric algebra axioms it follows that $\widehat{N} \,.\, d_{ij} \neq 0$. By lemma 26 there is a network M where $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{i\}$ such that $\widehat{M} \,.\widehat{N} \,.\, d_{ij} \neq 0$, so $\widehat{M} \neq 0$. By lemma 26 we have $M \supseteq N$ and $M(i,j) \leq 1'$. It follows that $M = N[i/j]$. Hence $\widehat{N[i/j]} = \widehat{M} \neq 0$.

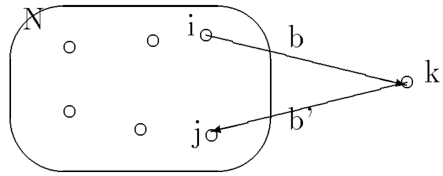


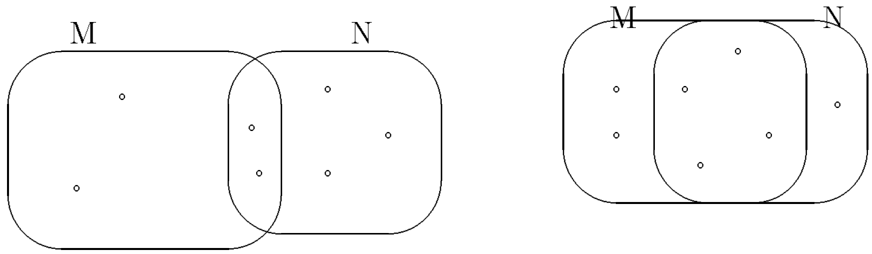
FIGURE 1. Triangle move

For the final part (cf. [6, lemma 13.29]), since there is $k \in n \setminus \mathsf{nodes}(N)$, θ can be expressed as a product $\sigma_0\sigma_1 \ldots \sigma_t$ of maps such that, for $s \leq t$, we have either $\sigma_s = Id_{-i}$ for some $i < n$ or $\sigma_s = [i/j]$ for some $i, j < n$ and where $i \notin \mathsf{nodes}(N\sigma_0 \ldots \sigma_{s-1})$. Now apply parts 2 and 3 of the lemma. $\dashv$

DEFINITION 28 (Games). *For any relation algebra atom structure α and $3 \leq n \leq \omega$, we define two-player games $F^n(\alpha)$, $G(\alpha)$ and $H(\alpha)$, each with ω rounds, and for $n < \omega$ we define $H_n(\alpha)$ with n rounds.*

- *Let $3 \leq n \leq \omega$. In a play of $F^n(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \ldots$ where $\mathsf{nodes}(N_i)$ is a finite subset of $n = \{j : j < n\}$, for each i. In the initial round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$ must play a network N_0 with $\mathsf{nodes}(N_0) \subseteq \{0, 1\}$, such that $N_0(i, j) = a$ for some $i, j \in \mathsf{nodes}(N_0)$. In a subsequent round of a play of $F^n(\alpha)$ $\forall$ can pick a previously played network N and $i, j \in \mathsf{nodes}(N)$, $k \in n\setminus\{i, j\}$, and atoms $b, b' \in \alpha$ such that $b; b' \geq N(i, j)$. This move is called a* triangle move *and is denoted (N, i, j, k, b, b'), see figure 1. In order to make a legal response, $\exists$ must play a network $M \supseteq N$ such that $M(i, k) = b$ and $M(k, j) = b'$ and $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$.*

  $\exists$ wins $F^n(\alpha)$ if she responds with a legal move in each of the ω rounds. If she fails to make a legal response in any round then $\forall$ wins.
- *$G(\alpha)$ is similar to $F^\omega(\alpha)$. For each i, the nodes of N_i are a finite subset of ω. The initial round in a play of $G(\alpha)$ is the same as in a play of $F^\omega(\alpha)$. In any subsequent round $\forall$ can play a triangle move, as in $F^\omega(\alpha)$ and the rules for $\exists$'s response are the same. In $G(\alpha)$, $\forall$ has the option of playing a* transformation move *(N, θ) by picking a previously played network N and a partial finite surjection $\theta : \omega \to \mathsf{nodes}(N)$. $\exists$ must respond with $N\theta$. Also, $\forall$ can play an* amalgamation move *(M, N) by picking previously played networks M, N such that $0 < |\mathsf{nodes}(M) \cap \mathsf{nodes}(N)| \leq 2$ and $M \equiv^{\mathsf{nodes}(M)\cap\mathsf{nodes}(N)} N$, see figure 2(a). To make a legal response, $\exists$ must respond with some network L extending M and N. If she fails to make a legal response in any of the ω rounds of the play, $\forall$ wins. If she succeeds in each round, she wins.*
- *Fix some hyperlabel λ_0. $H(\alpha)$ is similar to $G(\alpha)$, but in this game the play consists of a sequence of λ_0-neat hypernetworks $N_0, N_1, \ldots$ where $\mathsf{nodes}(N_i)$ is a finite subset of ω, for each $i < \omega$. The other main difference is that $\forall$ can play a more general kind of amalgamation move. In the initial round $\forall$ picks $a \in \alpha$ and $\exists$ must play a λ_0-neat hypernetwork N_0 with nodes contained in $\{0, 1\}$ and $N_0(i, j) = a$ for some nodes i, j. At*



(a) Game $G(\alpha)$ (b) Game $H(\alpha)$

FIGURE 2. Amalgamation moves

a later stage $\forall$ can make any triangle move (N, i, j, k, b, b') by picking a previously played hypernetwork N and $i, j \in \mathsf{nodes}(N)$, $k \in \omega \setminus \mathsf{nodes}(N)$ and $b; b' \geq N(i,j)$. [In H we require that $\forall$ chooses k as a 'new node', i.e. not in $\mathsf{nodes}(N)$, whereas in F^n for finite n it was necessary to allow $\forall$ to 'reuse old nodes'.] For a legal response, $\exists$ must play a λ_0-neat hypernetwork $M \equiv_k N$ where $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$ and $M(i,k) = b$ and $M(k,j) = b'$. Alternatively, $\forall$ can play a transformation move *by picking a previously played hypernetwork N and a partial, finite surjection $\theta : \omega \to \mathsf{nodes}(N)$, this move is denoted (N, θ). $\exists$ must respond with $N\theta$. Finally, $\forall$ can play an* amalgamation move *by picking previously played hypernetworks M, N such that $M \equiv^{\mathsf{nodes}(M) \cap \mathsf{nodes}(N)} N$ and $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) \neq \emptyset$, see figure 2(b). This move is denoted (M, N). To make a legal response, $\exists$ must play a λ_0-neat hypernetwork L extending M and N, where $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N)$.*

Again, $\exists$ wins $H(\alpha)$ if she responds legally in each of the ω rounds, otherwise $\forall$ wins.

- *For $n < \omega$ the game $H_n(\alpha)$ is similar to $H(\alpha)$ but play ends after n rounds, so a play of $H_n(\alpha)$ could be*

$$N_0, N_1, \ldots, N_n$$

If $\exists$ responds legally in each of these n rounds she wins, otherwise $\forall$ wins.

THEOREM 29. *Let $\mathcal{A}$ be a relation algebra. With reference to the four conditions below, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $\mathcal{A}$ is atomic with countably many atoms then (4) $\Rightarrow$ (1) and all conditions are equivalent.*

1. *$\mathcal{A}$ has a complete representation.*
2. *There is an atomic representable cylindric algebra $\mathcal{C} \in \mathbf{RCA}_\omega$ such that $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$.*
3. *$\mathcal{A}$ is atomic and $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega$.*
4. *$\mathcal{A}$ is atomic and $\exists$ has a winning strategy in $F^\omega(\mathsf{At}(\mathcal{A}))$.*

PROOF. The equivalence of (1) and (3), for countable algebras, is proved in [16, theorem 1].

(1) $\Rightarrow$ (2): Let $\mathcal{M}$ be a complete representation of $\mathcal{A}$. By lemma 16 $\mathcal{A}$ is atomic. The plan is to define an atomic representable cylindric algebra $\mathcal{C}$, to show that there is an embedding $I : \mathcal{A} \to \mathfrak{Ra}(\mathcal{C})$ and that for all non-zero

$x \in \mathfrak{Ra}(\mathcal{C})$ there is $a \in \mathsf{At}(\mathcal{A})$ such that $I(a)\ .\ x \neq 0$. We will then apply lemma 15 to get $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$.

$1^{\mathcal{M}}$ must be an equivalence relation over the domain of $\mathcal{M}$, as we saw earlier. Let E be the set of equivalence classes of $1^{\mathcal{M}}$. For each equivalence class $D \in E$ pick an arbitrary sequence $f_D \in {}^{\omega}D$. Let $W_D = \{f \in {}^{\omega}D : \{i < \omega : f(i) \neq f_D(i)\}$ is finite$\}$ and let $\mathcal{C}_D = (\wp(W_D), \emptyset, W_D, \cup, \setminus, D_{ij}, C_i : i, j < \omega)$. This is a weak cylindric algebra (see section 2.4) and by proposition 2 it belongs to $\mathbf{RCA}_\omega$. $\mathcal{C}_D$ is an atomic cylindric algebra — the atoms are the singleton sets $\{f\}$, for $f \in W_D$. Note, for $f, g \in W_D$ and $i < \omega$, that if $f\restriction_{\omega\setminus\{i\}} = g\restriction_{\omega\setminus\{i\}}$ then $\{f\} \leq C_i\{g\}$.

Let $x \in \mathfrak{Ra}(\mathcal{C}_D)$, i.e. $C_i x = x$ for $2 \leq i < \omega$. If $f \in x$ and $g \in W_D$ satisfies $g(0) = f(0),\ g(1) = f(1)$ then $g \in x$ since $\{2 \leq i < \omega : f(i) \neq g(i)\}$ is finite. It follows that $\mathfrak{Ra}(\mathcal{C}_D)$ is atomic and its atoms are $\{\{g \in W_D : g(0) = m,\ g(1) = n\} : m, n \in D\}$. There is a homomorphism $h_D : \mathcal{A} \to \mathfrak{Ra}(\mathcal{C}_D)$ given by $h_D(a) = \{f \in W_D : \exists a' \leq a,\ a' \in \mathsf{At}\mathcal{A},\ \mathcal{M} \models a'(f(0), f(1))\}$.

Let $\mathcal{C} = \prod_{D \in E} \mathcal{C}_D \in \mathbf{RCA}_\omega$. Let $\pi_D : \mathcal{C} \to \mathcal{C}_D$ be the D'th projection and let $\iota_D : \mathcal{C}_D \to \mathcal{C}$ be the D'th embedding. Since $\mathcal{C}$ is a product of atomic cylindric algebras, it is atomic and its atoms are $\{\iota_D(\beta) : D \in E,\ \beta \in \mathsf{At}(\mathcal{C}_D)\}$.

$\mathcal{A}$ embeds into $\mathfrak{Ra}(\mathcal{C})$ by $I : a \mapsto (h_D(a) : D \in E)$. If $x \in \mathfrak{Ra}(\mathcal{C})$ then for each D we have $\pi_D(x) \in \mathfrak{Ra}(\mathcal{C}_D)$ and if x is non-zero then $\pi_D(x) \neq 0$ for some D. By atomicity of $\mathcal{C}_D$ there are $m, n \in D$ such that $\{g \in W_D : g(0) = m,\ g(1) = n\} \subseteq \pi_D(x)$. By lemma 16 there is $a \in \mathsf{At}(\mathcal{A})$ such that $\mathcal{M} \models a(m, n)$. Hence, $\{g \in W_D : g(0) = m,\ g(1) = n\} \subseteq \pi_D((x)\ .\ I(a))$ and so $x\ .\ I(a) \neq 0$. By lemma 15, $\mathcal{A} \subseteq_c \mathfrak{Ra}(\mathcal{C})$.

(2) ⇒ (3): Trivial (use lemma 14 for atomicity of $\mathcal{A}$).

(3) ⇒ (4): Let $\mathcal{A} \subseteq_c \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. We have to show that $\exists$ has a winning strategy in $F^\omega(\mathsf{At}(\mathcal{A}))$.

$\exists$'s strategy is to always play networks N such that $\widehat{N} \neq 0$. In the initial round, let $\forall$ play $a \in \mathsf{At}(\mathcal{A})$. $\exists$ plays the network N_0 with nodes $\{0, 1\}$ and labelling determined by $N_0(0, 1) = a$. Then $\widehat{N_0} = a \neq 0$.

At a later stage suppose $\forall$ plays the triangle move (N, i, j, k, b, b'), where $k \neq i, j,\ b; b' \geq N(i, j)$ and N was previously played so $\widehat{N} \neq 0$. By proposition 11, $\mathsf{c}_k(s_{ik}b\ .\ s_{kj}b') = s_{ij}(b; b') \geq s_{ij}N(i, j) \geq \widehat{N}$. By lemma 27(1), $\mathsf{c}_k\widehat{N} = \widehat{N}$. Therefore $\mathsf{c}_k(s_{ik}b\ .\ s_{kj}b') \geq \mathsf{c}_k\widehat{N}$ and hence $s_{ik}b\ .\ s_{kj}b'\ .\ \mathsf{c}_k\widehat{N} \neq 0$, by fact 3.3.

By lemma 26, there is a network M where $\mathsf{nodes}(M) = \mathsf{nodes}(N) \cup \{k\}$ such that $\widehat{M}\ .\ \mathsf{c}_k\widehat{N}\ .\ s_{ik}b\ .\ s_{kj}b' \neq 0$. By lemma 26, $M \equiv_k N$. Lemma 26 also proves that $M(i, k) = b$ and $M(k, j) = b'$. To see why, consider a network B where $\mathsf{nodes}(B) = \{i, k\}$ and $B(i, k) = b$. It is not hard to show that $\widehat{B} = s_{ik}b$, so by lemma 26 we get $M \equiv^{\{i,k\}} B$ hence $M(i, k) = b$ and similarly $M(k, j) = b'$. This means that M is a legal response, so $\exists$ plays such a network M. Thus $\exists$ can preserve the conditions: M is a network and $\widehat{M} \neq 0$.

Now suppose $\mathcal{A}$ is atomic with countably many atoms. The implication (4) ⇒ (1) is essentially [6, theorem 11.7(2)], or see lemma 35 for a very similar proof.

$\dashv$

PROBLEM 30. *Let $\mathcal{A}$ be an atomic relation algebra. If $\mathcal{A} \in \bigcap_{n<\omega} \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ must $\exists$ have a winning strategy in $F^\omega(\mathsf{At}(\mathcal{A}))$?*

REMARK 31. *For atomic relation algebras with uncountably many atoms the four conditions in theorem 29 need not be equivalent. Let $\mathcal{A}$ be the atomic relation algebra with atoms $\{1', a_0^i, a_j : i < 2^\omega,\ 1 \leq j < \omega\}$, all symmetric, and where the forbidden triples of atoms are the permutations of $(1', x, y)$ for $x \neq y$, (a_j, a_j, a_j) for $1 \leq j < \omega$, and $(a_0^i, a_0^{i'}, a_0^{i^*})$ for $i, i', i^* < 2^\omega$. In other words, if you think of the subscript of a non-identity atom as its colour, then monochromatic triangles are forbidden. All other triples of atoms are consistent. Write a_0 for $\{a_0^i : i < 2^\omega\}$ and a_+ for $\{a_j : 1 \leq j < \omega\}$. Define $\mathcal{A}$ to be the subalgebra of the complex algebra over this atom structure generated by the atoms (this is called the* term algebra *of the atom structure). It is easy to check that each element of $\mathcal{A}$ has the form $F \cup A_0 \cup A_+$, where F is a finite set of atoms, A_0 is either empty of a cofinite subset of a_0 and A_+ is either empty of a cofinite subset of a_+. With this definition, we can prove:*

(2) *$\mathcal{A}$ has no complete representation.*

The proof of this is based on an infinite version of Ramsay's theorem (which requires continuum many atoms a_0^i).

(3) $$\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\omega.$$

The proof of this is more complicated, but here is an outline. Let S be the set of atomic $\mathcal{A}$-networks N with nodes ω such that $\{a_j : a_j$ labels some edge of $N\}$ is finite. We can show that S forms an ω-dimensional cylindric algebra atom structure and hence $\mathfrak{Cm}(S) \in \mathbf{CA}_\omega$. We have $\mathcal{A} \subseteq \mathfrak{Ra}(\mathfrak{Cm}(S))$, the embedding is $a \mapsto \{N \in S : N(0,1) \leq a\}$. We will identify $\mathcal{A}$ with its image under this embedding henceforth. The next step is to calculate the subalgebra of $\mathfrak{Cm}(S)$ generated by $\mathcal{A}$ using the cylindric algebra operations.

Let X be the set of finite labelled graphs N where the label of any edge of N is either an atom of $\mathcal{A}$, a cofinite subset of a_+ or a cofinite subset of a_0, such that for any nodes l, m, n of N we have $N(l,n) \leq N(l,m);N(m,n)$. For $N \in X$ let $N' \in \mathfrak{Cm}(S)$ be defined by $N' = \{L \in S : L(m,n) \leq N(m,n)$ for $m, n \in N\}$. For $i < \omega$ let $N{\restriction}_{-i}$ be the subgraph of N obtained by deleting the node i. We can show that

(4) $$\mathsf{c}_i(N') = (N{\restriction}_{-i})'$$

It follows that the subalgebra $\mathcal{C}$ of $\mathfrak{Cm}(S)$ generated by $\mathcal{A}$ consists of finite unions of elements of the form N', for $N \in X$. [Note that $\mathcal{C}$ is not *an atomic cylindric algebra, indeed it is atomless, because for any $N \in X$ we can add an extra node and extend N to $M \in X$ in such a way that $\emptyset \subsetneq M' \subsetneq N'$, so N' is not an atom.]*

Finally we show that $\mathcal{A} = \mathfrak{Ra}(\mathcal{C})$. The inclusion $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$ is easy. Conversely, let $z \in \mathfrak{Ra}(\mathcal{C})$. By definition of $\mathfrak{Ra}$, we have $\mathsf{c}_i z = z$ for $i > 1$. By the above, z is a finite union $\bigcup_{N \in F} N'$, where F is a finite subset of X. Let $i_0, \ldots, i_k$ enumerate all the nodes, other than 0 and 1, that occur in any labelled graph in F. Then for $N \in F$, by (4), $\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_k} N' = (N{\restriction}_{\{0,1\}})'$, hence

$\mathsf{c}_{i_0}\ldots\mathsf{c}_{i_k}N' \in \mathcal{A}$, *using our identification of* $\mathcal{A}$ *with its embedded image in* $\mathfrak{Ra}\mathcal{C}$. *So* $z = \mathsf{c}_{i_0}\ldots\mathsf{c}_{i_k}z = \bigcup_{N\in F}\mathsf{c}_{i_0}\ldots\mathsf{c}_{i_k}N' \in \mathcal{A}$. *This shows that* $\mathfrak{Ra}(\mathcal{C}) \subseteq \mathcal{A}$.

Thus $\mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_\omega$ *but* $\mathcal{A}$ *has no complete representation, so* $\mathcal{A}$ *satisfies condition 3 but not condition 1 of theorem 29.*

For a corollary to neat cylindric reducts, let $\mathcal{B} = \mathfrak{Nr}_n\mathcal{C}$ $(2 < n < \omega)$. *Then* $\mathcal{B} \in \mathfrak{Nr}_n\mathbf{CA}_\omega$ *but* $\mathcal{B}$ *has no complete representation (a complete representation of* $\mathcal{B}$ *would induce a complete representation of* $\mathcal{A} = \mathfrak{Ra}(\mathcal{B})$*).*

PROBLEM 32. *If* $\mathcal{A} \subseteq_c \mathfrak{Ra}\mathcal{C}$ *for some* atomic $\mathcal{C} \in \mathbf{CA}_\omega$ *does it follow that* $\mathcal{A}$ *has a complete representation? In other words, does (2)* $\Rightarrow$ *(1) in theorem 29? The remark, above, does not answer this question since the cylindric algebra* $\mathcal{C}$ *in that remark is not atomic.*

For finite $n < \omega$ an n-dimensional version of theorem 29 can also be obtained, but instead of classical representations we have to use 'n-square relativised representations' [6, definition 5.7]. But we do not have to follow that particular deviation, we only need the n-dimensional version of part of the preceding theorem.

THEOREM 33. *Let* $3 \leq n < \omega$ *and let* $\mathcal{A}$ *be an atomic relation algebra. If* $\mathcal{A} \in \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ *then* $\exists$ *has a winning strategy in* $F^n(\mathsf{At}\mathcal{A})$.

The proof is very similar to the proof of the implication (3) $\Rightarrow$ (4) of theorem 29. If $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_n$ then $\exists$ always plays hypernetworks N with $\mathsf{nodes}(N) \subseteq n$ such that $\widehat{N} \neq 0$. We omit the details.

The theorems above help us determine whether or not an atomic relation algebra is a strong subalgebra of a member of $\mathfrak{Ra}\mathbf{CA}_\omega$. The next theorem uses the game G and can be used to prove that an atom structure is not in $\mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega)$. This game and the theorem below will help us prove that the inclusion $\mathfrak{Ra}\mathbf{CA}_\omega \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\gamma$ is strict.

THEOREM 34. *Let* α *be a relation algebra atom structure. If* $\alpha \in \mathsf{At}(\mathfrak{Ra}\mathbf{CA}_\omega)$ *then* $\exists$ *has a winning strategy in* $G(\alpha)$.

PROOF. Assume $\alpha = \mathsf{At}(\mathfrak{Ra}\mathcal{C})$ for some $\mathcal{C} \in \mathbf{CA}_\omega$. For all $a \in \alpha$ and $x \in \mathfrak{Ra}\mathcal{C}$ if $a.x \neq 0$ then $a \leq x$, by atomicity of a. By considering $x = s_1^j s_0^i y$ and using facts 3.1 and 3.4–3.9, a non-trivial calculation shows, for all $i < j < \omega$, $a \in \alpha$ and $y \in \mathcal{C}$, that

$$[(\forall k \in \omega \setminus \{i,j\}\mathsf{c}_k y = y) \wedge y\ .\ s_{ij}a \neq 0] \rightarrow s_{ij}a \leq y \tag{5}$$

$\exists$'s strategy is to always play networks N such that $\widehat{N} \neq 0$.

As in the proof of theorem 29(3) $\Rightarrow$ (4), $\exists$ can always play N such that $\widehat{N} \neq 0$ in the initial round and in response to any triangle move by $\forall$. If $\forall$ plays the transformation move (N, θ) then $\exists$ responds with $N\theta$. Since the dimension set is ω and $\mathsf{nodes}(N)$ is finite, by lemma 27(4) we get $\widehat{N\theta} \neq 0$.

If $\forall$ plays an amalgamation move (M, N) where $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) = \{i,j\}$ then $M(i,j) = N(i,j)$. For now we suppose that $i \neq j$, without loss $i < j$.

Let $\mu = \mathsf{nodes}(M) \setminus \{i,j\}$ and let $\nu = \mathsf{nodes}(N) \setminus \{i,j\}$. By lemma 27(1),

$$\mathsf{c}_{(\nu)}\widehat{M} = \widehat{M}$$
$$\mathsf{c}_{(\mu)}\widehat{N} = \widehat{N}$$

and by facts 3.4 and 3.6,

$$\begin{aligned} 0 \neq \mathsf{c}_{(\mu)}\widehat{M} &\leq \mathsf{c}_{(\mu)} s_{ij} M(i,j) \\ &= s_{ij} M(i,j) \end{aligned}$$

Therefore, by (5) applied to $M(i,j)$, $s_{ij}M(i,j) \leq \mathsf{c}_{(\mu)}\widehat{M}$, so

$$\mathsf{c}_{(\mu)}\widehat{M} = s_{ij}M(i,j) = s_{ij}N(i,j) = \mathsf{c}_{(\nu)}\widehat{N}$$

Hence

$$\mathsf{c}_{(\nu)}\widehat{M} = \widehat{M} \leq \mathsf{c}_{(\mu)}\widehat{M} = \mathsf{c}_{(\nu)}\widehat{N}$$

By fact 3.3, it follows that

$$x = \widehat{M} \,.\, \widehat{N} \neq 0$$

If $i = j$ we can still deduce that $\widehat{M} \,.\, \widehat{N} \neq 0$. To see why, suppose $i = j$, so $\mathsf{nodes}(M) \cap \mathsf{nodes}(N) = \{i\}$. Let $M' \supseteq M$ be defined by $\mathsf{nodes}(M') = \mathsf{nodes}(M) \cup \{k\}$ and $M'(i,k) \leq 1'$ (here $k \in \omega \setminus (\mathsf{nodes}(M) \cup \mathsf{nodes}(N))$ is arbitrary) and let $N' \supseteq N$ be defined by $\mathsf{nodes}(N') = \mathsf{nodes}(N) \cup \{k\}$ and $N'(i,k) \leq 1'$. Since $M' \supseteq M$ we have $\widehat{M'} \leq \widehat{M}$ and similarly $\widehat{N'} \leq \widehat{N}$. By the previous case (where $|\mathsf{nodes}(M) \cap \mathsf{nodes}(N)| = 2$) we get $0 \neq \widehat{M'} \,.\, \widehat{N'} \leq \widehat{M} \,.\, \widehat{N} = x$, say.

By lemma 26 there is a network L with $\mathsf{nodes}(L) = \mathsf{nodes}(M) \cup \mathsf{nodes}(N) \neq 0$ and $\widehat{L} \,.\, x \neq 0$. This implies $\widehat{L} \,.\, \widehat{M} \neq 0$ so by lemma 26 $\widehat{L} \equiv^{\mathsf{nodes}(M)} \widehat{M}$. It follows that $L \supseteq M$ and similarly $L \supseteq N$, so L is a legal response to the amalgamation move.

$\dashv$

LEMMA 35. *If α is a countable relation algebra atom structure and $\exists$ has a winning strategy in $G(\alpha)$ then $\mathfrak{Cm}(\alpha)$ has a complete representation in which for any partial isomorphism ι of size two or less and any finite subset X of the domain of the representation there is a partial isomorphism θ extending ι with X contained within its range.*

PROOF. A minor complication arises due to the fact that α might be the atom structure of a non-simple relation algebra. Let C the the set of consistent triples of α. Define a binary relation $\sim$ over α by $a \sim b \iff [\exists c, d, f \in \alpha,\ (c,a,d), (d,f,b) \in C]$. The properties of relation algebra atom structures (see section 2.5) prove that $\sim$ is an equivalence relation (in fact $a \sim b$ iff a and b belong to the same simple component of a subdirect representation of $\mathfrak{Cm}\alpha$). Let $A \subseteq \alpha$ contain exactly one atom from each $\sim$-equivalence class. [This means that A has one representative atom from each of the simple components of $\mathfrak{Cm}\alpha$.]

Let $a \in A$. Next we define a nested sequence of networks $N_0 \subseteq N_1 \subseteq \ldots$. Let N_0 be $\exists$'s response, using her winning strategy, to the $\forall$-move a in the initial round. We have to schedule a sequence of extensions according to a fair system. Suppose $N_0 \subseteq \ldots \subseteq N_r$ has been defined and that each network $N_i \quad (i \leq r)$ occurs in a play of $G(\alpha)$ in which $\exists$ uses her winning strategy. Consider the following requirements to extend N_r.

1. If $N_r(i,j) \leq b; b'$, for some $i \leq j \in \mathsf{nodes}(N_r)$, some $b, b' \in \alpha$, we seek $N_s \supseteq N_r$ (some $s \geq r$) with a node $k \in \omega \setminus \mathsf{nodes}(N_r)$ such that $N_s(i,k) = b,\ N_s(k,j) = b'$.

2. If there are $i, j, i', j' \in \mathsf{nodes}(N_r)$ such that $N_r(i,j) = N_r(i',j')$ (equivalently $\iota = \{(i', i), (j', j)\}$ is a partial isomorphism of N_r), we seek a finite surjection θ extending ι, mapping onto $\mathsf{nodes}(N_r)$ such that $\mathsf{dom}(\theta) \cap \mathsf{nodes}(N_r) = \{i', j'\}$, and we seek an extension $N_s \supseteq N_r, N_r\theta$ (some $s \geq r$).

Since α is countable there are countably many of these requirements to extend. Since our sequence of networks is nested, these requirements to extend remain in all subsequent rounds. So we can schedule these requirements to extend so that eventually, every requirement gets dealt with.

Now, if we are required to find $k \in \omega \setminus \mathsf{nodes}(N_r)$ and $N_{r+1} \supseteq N_r$ such that $N_{r+1}(i,k) = b$, $N_{r+1}(k,j) = b'$ (case 1), then let $k \in \omega \setminus \mathsf{nodes}(N_r)$ be least possible (for definiteness) and let N_{r+1} be $\exists$'s response, using her winning strategy, to the $\forall$-move (N_r, i, j, k, b, b'). For an extension of type 2, let ι be a partial isomorphism of N_r of size two and let θ be any finite surjection onto $\mathsf{nodes}(N_r)$ such that $\mathsf{dom}(\theta) \cap \mathsf{nodes}(N_r) = \{i', j'\}$. $\exists$'s response to the $\forall$-move (N_r, θ) is necessarily $N_r\theta$. Let N_{r+1} be her response, using her winning strategy, to the subsequent $\forall$-move $(N_r, N_r\theta)$. Observe that in this latter case, θ is a partial isomorphism of N_{r+1} with $\mathsf{rng}(\theta) = \mathsf{nodes}(N_r)$ and $\mathsf{dom}(\theta) = \mathsf{nodes}(N_r\theta)$.

This defines how we construct the sequence $N_0 \subseteq N_1 \subseteq \ldots$. Let N_a be the limit of this sequence (see definition 23, this is well-defined since the sequence is nested). Observe that if $\iota = \{(i', i), (j', j)\}$ is any partial isomorphism of N_a and X is any finite subset of $\mathsf{nodes}(N_a)$ then

$$\text{there is a partial isomorphism } \theta \supseteq \iota,\ \mathsf{rng}(\theta) \supseteq X \tag{6}$$

Also note that for $b \in \alpha$,

$$b \text{ occurs as the label of some edge of } N_a \iff b \sim a \tag{7}$$

Rename the nodes, if necessary, so that $a \neq b \in A$ implies $\mathsf{nodes}(N_a) \cap \mathsf{nodes}(N_b) = \emptyset$.

Now define a representation $\mathcal{N}$ of $\mathfrak{Cm}(\alpha)$ with base $\bigcup_{a \in A} \mathsf{nodes}(N_a)$, by

$$S^{\mathcal{N}} = \{(i,j) : \exists a \in A,\ \exists s \in S,\ N_a(i,j) = s\}$$

for any subset S of α. By lemma 16, $\mathcal{N}$ is a complete representation of $\mathfrak{Cm}\alpha$. By (7), any partial isomorphism of $\mathcal{N}$ fixes each component N_a setwise. By (6), for every partial isomorphism ι of size two or less and every finite subset X of the domain of $\mathcal{N}$ there is a partial isomorphism $\theta \supseteq \iota$ with $\mathsf{rng}(\theta) \supseteq X$. $\dashv$

THEOREM 36. *The inclusion $\mathfrak{Ra}\mathbf{CA}_\omega \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega$ is strict.*

PROOF. A relation algebra is integral if its identity is an atom. A permutational representation of an integral relation algebra is one in which, for any pair of points x, y, there is an automorphism of the representation taking x to y (in model theory this kind of representation is called *transitive*). An integral relation algebra is called non-permutational if none of its representations is permutational. In [1] a finite, integral, representable, non-permutational relation algebra $\mathcal{A}$ is defined and it is shown that the representations of $\mathcal{A}$ are all finite (they have size 45). Since $\mathcal{A}$ is finite and representable it is completely representable, so by theorem 29 it belongs to $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega$.

Since all representations of $\mathcal{A}$ are finite and not permutational, in any representation of $\mathcal{A}$ there is a partial isomorphism of size one that does not extend to an automorphism of the representation. Hence, by lemma 35, $\forall$ has a winning strategy in $G(\mathsf{At}(\mathcal{A}_n))$, so by theorem 34 it does not belong to $\mathfrak{Ra}\mathbf{CA}_\omega$. This proves that the inclusion in the theorem is strict. $\dashv$

PROBLEM 37. *For which finite values n is it the case that the inclusion $\mathfrak{Ra}\mathbf{CA}_n \subseteq \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ is strict?*

One suggestion here is the following. For $n < \omega$, define a game G^n, like G but played on networks N with $\mathsf{nodes}(N) \subseteq n$, show that $\alpha \in \mathfrak{Ra}\mathbf{CA}_n$ implies $\exists$ has a winning strategy in $G^n(\alpha)$. Now use the fact that $\mathcal{A}$ (above) has only non-permutational representations and they all have size 45 to show for $n \geq 45$ that the inclusion $\mathfrak{Ra}\mathbf{CA}_n \subset \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ is strict.

PROBLEM 38. *In fact [1] define a whole sequence $\mathcal{A}_n$ of finite, non-permutational relation algebras and prove that a non-principal ultraproduct $\mathcal{B}$ of the $\mathcal{A}_n$ has a permutational representation. If it could be shown that $\mathcal{B}$ has a homogeneous representation, where arbitrary finite partial isomorphisms extend to full automorphisms, then it would follow that $\exists$ has a winning strategy in $H(\mathsf{At}(\mathcal{B}))$ so a countable elementary subalgebra of $\mathcal{B}$ would belong to $\mathfrak{Ra}\mathbf{CA}_\omega$, by theorem 39, below. This would show that $\mathfrak{Ra}\mathbf{CA}_\omega$ cannot be defined by finitely many axioms over $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_\omega$.*

We have now established techniques to determine whether a relation algebra is in $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ (theorem 29) and to prove that a relation algebra is not in $\mathfrak{Ra}\mathbf{CA}_n$ (theorem 34). The next theorem will be useful to prove that an atomic relation algebra is in $\mathfrak{Ra}\mathbf{RCA}_\omega$. Recall that $H(\alpha)$ is the hypernetwork game of definition 28 where the nodes of any hypernetwork played form a finite subset of ω.

THEOREM 39. *Let α be a countable relation algebra atom structure. If $\exists$ has a winning strategy in $H(\alpha)$ then there is $\mathcal{C} \in \mathbf{CA}_\omega$ such that $\mathfrak{Ra}(\mathcal{C})$ is atomic and $\mathsf{At}(\mathfrak{Ra}(\mathcal{C})) \cong \alpha$.*

PROOF. In fact we'll construct $\mathcal{C} \in \mathbf{RCA}_\omega$. Suppose $\exists$ has a winning strategy in $H(\alpha)$. Fix some $a \in \alpha$. As in the proof of lemma 35 we can define a nested sequence $N_0 \subseteq \ldots$ (but here they are hypernetworks) where N_0 is $\exists$'s response to the initial $\forall$-move a, so that:

1. If N_r is in the sequence and $N_r(i,j) \leq b; b'$ then there is $s \geq r$ and $k \in \mathsf{nodes}(N_s)$ such that $N_s(i,k) = b$, $N_s(k,j) = b'$.
2. If N_r is in the sequence and θ is any partial isomorphism of N_r then there is $s \geq r$ and a partial isomorphism θ^+ of N_s extending θ such that $\mathsf{rng}(\theta^+) \supseteq \mathsf{nodes}(N_r)$.

The difference is that here we extend arbitrary finite partial isomorphisms whereas in lemma 35 we only extended partial isomorphisms of size one or two. The more general kind of amalgamation move in $H(\alpha)$ means that this can be done. We omit the details which are very similar to the proof of lemma 35. Now let N_a be the limit of this sequence. This limit is well-defined since the hypernetworks are nested. Note, for $b \in \alpha$, that

(8) $$(\exists i, j \in \mathsf{nodes}(N_a),\ N_a(i,j) = b) \iff b \sim a$$

Let θ be any finite partial isomorphism of N_a and let X be any finite subset of $\mathsf{nodes}(N_a)$. Since θ, X are finite, there is $i < \omega$ such that $\mathsf{nodes}(N_i) \supseteq X \cup \mathsf{dom}(\theta)$. There is a bijection $\theta^+ \supseteq \theta$ onto $\mathsf{nodes}(N_i)$ and $j \geq i$ such that $N_j \supseteq N_i, N_i\theta^+$. Then θ^+ is a partial isomorphism of N_j and $\mathsf{rng}(\theta^+) = \mathsf{nodes}(N_i) \supseteq X$. Hence, if θ is any finite partial isomorphism of N_a and X is any finite subset of $\mathsf{nodes}(N_a)$ then

(9) $$\exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_a \text{ where } \mathsf{rng}(\theta^+) \supseteq X$$

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of $\mathsf{nodes}(N_a)$ within its domain.

We will use the networks $N_a : a \in \alpha$ as the base of a cylindric algebra $\mathcal{C} \in \mathbf{RCA}_\omega$. Let L be the signature with one binary predicate symbol (b) for each $b \in \alpha$, and one k-ary predicate symbol (λ) for each k-ary hyperlabel λ. The set of variables for L-formulas is $\{x_i : i < \omega\}$. Pick $f_a \in {}^\omega\mathsf{nodes}(N_a)$. Let $U_a = \{f \in {}^\omega\mathsf{nodes}(N_a) : \{i < \omega : g(i) \neq f_a(i)\} \text{ is finite}\}$.

We can make U_a into the base of an L-structure $\mathcal{N}_a$ and evaluate L-formulas at $f \in U_a$ as follow. For $b \in \alpha,\ i, j, i_0 \ldots, i_{k-1} < \omega,\ k$-ary hyperlabels λ, and all L-formulas ϕ, ψ, let

$$\begin{aligned}
\mathcal{N}_a, f \models b(x_i, x_j) &\iff N_a(f(i), f(j)) = b \\
\mathcal{N}_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) &\iff N_a(f(i_0), \ldots, f(i_{k-1})) = \lambda \\
\mathcal{N}_a, f \models \neg\phi &\iff \mathcal{N}_a, f \not\models \phi \\
\mathcal{N}_a, f \models (\phi \vee \psi) &\iff \mathcal{N}_a, f \models \phi \text{ or } \mathcal{N}_a, f \models \psi \\
\mathcal{N}_a, f \models \exists x_i \phi &\iff \mathcal{N}_a, f[i/m] \models \phi, \text{ some } m \in \mathsf{nodes}(N_a)
\end{aligned}$$

For any L-formula ϕ, write $\phi^{\mathcal{N}_a}$ for $\{f \in {}^\omega\mathsf{nodes}(N_a) : \mathcal{N}_a, f \models \phi\}$. Let $Form^{\mathcal{N}_a} = \{\phi^{\mathcal{N}_a} : \phi \text{ is an } L\text{-formula}\}$ and define a cylindric algebra

$$\mathcal{C}_a = (Form^{\mathcal{N}_a}, \emptyset, U_a, \cup, \setminus, D_{ij}, C_i : i, j < \omega)$$

where $D_{ij} = 1'(x_i, x_j)^{\mathcal{N}_a}$, $C_i(\phi^{\mathcal{N}_a}) = (\exists x_i \phi)^{\mathcal{N}_a}$. Observe that $\top^{\mathcal{N}_a} = U_a$, $(\phi \vee \psi)^{\mathcal{N}_a} = \phi^{\mathcal{N}_a} \cup \psi^{\mathcal{N}_a}$, etc. Note also that $\mathcal{C}_a$ is a subalgebra of the ω-dimensional cylindric set algebra on the base $\mathsf{nodes}(N_a)$, hence $\mathcal{C}_a \in \mathbf{RCA}_\omega$.

Let $\phi(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ be an arbitrary L-formula using only variables belonging to $\{x_{i_0}, \ldots, x_{i_k}\}$. Let $f, g \in U_a$ (some $a \in \alpha$) and suppose $\{(f(i_0), g(i_0)), (f(i_1), g(i_1)), \ldots, (f(i_k), g(i_k))\}$ is a partial isomorphism of N_a. We can prove by induction over the quantifier depth of ϕ and using (9), that

(10) $$\mathcal{N}_a, f \models \phi \iff \mathcal{N}_a, g \models \phi$$

Let $\mathcal{C} = \prod_{a \in \alpha} \mathcal{C}_a$. By proposition 1, $\mathcal{C} \in \mathbf{RCA}_\omega$. It remains to show that $\alpha \cong \mathsf{At}(\mathfrak{Ra}\mathcal{C})$. An element x of $\mathcal{C}$ has the form $(x_a : a \in \alpha)$, where $x_a \in \mathcal{C}_a$. For $b \in \alpha$ let $\pi_b : \mathcal{C} \to \mathcal{C}_b$ be the projection defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : \mathcal{C}_a \to \mathcal{C}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Evidently $\pi_b(\iota_b(y)) = y$ for $y \in \mathcal{C}_b$ and $\pi_b(\iota_a(y)) = 0$ if $a \neq b$.

Suppose $x \in \mathfrak{Ra}(\mathcal{C}) \setminus \{0\}$. Since $x \neq 0$ it must have a non-zero component $\pi_a(x) \in \mathcal{C}_a$, for some $a \in \alpha$. Say $\emptyset \neq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a} = \pi_a(x)$ for some L-formula $\phi(x_{i_0}, \ldots, x_{i_k})$. We have $\phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a} \in \mathfrak{Ra}(\mathcal{C}_a)$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a}$ and let $b = N_a(f(0), f(1)) \in \alpha$. We will show that $b(x_0, x_1)^{\mathcal{C}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a}$. For this, take any $g \in b(x_0, x_1)^{\mathcal{C}_a}$, so $N_a(g(0), g(1)) = b$. The map $\{(f(0), g(0)), (f(1), g(1))\}$ is a partial isomorphism of N_a — here it is crucial that short hyperedges have constant label λ_0. By (9) this extends to a finite partial isomorphism θ of N_a whose domain includes $f(i_0), \ldots, f(i_k)$. Let $g' \in U_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \mathsf{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

By (10), $\mathcal{N}_a, g' \models \phi(x_{i_0}, \ldots, x_{i_k})$. Observe that $g'(0) = \theta(0) = g(0)$ and similarly $g'(1) = g(1)$, so g is identical to g' over $\{0, 1\}$ and it differs from g' on only a finite set of coordinates. Since $\phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a} \in \mathfrak{Ra}(\mathcal{C})$ we deduce $\mathcal{N}_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a}$. This proves that $b(x_0, x_1)^{\mathcal{C}_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a} = \pi_a(x)$, and so $\iota_a(b(x_0, x_1)^{\mathcal{C}_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_k})^{\mathcal{C}_a}) \leq x \in \mathcal{C} \setminus \{0\}$. Hence every non-zero element x of $\mathfrak{Ra}\mathcal{C}$ is above a non-zero element $\iota_a(b(x_0, x_1)^{\mathcal{C}_a})$ (some $a, b \in \alpha$) and these latter elements are the atoms of $\mathfrak{Ra}\mathcal{C}$. So $\mathfrak{Ra}\mathcal{C}$ is atomic and $\alpha \cong \mathsf{At}(\mathfrak{Ra}\mathcal{C})$ — the isomorphism is $b \mapsto (b(x_0, x_1)^{\mathcal{C}_a} : a \in A)$. ⊣

§5. Rainbow algebra.

DEFINITION 40. *We define a* rainbow algebra atom structure α *(in the terminology of [6, §16.2] it is very similar, though not identical, to* $\mathsf{At}(\mathcal{A}_{\mathbb{Z},\mathbb{N}})$*).*

Let F be the set of partial, order preserving functions $f : \mathbb{Z} \to \mathbb{N}$ where $|\mathsf{dom}(f)| \leq 2$. The atoms of α are $\{1', \mathsf{y}, \mathsf{b}, \mathsf{w}\} \cup \{\mathsf{g}_i : i \in \mathbb{Z}\} \cup \{\mathsf{w}_f : f \in F\} \cup \{\mathsf{r}_{ij} : i, j \in \mathbb{N}\}$. Non-identity atoms have colours: y is yellow, b is black, w, w_f are white, g_i is green and r_{ij} is red. All atoms are self-converse except the red atoms, for these $\mathsf{r}_{ij}^{\smile} = \mathsf{r}_{ji}$. Composition of atoms is defined by listing the forbidden triples of atoms *(the set of consistent triples of atoms is the complement in $\alpha \times \alpha \times \alpha$ of the set of forbidden triples). The forbidden triples (a, b, c) are those where $a, b, c \in \alpha$ and $a; b \not\geq c$. If (a, b, c) is a forbidden triple of atoms, its* Peircean transforms $(a, b, c), (b, c^{\smile}, a^{\smile}), (c^{\smile}, a, b^{\smile}), (b^{\smile}, a^{\smile}, c^{\smile}), (a^{\smile}, c, b), (c, b^{\smile}, a)$ *are also forbidden. The forbidden triples of atoms of α are the Peircean transforms of the following.*

(11) $(1', x, y)$ *unless* $x = y$

(12) $(\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{g}_{i^*}), (\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{w})$, *any* $i, i', i^* \in \mathbb{Z}$, *any* $f \in F$
$(\mathsf{g}_i, \mathsf{g}_{i'}, \mathsf{w}_f)$

(13) $(\mathsf{y}, \mathsf{y}, \mathsf{y}), (\mathsf{y}, \mathsf{y}, \mathsf{b})$

(14) $(\mathsf{g}_i, \mathsf{y}, \mathsf{w}_f)$ *unless* $i \in \mathsf{dom}(f)$

(15) $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{r}_{kl})$ *unless* $\{(i, k), (j, l)\}$ *is an order-preserving partial function* $\mathbb{Z} \to \mathbb{N}$

(16) $(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*})$ *unless* $i = i^*$, $j = j'$ *and* $k' = k^*$

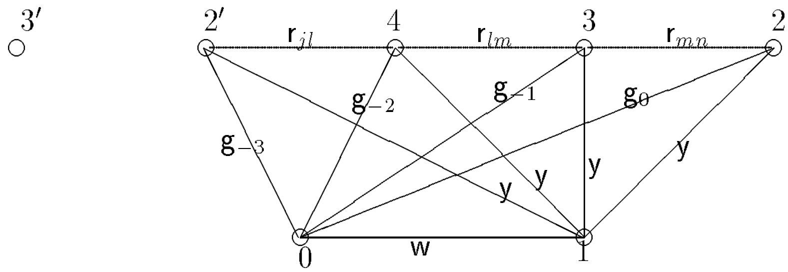


FIGURE 3. How $\forall$ can win $F^5(\alpha)$

and no other triple of atoms is forbidden.

Let $\mathcal{A}$ be the complex algebra over α (so the domain of $\mathcal{A}$ consists of arbitrary sets of atoms).

We will show that $\mathcal{A} \notin \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$, but an elementary extension $\mathcal{A}'$ of $\mathcal{A}$ belongs to $\mathfrak{Ra}\mathbf{RCA}_\omega$.

LEMMA 41. *For any relation algebra $\mathcal{B}$ such that $\mathsf{At}(\mathcal{B}) = \alpha$, we have $\mathcal{B} \notin \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$. The rainbow algebra $\mathcal{A}$ (definition above) is not in $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$.*

PROOF. We prove that $\forall$ has a winning strategy in $F^5(\alpha)$, see figure 3. In the initial round $\forall$ plays w and $\exists$ must play a network N_0 with $N_0(0,1) = \mathsf{w}$. In the next round $\forall$ plays the triangle move $(N_0, 0, 1, 2, \mathsf{g}_0, \mathsf{y})$ and $\exists$ must play a network $N_1 \equiv_2 N_0$ with $N_1(0,2) = \mathsf{g}_0$, $N_1(2,1) = \mathsf{y}$. In the following round $\forall$ plays the triangle move $(N_1, 0, 1, 3, \mathsf{g}_{-1}, \mathsf{y})$ and $\exists$ must play $N_2 \equiv_3 N_1$ with $N_2(0,3) = \mathsf{g}_{-1}$, $N_2(3,1) = \mathsf{y}$. $\exists$ must choose an atomic label for the edge $(3,2)$ of N_2. By considering the triangle $(2,3,0)$ we see that the identity, a green atom or a white atom are impossible (see forbidden triples 11, 12). From the triangle $(2,3,1)$ we see that the yellow atom or the black atom are impossible (forbidden triple 13). So $\exists$ must let $N_2(3,2)$ be a red atom, say r_{mn} (some $m, n \in \mathbb{N}$) and since $-1 < 0$ we must have $m < n$ (forbidden triple 15). In the next move $\forall$ plays the triangle move $(N_3, 0, 1, 4, \mathsf{g}_{-2}, \mathsf{y})$ and $\exists$ must play $N_3 \equiv_4 N_2$ such that $N_3(0,4) = \mathsf{g}_{-2}$, $N_3(4,1) = \mathsf{y}$. As before we must have $N_3(4,3)$ and $N_3(4,2)$ both being red atoms and from the triangle $(2,3,4)$ we see that the indices of these red atoms must match (forbidden triple 16), so we have $N_3(4,3) = \mathsf{r}_{ln}$, $N_3(4,2) = \mathsf{r}_{lm}$, for some $l < m \in \mathbb{N}$.

In the next round $\forall$ plays $(N_3, 0, 1, 2, \mathsf{g}_{-3}, \mathsf{y})$ and $\exists$ must play $N_4 \equiv_2 N_3$ with $N_4(0,2) = \mathsf{g}_{-3}$, $N_4(2,1) = \mathsf{y}$. In figure 3, node 2 of N_4 is marked $2'$ to distinguish it from node 2 of N_3. This time we get $N_4(3,2) = \mathsf{r}_{jl}$ for some $j < l \in \mathbb{N}$. In this way $\forall$ can force an infinite descending sequence of natural numbers $n > m > l > j > \ldots$. This is impossible. Hence $\exists$ has no winning strategy.

By theorem 33, $\alpha \notin \mathsf{At}(\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5)$. ⊣

Recall from definition 28 that $H_n(\alpha)$ is the hypernetwork game with n rounds.

REMARK 42. *It will simplify things a bit if we alter the rules of the game $H(\alpha)$ slightly so that only strict hypernetworks are played. In the initial round if $\forall$ plays a then $\exists$ can always play a strict hypernetwork N_0 where $\mathsf{nodes}(N_0) = \{0\}$*

if $a \leq 1'$ *and* $\mathsf{nodes}(N_0) = \{0,1\}$ *otherwise. In the former case* $N_0(0,0) = a$ *and in the latter case the edge labelling is completely determined by* $N_0(0,1) = a$.

The restrictions we impose on $\forall$*'s moves are*

- $\forall$ *is only allowed to play a triangle move* (N, i, j, k, a, b) *if there does not exist* $l \in \mathsf{nodes}(N)$ *such that* $N(i,l) = a$ *and* $N(l,j) = b$.
- $\forall$ *is only allowed to play transformation moves* (N, θ) *if* θ *is injective.*
- $\forall$ *is only allowed to play an amalgamation move* (M, N) *if for all* $m \in \mathsf{nodes}(M) \setminus \mathsf{nodes}(N)$ *and all* $n \in \mathsf{nodes}(N) \setminus \mathsf{nodes}(M)$ *the map* $\{(m,n)\} \cup \{(x,x) : x \in \mathsf{nodes}(M) \cap \mathsf{nodes}(N)\}$ *is not a partial isomorphism. I.e. he can only play* (M, N) *if the amalgamated part is 'as large as possible'.*

If, as a result of these restrictions, $\forall$ *cannot move at some stage then he loses and the game halts.*

It is easy to check that $\forall$ *has a winning strategy in* $H(\alpha)$ *iff he has a winning strategy with these restrictions to his moves. Also, if* $\forall$ *plays with these restrictions to his moves, if* $\exists$ *has a winning strategy then she has a winning strategy which only directs her to play strict hypernetworks. The same holds when we consider* $H_n(\alpha)$*. We will assume that* $\forall$ *plays according to these restrictions and* $\exists$ *only plays strict hypernetworks in* $H(\alpha)$ *and* $H_n(\alpha)$.

LEMMA 43. $\exists$ *has a winning strategy in* $H_n(\alpha)$*, for any* $n < \omega$.

PROOF. In a play of $N_n(\alpha)$, $\exists$ is required to play λ_0-neat hypernetworks, so she has no choice about the hyperlabels used for short edges — she must label these with λ_0. $\exists$ uses the *default strategy* for choosing hyperlabels for long hyperedges, as follows. In response to a triangle move (N, i, j, k, a, b), all long hyperedges not incident with k necessarily keep the hyperlabel they had in N. By remark 42, we are assuming $a \neq 1'$ and $b \neq 1'$. All long hyperedges incident with k in M are given unique hyperlabels, not occurring as the hyperlabel of any previously played hypernetwork and not occurring as the hyperlabel of any other hyperedge in M. We assume we have an infinite supply of hyperlabels of all finite arities, so this is possible. In response to an amalgamation move (M, N) all long hyperedges whose range is contained in $\mathsf{nodes}(M)$ have hyperlabel determined by M, and those whose range is contained in $\mathsf{nodes}(N)$ have hyperlabel determined by N. If $\bar{x}$ is a long hyperedge of $\exists$'s response L where $\mathsf{rng}(\bar{x}) \not\subseteq \mathsf{nodes}(M), \mathsf{nodes}(N)$ then $\bar{x}$ is given a new hyperlabel, not used in any previously played hypernetwork and not used within L as the label of any hyperedge other than $\bar{x}$. This completes the definition of her strategy for labelling hyperedges. Condition IV in definition 23 is clearly satisfied by this. [In fact, the only function served by these hyperlabels is to restrict the possible amalgamation moves that $\forall$ can make in future rounds.]

Before we give $\exists$'s strategy for edge labelling, we need some more notation and terminology. Every irreflexive edge of any hypernetwork played in the game has an owner, $\forall$ or $\exists$. We call such edges $\forall$-edges or $\exists$-edges, as appropriate. And a long hyperedge $\bar{x}$ in a hypernetwork N occurring in the play has an *envelope* $\nu_N(\bar{x}) \subseteq \mathsf{nodes}(N)$. We will see that although our hypernetworks are all strict, it is not necessarily the case that hyperlabels label unique hyperedges — amalgamation moves can force the same hyperlabel to label more than one hyperedge. However, we will be able to prove that *within the envelope* of a hyperedge $\bar{x}$ of a N, the hyperlabel $N(\bar{x})$ is unique (see the claim, below). Lets explain this more carefully.

In the initial round, if $\forall$ plays $a \in \alpha$ and $\exists$ plays N_0 then all irreflexive edges of N_0 belong to $\forall$. There are no long hyperedges in N_0. If, in a later round, $\forall$ plays the transformation move (N, θ) and $\exists$ responds with $N\theta$ then owners and envelopes are inherited in the obvious way: $(\theta(m), \theta(n))$ is a $\forall$-edge of N iff (m, n) is a $\forall$-edge of $N\theta$ (any $m \neq n \in \mathsf{dom}(\theta)$), and $\nu_N(\theta(\bar{x})) = \nu_{N\theta}(\bar{x})$ (any long hyperedge $\bar{x}$ of $N\theta$). If $\forall$ plays a triangle move (N, i, j, k, a, b) and $\exists$ responds with M then the owner in M of an edge not incident with the new node k is the same as it was in N and the envelope in M of a long hyperedge not incident with k is the same as it was in N. By remark 42 we know that $a \neq 1'$ and $b \neq 1'$. The edges $(i, k), (k, i), (j, k), (k, j)$ belong to $\forall$ in M, all edges $(l, k), (k, l)$ for $l \in \mathsf{nodes}(N) \setminus \{i, j\}$ belong to $\exists$ in M. If $\bar{x}$ is any long hyperedge of M with $k \in \mathsf{rng}(\bar{x})$ then $\nu_M(\bar{x}) = \mathsf{nodes}(M)$.

If $\forall$ plays the amalgamation move (M, N) and $\exists$ responds with L then, for $m \neq n \in \mathsf{nodes}(L)$, the owner in L of an edge (m, n) is $\forall$ if it belongs to $\forall$ in either M or N; in all other cases (either it belongs to $\exists$ in M or it is not an edge of M, and either it belongs to $\exists$ in N or it is not an edge of N) it belongs to $\exists$ in L. If $\bar{x}$ is a long hyperedge of L then

$$\nu_L(\bar{x}) = \begin{cases} \nu_M(\bar{x}) & \text{if } \mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(M) \\ \nu_N(\bar{x}) & \text{if } \mathsf{rng}(\bar{x}) \subseteq \mathsf{nodes}(N),\ \mathsf{rng}(\bar{x}) \not\subseteq \mathsf{nodes}(M) \\ \mathsf{nodes}(M) & \text{otherwise} \end{cases}$$

In fact the first two parts of the following claim show that if $\bar{x} \subseteq \mathsf{nodes}(M) \cap \mathsf{nodes}(N)$ then $\nu_M(\bar{x}) = \nu_N(\bar{x})$. This completes the definition of owners and envelopes.

CLAIM: Let M, N occur in a play of $H(\alpha)$ in which $\exists$ uses the default labelling for hyperedges. Let $\bar{x}$ be a long hyperedge of M and let $\bar{y}$ be a long hyperedge of N.

1. For any hyperedge $\bar{x}'$ with $\mathsf{rng}(\bar{x}') \subseteq \nu_M(\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$.
2. If $\bar{x}$ is a long hyperedge of M and $\bar{y}$ is a long hyperedge of N and $M(\bar{x}) = N(\bar{y})$ then there is a local isomorphism $\theta : \nu_M(\bar{x}) \to \nu_N(\bar{y})$ such that $\theta(x_i) = y_i$, for $i < |\bar{x}|$.
3. For any $x \in \mathsf{nodes}(M) \setminus \nu_M(\bar{x})$ and $S \subseteq \nu_M(\bar{x})$, if (x, s) belongs to $\forall$ in M, for all $s \in S$, then $|S| \leq 2$.

The claim can be proved by a simple induction over the number of rounds taken before M and N are played.

Now we define $\exists$'s strategy for choosing the labels for edges in response to $\forall$-moves. Let $N_0, N_1, \ldots, N_r$ be the start of a play of $H_n(\alpha)$ just before round $r + 1$ (where $r < n$). $\exists$ computes partial functions $\rho_s : \mathbb{Z} \to \mathbb{N}$, for $s \leq r$. Inductively, for each $s \leq r$, suppose:

I. If $N_s(x, y)$ is green or yellow then (x, y) belongs to $\forall$ in N_s.
II. $\rho_0 \subseteq \ldots \subseteq \rho_r$,
III. $\mathsf{dom}(\rho_s) = \{i \in \mathbb{Z} : \exists t \leq s,\ x, y \in \mathsf{nodes}(N_t),\ N_t(x, y) = \mathsf{g}_i\}$.

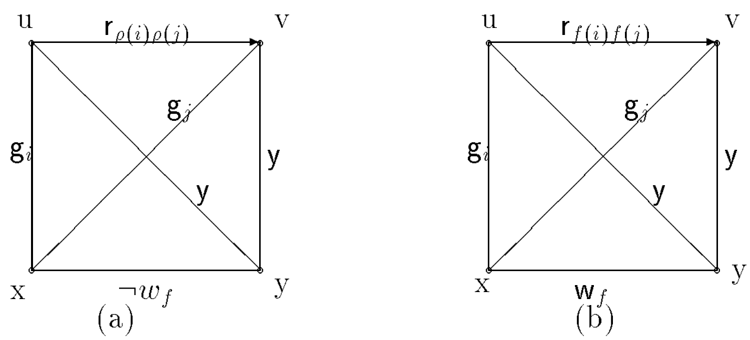


FIGURE 4. Property V and red indices

IV. ρ_s is order preserving: if $i < j \in \mathsf{dom}(\rho_s)$ then $\rho_s(i) < \rho_s(j)$. The range of ρ_s is 'widely spaced': if $i < j \in \mathsf{dom}(\rho_s)$ then $\rho_s(i), (\rho_s(j) - \rho_s(i)) \geq 3^{n-r}$ ($n - r$ is the number of rounds remaining in the game).

V. For $u, v, x, y \in \mathsf{nodes}(N_s)$, if $N_s(u,v) = \mathsf{r}_{\gamma,\delta}$, $N_s(x,u) = \mathsf{g}_i$, $N_s(x,v) = \mathsf{g}_j$, $N_s(y,u) = N_s(y,v) = \mathsf{y}$ then
 (a) if $N_s(x,y) \neq \mathsf{w}_f$ (all $f \in F$) then $\rho_s(i) = \gamma$, $\rho_s(j) = \delta$,
 (b) if $N_s(x,y) = \mathsf{w}_f$ (some $f \in F$) then $\gamma = f(i)$, $\delta = f(j)$.
 See figure 4.

VI. N_s is a strict λ_0-neat hypernetwork.

To start with if $\forall$ plays $a \neq 1'$ in the initial round then $\mathsf{nodes}(N_0) = \{0, 1\}$, the edge labelling of N_0 is determined by $N_0(0,1) = a$. If $\forall$ plays $1'$ then $\mathsf{nodes}(N_0) = \{0\}$ and $N_0(0,0) = 1'$. If $a = \mathsf{g}_p$ (some $p \in \mathbb{Z}$) let $\rho_0 = \{(p, 3^n)\}$, otherwise let $\rho_0 = \emptyset$. All properties hold when $r = 0$.

Suppose the properties hold after round r (some $r < n$). We'll define how $\exists$ chooses atoms for new edges and maintains the properties above in response to a $\forall$-move in round $r+1$. In response to a transformation move (N, θ) $\exists$ has nothing to do: her response, $N_{r+1} = N\theta$, is forced. There are no new edge labels, so she lets $\rho_{r+1} = \rho_r$.

In response to a triangle move $(N_s, i, j, k, \mathsf{g}_p, \mathsf{g}_q)$ by $\forall$ (some $s \leq r$ and some $p, q \in \mathbb{Z}$), $\exists$ must extend ρ_r to ρ_{r+1} so that $p, q \in \mathsf{dom}(\rho_{r+1})$ (property III) and the gap between elements of its range is at least 3^{n-r-1} (property IV). Inductively, ρ_r is order-preserving and the gap between elements of its range is at least 3^{n-r}, so this can be maintained. If $\forall$ chooses non-green atoms, green atoms with the same suffix, or green atoms whose suffices already belong to $\mathsf{dom}(\rho_r)$, there would be fewer elements to add to the domain of ρ_{r+1} so it only makes it easier for $\exists$ to define ρ_{r+1}. This establishes properties (II–IV) for round $r+1$.

To choose edge labels in response to a triangle move by $\forall$, $\exists$ uses her normal strategy for rainbow algebras. In rough outline: she chooses a white atom if possible, else the black atom, and if neither of these is consistent then she chooses a red atom. In the first of these cases she chooses a white atom for the new edge under the circumstances that this does not complete a triangle where a forbidden triple of atoms listed under (12) would result. In this case, she could easily choose

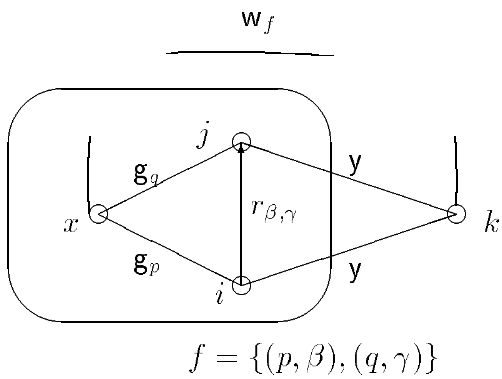


FIGURE 5. Defining the suffix f

the white atom w and avoid all inconsistencies in that round, but because she has an eye to future $\forall$ moves, she very carefully selects an appropriate atom w_f for some $f \in F$, avoiding forbidden triples of atoms (14), so as to restrict $\forall$'s moves in later rounds. In the second of these cases it is not consistent to choose a white atom but the black atom is consistent because it does not complete a triangle where a forbidden triple of atoms listed under (13) is exhibited. This case is straight-forward. Finally, if a white atom and the black atom are both inconsistent then she chooses a red atom. This case is tricky, but she uses the functions ρ_s and the suffix f in a label w_f to help her choose the suffices of red atoms for this case.

Now we explain this strategy in more detail. Let $\forall$ play the triangle move (N_s, i, j, k, a, b) in round $r+1$. $\exists$ has to choose labels for the edges $\{(x, k), (k, x) : x \in \mathsf{nodes}(N_s) \setminus \{i, j\}\}$. She chooses the labels for the edges (x, k) one at a time, this then determines the labels of the reverse edges (k, x) uniquely. She selects the first permissible option below. Property I is clear in all cases since the only atoms $\exists$ chooses are white, black or red.

1. Suppose it is not the case that $N_s(x, i)$ and a are both green, and it is not the case that $N_s(x, j)$ and b are both green. Let $S = \{p \in \mathbb{Z} : (N_s(x, i) = \mathsf{g}_p \wedge a = \mathsf{y}) \vee (N_s(x, i) = \mathsf{y} \wedge a = \mathsf{g}_p) \vee (N_s(x, j) = \mathsf{g}_p \wedge b = \mathsf{y}) \vee (N_s(x, j) = \mathsf{y} \wedge b = \mathsf{g}_p)\}$. Clearly $|S| \leq 2$. $\exists$ lets $N_{s+1}(x, k) = \mathsf{w}_f$ for some $f \in F$ with $\mathsf{dom}(f) = S$, which we define next. Since $\mathsf{dom}(f) \supseteq S$ and since $\exists$ does not choose green or yellow for her edges, this will avoid all forbidden triples of atoms (12) and (14) and these are the only forbidden triples including a white atom.

   Suppose $N_s(i, j) = \mathsf{r}_{\beta,\gamma}$ (some $\beta, \gamma \in \mathbb{N}$), $N_s(x, i) = \mathsf{g}_p$, $N_s(x, j) = \mathsf{g}_q$ (some $p, q \in \mathbb{Z}$) and $a = b = \mathsf{y}$, see figure 5. By property VI and forbidden triple (15), $f = \{(p, \beta), (q, \gamma)\}$ is an order-preserving function. $\exists$ lets $N_{s+1}(x, k) = \mathsf{w}_f$ in this case. Similarly, if $N_s(i, j) = \mathsf{r}_{\beta,\gamma}$, $N_s(x, i) = N_s(x, j) = \mathsf{y}$, $a = \mathsf{g}_p$, $b = \mathsf{g}_q$ then $\exists$ lets $f = \{(p, \beta), (q, \gamma)\}$ and $N_{s+1}(x, k) = \mathsf{w}_f$ (here we use the fact that $a; b \geq \mathsf{r}_{\beta,\gamma}$ to prove that f is order-preserving). By definition, $\mathsf{dom}(f) = \{p, q\} = S$, as promised.

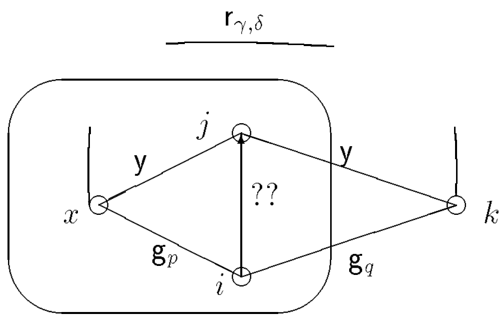


FIGURE 6. (x,k) is given a red label

In all other cases (either $N_s(i,j)$ is not red or if it is then is not the case that $N_s(x,i), N_s(x,j)$ are both green and $a = b = \mathsf{y}$ and it is not the case that $N_s(x,i) = N_s(x,j) = \mathsf{y}$ and a, b are both green) $\exists$ lets $f : S \to \mathbb{N}$ be an arbitrary order-preserving function (e.g. if $S = \{p, q\}$ and $p < q$ let $f(p) = 0,\ f(q) = 1$).

Having defined f $\exists$ lets $N_{r+1}(x,k) = \mathsf{w}_f$. This maintains property V for round $r+1$.

The only forbidden triples of atoms involving w_f are (12) and (14) of definition 40. Since $\exists$ does not choose green or yellow atoms to label new edges and $N_{r+1}(x,k) = \mathsf{w}_f$, all triangles involving the new edge (x,k) are consistent in N_{r+1}, so property VI holds after round $r+1$.

2. Else, if it is not the case that $N_s(x,i) = a = \mathsf{y}$ and it is not the case that $N_s(x,j) = b = \mathsf{y}$, $\exists$ lets $N_{r+1}(x,k) = \mathsf{b}$. Property V is not applicable in this case. The only forbidden triple involving the atom b is (13), so all triangles (x,y,k) are consistent in N_{r+1} and property VI holds after round $r+1$.
3. If neither case above apply, then either $N_s(x,i) = \mathsf{g}_p,\ a = \mathsf{g}_q$ (some p, q) and $N_s(x,j) = b = \mathsf{y}$ or $N_s(x,i) = a = \mathsf{y}$ and $N_s(x,j) = \mathsf{g}_p,\ b = \mathsf{g}_q$. Assume the first alternative, see figure 6. $\exists$ lets $N_{r+1}(x,k) = \mathsf{r}_{\gamma,\delta}$, where γ, δ remain to be specified. There are two subcases.
   (a) $N_s(i,j) \neq \mathsf{w}_f$ (all $f \in F$). $\exists$ lets $\gamma = \rho_{r+1}(p),\ \delta = \rho_{r+1}(q)$, maintaining property Va. The only forbidden triples of atoms involving $\mathsf{r}_{\gamma,\delta}$ are (15) and (16) of definition 40. The triple of atoms from a triangle (x,y,k) will not be forbidden by (15) since the only green edge incident with k is (i,k) and since ρ_{r+1} is order preserving. To check forbidden triple (16) suppose $N_s(x,y), N_{r+1}(y,k)$ are both red (some $y \in \mathsf{nodes}(N_r)$). We have $y \notin \{i,j\}$ so $\exists$ chose the red label $N_{r+1}(y,k)$. By her strategy, we must have $N_s(i,y) = \mathsf{g}_t$ (some t, else she would have chosen a white atom) and $N_s(j,y) = \mathsf{y}$ (else she would have chosen the black atom). By property (Va) for N_{r+1} we have $N_{r+1}(x,y) = \mathsf{r}_{\rho_{r+1}(p),\rho_{r+1}(t)}$ and by her strategy $N_{r+1}(y,k) = \mathsf{r}_{\rho_{r+1}(t),\rho_{r+1}(q)}$, hence the triple of atoms from the triangle (x,y,k) is not forbidden by (16). Thus property VI holds for N_{r+1}.
   (b) $N_s(i,j) = \mathsf{w}_f$ (some $f \in F$). By consistency of N_s and forbidden triple (14) we have $p \in \mathsf{dom}(f)$ and since $\forall$'s move was legal $a;b =$

$\mathsf{g}_q;\mathsf{y} \geq N_s(i,j) = \mathsf{w}_f$ so $q \in \mathsf{dom}(f)$. $\exists$ lets $\gamma = f(p), \delta = f(q)$, maintaining property Vb for round $r+1$. As above, the only forbidden triples of atoms involving $\mathsf{r}_{\gamma,\delta}$ are (15) and (16) of definition 40. Since f is order preserving and since the only green edge incident with k is (i,k) in N_{r+1}, triangles involving the new edge (x,k) cannot give a forbidden triple of the form (15). For forbidden triple (16), let $y \in \mathsf{nodes}(N_s)$ and suppose $N_{r+1}(x,y), N_{r+1}(y,k)$ are both red. As above, by her strategy, we must have $N_s(y,i) = \mathsf{g}_t$ for some t and $N_s(y,j) = \mathsf{y}$. By consistency of N_s we have $t \in \mathsf{dom}(f)$ and by the current part of her strategy she let $N_{r+1}(y,k) = \mathsf{r}_{f(t),f(q)}$. By property Vb for N_s we have $N_{r+1}(x,y) = \mathsf{r}_{f(p),f(t)}$. So the triple of atoms from the triangle (x,y,k) is not forbidden by (16). This establishes property VI for N_{r+1}.

Thus $\exists$ can maintain all the properties in round $r+1$ in response to a triangle move by $\forall$.

Finally we consider an amalgamation move (N_s, N_t) by $\forall$ in round $r+1$. Essentially, the claim above, particularly the third part, reduces this case to a case very similar to the triangle move case. $\exists$ has to choose a label for each edge (i,j) where $i \in \mathsf{nodes}(N_s) \setminus \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \setminus \mathsf{nodes}(N_s)$ (this then determines the label for the reverse edge (j,i)).

Let $\bar{x}$ enumerate $\mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$. If $\bar{x}$ is short then, by strictness of the hypernetworks, there are at most two nodes in $\mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ and this case is already quite similar to the triangle move case. If $\bar{x}$ is long in N_s then by the claim (2) there is a partial isomorphism $\theta : \nu_{N_s}(\bar{x}) \to \nu_{N_t}(\bar{x})$ fixing $\bar{x}$. By remark 42, since we are assuming that $\forall$ only plays 'maximal amalgamations', we see that $\nu_{N_s}(\bar{x}) = \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t) = \mathsf{rng}(\bar{x}) = \nu_{N_t}(\bar{x})$.

It remains to label the edges (i,j) in N_{r+1} where $i \in \mathsf{nodes}(N_s) \setminus \mathsf{nodes}(N_t)$ and $j \in \mathsf{nodes}(N_t) \setminus \mathsf{nodes}(N_s)$. Her strategy for labelling these edges is similar to her strategy for dealing with triangle moves. She chooses the labels for edges (i,j) one at a time. As before she chooses a white atom if possible, else the black atom if possible, otherwise a red atom. Since she never chooses a green atom, she lets $\rho_{r+1} = \rho_r$ and properties II, III and IV remain true after round $r+1$. She uses the first possible of the cases below.

1. There is no $x \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x)$ and $N_t(x,j)$ are both green. If there are $u, v \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(u,v) = \mathsf{r}_{\beta,\gamma}$, $N_s(i,u) = \mathsf{g}_p$, $N_s(i,v) = \mathsf{g}_q$, $N_t(u,j) = N_t(v,j) = \mathsf{y}$ (some $\beta, \gamma \in \mathbb{N}$, some $p, q \in \mathbb{Z}$) or the roles of i and j are swapped, she lets $f = \{(p,\beta),(q,\gamma)\}$ and sets $N_{r+1}(i,j) = \mathsf{w}_f$. Since all the edges labelled by green or yellow atoms belong to $\forall$ (property I), we can apply the claim (3) to show that the points u, v are unique, so f is well-defined. This is also true if $\bar{x}$ is short, since in this case there are only two nodes in $\mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$.

   If there are no such points u, v as just described then let $S = \{p \in \mathbb{Z} : \exists y \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t),\ (N_s(i,y) = \mathsf{g}_p \wedge N_t(y,j) = \mathsf{y}) \vee (N_s(i,y) = \mathsf{y} \wedge N_t(y,j) = \mathsf{g}_p)\}$. By the claim (3), $|S| \leq 2$. Let f be any order preserving function from S into $\mathbb{N}$. $\exists$ lets $N_{r+1}(i,j) = \mathsf{w}_f$. Property VI holds for N_{r+1}, as for triangle moves.

2. Otherwise, if there is no $x \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x) = N_t(x,j) = \mathsf{y}$, then she lets $N_r(i,j) = \mathsf{b}$. As with triangle moves, all properties are maintained.
3. Otherwise, there are $x, y \in \mathsf{nodes}(N_s) \cap \mathsf{nodes}(N_t)$ such that $N_s(i,x) = \mathsf{g}_k$, $N_t(x,j) = \mathsf{g}_l$ (some $k, l \in \mathbb{N}$) and $N_s(i,y) = N_t(y,j) = \mathsf{y}$. By the claim (3), x, y are unique. She labels (i,j) in N_r with a red atom $\mathsf{r}_{\beta,\gamma}$ where:
   (a) If $N_s(x,y) \neq \mathsf{w}_f$, all $f \in F$), then $\beta = \rho_{r+1}(k)$, $\gamma = \rho_{r+1}(l)$. This maintains property Va.
   (b) Otherwise $N_s(x,y) = \mathsf{w}_f$, for some $f \in F$, and $\beta = f(k)$, $\gamma = f(l)$. This maintains property Vb.

   In either case, we can show that property VI holds for N_{r+1}, as in the case of triangle moves.

This proves that $\exists$ has a winning strategy in $H_n(\alpha)$. ⊣

§6. Non-elementary classes.

LEMMA 44. *Let $\mathcal{A}$ be the rainbow algebra of definition 40. There is a countable relation algebra $\mathcal{A}'$ such that $\mathcal{A}' \equiv \mathcal{A}$ and $\exists$ has a winning strategy in $H(\mathcal{A}')$.*

PROOF. We have seen that for $n < \omega$ $\exists$ has a winning strategy σ_n in $H_n(\mathcal{A})$. We can assume that σ_n is deterministic. Let $\mathcal{B}$ be a non-principal ultrapower of $\mathcal{A}$. We can show that $\exists$ has a winning strategy σ in $H(\mathcal{B})$ — essentially she uses σ_n in the n'th component of the ultraproduct so that at each round of $H(\mathcal{B})$ $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. Now use an elementary chain argument to construct countable elementary subalgebras $\mathcal{A} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \mathcal{B}$. For this, let $\mathcal{A}_{i+1}$ be a countable elementary subalgebra of $\mathcal{B}$ containing $\mathcal{A}_i$ and all elements of $\mathcal{B}$ that σ selects in a play of $H_\omega(\mathcal{B})$ in which $\forall$ only chooses elements from $\mathcal{A}_i$. Now let $\mathcal{A}' = \bigcup_{i<\omega} \mathcal{A}_i$. This is a countable elementary subalgebra of $\mathcal{B}$ and $\exists$ has a winning strategy in $H(\mathcal{A}')$.

⊣

THEOREM 45. *Let K be any class of relation algebras with $\mathfrak{Ra}(\mathbf{CA}_\omega) \subseteq K \subseteq \mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_5$. Then K is not closed under elementary subalgebra, hence K is not an elementary class.*

PROOF. Let $\mathcal{A}$ be the rainbow algebra of definition 40 and let $\mathcal{A}' \succ \mathcal{A}$ be the countable elementary extension given in the previous lemma. $\mathcal{A}'$ must belong to $\mathfrak{Ra}(\mathbf{CA}_\omega)$, by lemma 44 and theorem 39, hence $\mathcal{A}' \in K$. But $\mathcal{A} \notin K$ (lemma 41) and $\mathcal{A} \preceq \mathcal{A}'$. ⊣

PROBLEM 46. *For $n = 3$ or 4, is $\mathfrak{Ra}\mathbf{CA}_n$ elementary? Is $\mathbf{S_c}\mathfrak{Ra}\mathbf{CA}_n$ elementary?*

PROBLEM 47. *For $2 \leq n < m \leq \omega$ and if $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq \mathbf{S_c}\mathfrak{Nr}_n\mathbf{CA}_m$ is it always the case that K is not elementary?*

We expect a positive answer to this problem (i.e. K is not elementary), at least for $m \geq 5$. Some partial results are known: $\mathfrak{Nr}_n L$ is not elementary, for various subclasses L of $\mathbf{CA}_m$; $\mathbf{S_c}\mathfrak{Nr}_n\mathbf{CA}_\omega$ is not elementary; the inclusion $\mathfrak{Nr}_n\mathbf{CA}_m \subset \mathbf{S_c}\mathfrak{Nr}_n\mathbf{CA}_m$ is strict. See [17].

PROBLEM 48. **[Andréka and Németi]** *For which n with $3 < n < \omega$ is it the case that*

$$\mathfrak{Ra}\mathbf{RCA}_n = \mathfrak{Ra}\mathbf{CA}_n \cap \mathbf{RRA}$$

Andréka and Németi point out the the equation is true for $n = 3$ and $n \geq \omega$.

REFERENCES

[1] H ANDRÉKA, I DÜNTSCH, and I NÉMETI, *A nonpermutational integral relation algebra*, ***Michigan Mathematics Journal***, vol. 39 (1992), pp. 371–384.

[2] H Andréka, J Monk, and I Németi (editors), ***Algebraic logic***, Colloq. Math. Soc. J. Bolyai, vol. 54, North-Holland, Amsterdam, 1991.

[3] I ANELLIS and N HOUSER, *Nineteenth century roots of algebraic logic and universal algebra*, In Andréka et al. [2], pp. 1–36.

[4] L HENKIN, J MONK, and A TARSKI, ***Cylindric algebras part i***, North-Holland, 1971.

[5] ———, ***Cylindric algebras part ii***, North-Holland, 1985.

[6] R HIRSCH and I HODKINSON, ***Relation algebras by games***, North-Holland, Elsevier Science, Amsterdam, 2002.

[7] R MADDUX, *Topics in relation algebra*, ***Ph.D. thesis***, University of California, Berkeley, 1978.

[8] ———, *A relation algebra which is not a cylindric reduct*, ***Algebra Universalis***, vol. 27 (1990), pp. 279–288.

[9] ———, *Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections*, In Andréka et al. [2], pp. 361–392.

[10] ———, *The origin of relation algebras in the development and axiomatization of the calculus of relations*, ***Studia Logica***, vol. 3/4 (1991), pp. 421–455.

[11] ———, *Pair-dense relation algebras*, ***Trans. Amer. Math. Soc.***, vol. 328 (1991), no. 1, pp. 83–131.

[12] J MONK, *Studies in cylindric algebra*, ***Ph.D. thesis***, University of California, Berkeley, 1961.

[13] I NÉMETI, *The class of neat reducts is not a variety but is closed w.r.t HP*, ***Notre Dame J. Formal Logic***, vol. 24 (1983), pp. 399–409.

[14] ———, *Free algebras and decidability in algebraic logic*, 1986, Doctor of Science dissertation, Hungarian Academy of Sciences.

[15] I NÉMETI and A SIMON, *Relation algebras from cylindric and polyadic algebras*, ***Logic J. IGPL***, vol. 5 (1997), pp. 575–588.

[16] T SAYED AHMED, *Martin's axiom, omitting types, and complete representations in algebraic logic.*, ***Studia Logica***, vol. 72 (2002), no. 2, pp. 285–309.

[17] T SAYED AHMED, *The class of neat reducts is not elementary*, ***Logic J. IGPL***, vol. 9 (2001), pp. 625–660.

[18] T SAYED AHMED and I NÉMETI, *On neat reducts of algebras of logics*, ***Studia Logica***, vol. 68 (2001), no. 2, pp. 229–262.

[19] A SIMON, *Nonrepresentable algebras of relations*, ***Ph.D. thesis***, Math. Inst. Hungar. Acad. Sci., Budapest, 1997, http://www.renyi.hu/pub/algebraic-logic/simthes.html.

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY COLLEGE
GOWER STREET
LONDON WC1E 6BT
E-mail: R.Hirsch@cs.ucl.ac.uk
URL: http://www.cs.ucl.ac.uk/staff/R.Hirsch