Relation Algebras from Cylindric Algebras, I

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Abstract

We characterise the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ of subalgebras of relation algebra reducts of n-dimensional cylindric algebras (for finite $n \geq 5$) by the notion of a 'hyperbasis', analogous to the cylindric basis of Maddux, and by relativised representations. We outline a game-theoretic approximation to the existence of a representation, and how to use it to obtain a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

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1 Introduction

Relation algebras were defined by Jónsson and Tarski in 1948 [20]. One can see them as an attempt to extend the successful algebraisation of unary relations, by boolean algebras, to binary relations, a project that can be traced back at least to De Morgan in the last century. The equations defining relation algebras express certain simple properties that are true of any field of binary relations on a base set that includes the equality relation and is closed under the operations of converse and composition and the boolean operations. It quickly transpired, however, that these equations were not sufficient to characterise the 'representable' algebras isomorphic to genuine fields of binary relations. Lyndon exhibited the first non-representable relation algebra in [23], and gave a new approach to building representations of relation algebras, in which the base set is built 'point by point', the relations between base points being specified as the points are introduced. Later examples of his led to Monk's famous proof [34] that the class RRA of representable relation algebras is not finitely axiomatisable in first-order logic.

We can view Lyndon's examples as demonstrating that a relation algebra can contain 'contradictions'. These are invisible in small parts (with at most four points) of (the base set of) a representation of the algebra, which are well-controlled by the axioms governing the basic operations (associativity of composition etc.); but they show up when we try to build, in Lyndon's way, larger fragments of a representation and can destroy the possibility of a representation existing at all. The size of fragment needed to detect them can be arbitrarily

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large, and Monk used this to prove that no one first-order sentence can control them — whence the non-finite axiomatisability. Monk extended his result to cylindric algebras, the higher-arity analogue of relation algebras, in [35].

Monk's arguments were very influential. One reaction to them was to try to control the behaviour of larger fragments of a representation. We may try to characterise those relation algebras where we have no problems in building up to n points of a potential representation, for some finite $n \geq 4$. (We say 'potential' because of course the relation algebra may turn out not to have a representation.) As n increases, the class of such algebras approaches RRA.

Two striking ways of doing this were found. In [26, 28, 31], Maddux used 'n-dimensional relational and cylindric bases' — sets of 'networks' of n points, with certain closure properties. A network describes which elements of the relation algebra hold between each pair of points; the description should respect the relation algebra operations. We may think of a network as the isomorphism type of an n-point fragment of a potential representation. The class RA_n denotes those relation algebras that have an n-dimensional relational basis or embed in an algebra that does. For such a relation algebra, we can use the basis to build up to n points of a representation in Lyndon's manner; and we can even keep on going, except that for every new point we add, we must forget some other point in order to limit the total size of the fragment to n. The basis always shows us how to continue, because of its closure properties.

If the algebra has a cylindric basis, we can do even more, in that the different ways that the basis dictates to add points are guaranteed to be compatible with each other, again so long as no more than n points are simultaneously considered. This compatibility property is a very strong 'Church–Rosser condition' that is obtained from an amalgamation closure property in the definition of cylindric bases, and corresponds closely to commutativity of quantifiers in first-order logic: the equivalence of $\exists x \exists y \varphi$ and $\exists y \exists x \varphi$. Maddux proved that the networks in an n-dimensional cylindric basis can be viewed as the atoms of an n-dimensional cylindric algebra, which by definition has a property corresponding to commutativity of quantifiers. The original relation algebra can be recovered from this cylindric algebra by taking its 'relation algebra reduct', a term-definable relation algebra on the 2-dimensional elements of the cylindric algebra, and possibly then taking a subalgebra of that.

This leads us to the second way, essentially Monk's, to control larger fragments of a potential representation of a relation algebra. If we insist that the relation algebra is the 'relation algebra reduct' of an n-dimensional cylindric algebra, or is a subalgebra of such a reduct, then we can hope that fragments of a potential representation of up to n points are controlled by the cylindric algebra axioms, just as the smaller fragments (size four or less) are controlled by the axioms of relation algebras. The class of relation algebras that arise in this way is denoted $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Both these ways of ensuring good behaviour of large fragments of a potential representation have the basic properties we might expect. See, e.g., [6, 7, 28]. It is known that RA, the class of relation algebras, is equal to RA_4 and to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4$. We have $\mathsf{RA}_4 \supseteq \mathsf{RA}_5 \supseteq \cdots$; these inclusions were proven strict in [30]. Similarly, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supseteq \cdots$; infinitely many of these inclusions were shown to be strict in [31], and in [15] it is shown that they are all strict. Also, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$ for each n; by results in part II of this paper [12], this inclusion is strict for $n \geq 5$. For each n, the class of subalgebras of relation algebras with an n-dimensional cylindric basis is a subclass of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$; it is not known whether the inclusion is proper, though for each finite $n \geq 5$ there do exist finite relation algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ with no n-dimensional cylindric basis (theorem 60 below). Finally, it is known that $\bigcap_{4 \leq n < \omega} \mathsf{RA}_n = \bigcap_{4 \leq n < \omega} \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \mathsf{RRA}$.

At first sight, the definition of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ is more abstract than for RA_n, but in this paper we prove that it can be equivalently defined in the same way, using *n-dimensional hyperbases*. Hyperbases generalise cylindric bases by using 'hypernetworks' instead of networks. Hypernetworks contain more information than simply the relation algebra structure on n points of a potential representation: each sequence of up to n points in an n-dimensional hypernetwork carries a 'label'. The idea is that the points of the hypernetwork are controlled by an element of an n-dimensional cylindric algebra, and the additional labelling structure describes this n-ary relation more fully than the merely binary relation algebra structure on the points can do. We believe that this extra information is essential to capture $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ and cannot be eliminated, but further research is needed.[1]

Further, we provide a representation theory for the algebras of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. These algebras are not always in RRA and so are not necessarily representable in the classical sense. None the less, they have useful non-classical 'relativised' representations. In [14], we gave relativised 'n-square' semantics for the algebras of RA_n. An n-square relativised representation is 'locally classical' in that without simultaneously considering more than n points, one cannot tell at first sight that it is not classical. Here, we define the related but stronger notion of *n-flat relativised representation,* in which detecting its non-classical nature is even harder. We show that a relation algebra has such a representation if and only if it belongs to $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. We then give an alternative kind of relativised representation, which we call *n-smooth.* Its defining property is reminiscent of 'bisimulations', the rough idea being that certain local isomorphisms of size less than $n-1$ can be extended so that their domain includes an arbitrary additional point, provided the extended domain forms a 'clique' in the relativised representation. We show that the relation algebras with such a representation are again precisely those in $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.

Game-theoretic approximations to n-smooth relativised representations can be used to obtain a recursive axiomatisation of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$, and, using the fact that this class is an equational variety, we can turn such axioms into equations. We will outline how to do this, but we will not go into full detail.

Various other results will be stated in remarks in the text. Proofs are omitted through lack of space. Some familiarity with the theory of relation algebras and cylindric algebras is assumed. Readers who would like an introduction to these matters are referred to [29]. The standard references for cylindric algebras are [6, 7].

Part II We said that for $n \geq 4$ (or even 3, if we generalise from relation algebras to relation-type algebras), $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n \supseteq \mathsf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. In [15], we showed that this inclusion is strict, for each n. Using game-theoretic techniques, we will show in Part II of this paper [12] that for $n \geq 4$, the gap cannot be finitely axiomatised. A similar result for cylindric algebras, with applications to finite-variable proof theory, is also proved.

Plan of paper

The main aim of this paper is thus to provide an intrinsic characterisation of the approximations $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ $(4 \leq n < \omega)$ of the class RRA of representable relation algebras, using

[1] Test questions include: for $n \geq 5$, can a finite relation algebra have an infinite n-dimensional hyperbasis but no finite one? (Yes.) Is it decidable whether a finite relation algebra has an n-dimensional hyperbasis? (No; this is stronger, since $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ is recursively axiomatisable.) Does every relation algebra with an n-dimensional hyperbasis embed in one with an n-dimensional cylindric basis? (We believe not, but this is open.)

hyperbases and relativised representations. We will prove:

THEOREM 1 *Let $\mathcal{A}$ be a relation algebra and let $5 \leq n < \omega$. Then the following are equivalent:*

1. $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.
2. *The canonical embedding algebra $\mathcal{A}^+$ of $\mathcal{A}$ has an n-dimensional hyperbasis.*
3. *$\mathcal{A}$ embeds in some atomic relation algebra $\mathcal{B}$ that has an n-dimensional hyperbasis.*
4. *$\mathcal{A}$ has an n-flat relativised representation.*
5. *$\mathcal{A}$ has an n-smooth relativised representation.*
6. *$\mathcal{A}$ satisfies certain explicitly-given equations ε_k^n $(k < \omega)$.*

The theorem also holds (degenerately) for $n = 4$: see remark 55. Corollary 43 gives further extensions. Definitions of the terms in the theorem will be given at the appropriate places: in definitions 7, 15, 44, and immediately before theorem 54.

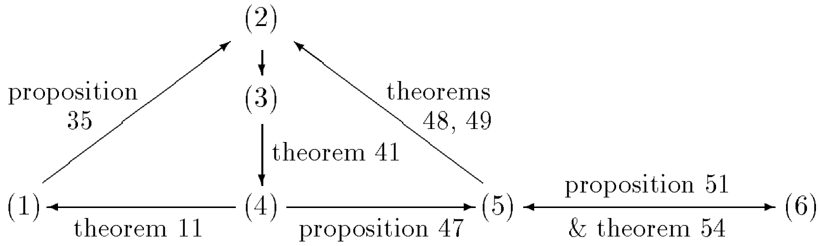

Figure 1: Summary of the proof of theorem 1

The proof will proceed as follows. First, we recall the definition of a relativised representation of a relation algebra, though instead of the usual definition using homomorphisms on algebras, we define our representations as models of a certain first-order theory. In such a representation, n-variable first-order formulas can be interpreted, roughly by relativising existential quantifiers to the unit of the relation algebra. We then define an n-flat relativised representation to be a relativised representation with the additional property that these relativised quantifiers commute ($\exists x_i \exists x_j \varphi$ is always equivalent to $\exists x_j \exists x_i \varphi$). It follows that the definable sets form an n-dimensional cylindric algebra, and so we prove in theorem 11 the implication (4) $\Rightarrow$ (1) of theorem 1.

In section 3, we introduce *hypernetworks* and *hyperbases*. We then develop some results on substitutions in cylindric algebras, based on results of [38] showing that the effect of a string of substitutions in an n-dimensional cylindric algebra is in many cases determined by an associated map on $\{0, 1, \ldots, n-1\}$. This work is used in proposition 35, which proves (1) $\Rightarrow$ (2) of theorem 1. The method is to take an atomic n-dimensional cylindric algebra whose relation algebra reduct embeds a given $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$, and to use its atoms to construct an n-dimensional hyperbasis for the canonical embedding algebra of $\mathcal{A}$.

(2) $\Rightarrow$ (3) is trivial. In section 4, we prove (3) $\Rightarrow$ (4) by showing that any atomic relation algebra with an n-dimensional hyperbasis has a (complete) n-flat relativised representation (theorem 41). This is done in a 'step-by-step' fashion. This completes the proof of the

equivalence of parts (1) to (4) in theorem 1. The proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (4) has some similarity to that in [3]. A direct proof of (3) $\Rightarrow$ (1) can also be given, along the lines of [31, theorem 10].

In section 5 we introduce *n-smooth* relativised representations. We show that the existence of an n-smooth relativised representation is equivalent to the previous parts of theorem 1 by proving (4) $\Rightarrow$ (5) (proposition 47, using an ω-saturated relativised representation), and (5) $\Rightarrow$ (2) (theorem 48, again using ω-saturation, and theorem 49, by taking a complete n-smooth relativised representation and considering the set of all hypernetworks that embed in it).

In section 6, we outline how to use n-smooth relativised representations to obtain a recursive equational axiomatisation of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. We use a two-player game where the players construct better and better 'approximations' to an n-smooth relativised representation. Proposition 51 uses these games to find a first-order characterisation of the countable relation algebras with n-smooth relativised representations. It states that in the infinite length game, at least for countable relation algebras, a winning strategy for the second player is equivalent to the existence of an n-smooth relativised representation, and that a winning strategy in all finite length games is equivalent (by König's tree lemma) to a winning strategy in the infinite length game. Moreover, the existence of a winning strategy for the second player in the game of finite length k can be expressed by a universal sentence σ_k^n. This gives a recursive set of universal first-order axioms σ_k^n $(k < \omega)$ which are true in a countable relation algebra $\mathcal{A}$ iff $\mathcal{A}$ has an n-smooth relativised representation. By the other parts of theorem 1, this holds iff $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.

In this paper we will only sketch the construction of the axiomatisation, as it is becoming a standard method and also for the following reason. The definition of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ and each of the equivalent characterisations (2)–(5) of theorem 1 fit the definition of a *pseudo-elementary* (PC'_Δ) *class* [16, chapters 5, 6]. Roughly, a pseudo-elementary class consists of every L-structure (for some first-order signature L) that arises in a uniform way as a definable part of a model of some first-order theory in a language extending L. In [8] it will be shown how to obtain explicit universal axioms for any pseudo-elementary class that is closed under subalgebras (for non-explicit axioms cf. [16, theorem 6.6.7]). This method of axiomatising a pseudo-elementary class generalises the axiomatisation we provide here. Also see [17], where the class of complex algebas over an arbitrary variety is axiomatised by games.

We continue with a proof (proposition 52) that $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety. This eliminates the countability restriction. Further, since RA is a discriminator class, for each universal formula σ_k^n we can find an equation ε_k^n which is equivalent to σ_k^n over simple relation algebras. Since every relation algebra $\mathcal{A}$ is a subdirect product of simple relation algebras which are quotients of $\mathcal{A}$, we deduce in theorem 54 that the equations $\{\varepsilon_k^n : k < \omega\}$, together with the basic Tarski equations for relation algebras, exactly define the class $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. This completes the proof of theorem 1.

In section 7, we discuss matters arising from the theorem. Section 8 investigates the connections between hyperbases, cylindric bases and relational bases. We outline how cylindric bases correspond to homogeneity of a representation (theorem 63).

Notation 2

Ordinals. Most ordinals in this paper are finite. For an ordinal $n = \{0, 1, \ldots, n-1\}$ and a set A, we write ${}^nA$ for the set of maps from n to A, and ${}^{<n}A$ for $\bigcup_{m<n} {}^mA$. We use ${}^{\leq n}A$ to denote ${}^{<n+1}A$.

Tuples. We often view a map in ${}^nA$ as an n-tuple, and write it as $\bar{a} = (a_0, \ldots, a_{n-1})$ and its length n as $|\bar{a}|$. We will switch between the map view and the tuple view whenever appropriate: so, for example, $rng(\bar{a})$ denotes $\{a_0, \ldots, a_{n-1}\}$. We may specify an n-tuple $\bar{a}$ by defining the elements a_i for $i < n$.

If $\bar{a}, \bar{b}$ are tuples, we write $\bar{a}\bar{b}$ for their concatenation $(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1})$. Thus, $\bar{a}b$ denotes $\bar{a}$ concatenated with the 1-tuple b. For n-tuples $\bar{a}, \bar{b}$ and $i_0, \ldots, i_{k-1} < n$, we write $\bar{a} \equiv_{i_0,\ldots,i_{k-1}} \bar{b}$ if $a_j = b_j$ for all $j \in n \setminus \{i_0, \ldots, i_{k-1}\}$. We also write $(\bar{a} \mapsto \bar{b})$ for the relation $\{(a_i, b_i) : i < n\}$; this may or may not be a well-defined map or function.

For any m-tuple $\bar{a}$ and any map θ with $rng(\bar{a}) \subseteq dom(\theta)$, we write $\theta(\bar{a})$ for the m-tuple $(\theta(a_0), \ldots, \theta(a_{m-1}))$. As a map, this is $\theta \circ \bar{a}$. On the other side, if l is an ordinal and $\theta : l \to m$ a map, we let $\bar{a} \circ \theta$ denote the l-tuple $(a_{\theta(0)}, \ldots, a_{\theta(l-1)})$.

Structures. If L is a signature and M an L-structure, we write S^M for the interpretation in M of a symbol $S \in L$. For example, $1'^{\mathcal{A}}$ is the identity of the relation algebra $\mathcal{A}$. We usually identify (notationally) a structure with its domain.

2 Representation theory

In [27], it was shown that the weakly associative algebras are precisely the algebras of the signature of relation algebras that have relativised representations in which the unit is a reflexive and symmetric relation. We will extend this to provide a representation theory for algebras in $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$: the unit remains reflexive and symmetric, but the representation is rather more complicated. In [14], we did this for the variety RA_n of subalgebras of relation algebras with n-dimensional relational bases. The approach here is rather similar.

Let $\mathcal{A}$ be a relation algebra. Let $\mathcal{L}(\mathcal{A})$ be the first-order language in a signature consisting of one binary predicate symbol for each element of $\mathcal{A}$. That is, each element of (the domain of) the algebra $\mathcal{A}$ will be regarded as a binary relation symbol. (This will not lead to ambiguity: for $r \in \mathcal{A}$, if we write $r(x, y)$, we are thinking of r as a relation symbol, but if we write simply r, we are thinking of r as an element of $\mathcal{A}$.)

2.1 Relativised representations

Definition 3

1. $S_{\mathcal{A}}$ is the $\mathcal{L}(\mathcal{A})$-theory consisting of the following axioms:

$$\begin{array}{rl}
\forall xy[1'(x,y) \leftrightarrow (x = y)] & \\
\forall xy[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & \text{for all } r,s,t \in \mathcal{A} \text{ with } r = s + t \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \neg s(x,y))] & \text{for all } r,s \in \mathcal{A} \text{ with } r = -s \\
\forall xy[r(x,y) \leftrightarrow s(y,x)] & \text{for all } r,s \in \mathcal{A} \text{ with } r = \breve{s} \\
\forall xy[1(x,y) \to (r(x,y) \leftrightarrow \exists z(s(x,z) \wedge t(z,y)))] & \text{for all } r,s,t \in \mathcal{A} \text{ with } r = s\,;t \\
\exists xy\, r(x,y) & \text{for all } r \in \mathcal{A} \text{ with } r \neq 0.
\end{array}$$

2. A *relativised representation* of $\mathcal{A}$ is a model of $S_{\mathcal{A}}$.

3. A *complete relativised representation* of $\mathcal{A}$ is a model of $S_{\mathcal{A}}$ satisfying the (potentially infinitary) axiom

$$\forall xy[(\bigwedge_{s \in S} s(x,y)) \leftrightarrow (\prod S)(x,y)]$$

for each set S of elements of $\mathcal{A}$ whose infimum $\prod S$ exists in $\mathcal{A}$.

The reader should check the next lemma. For the second part, see [10, theorem 5].

Lemma 4 *Let M be a relativised representation of the relation algebra $\mathcal{A}$. Then as boolean algebras, we have*

$$(\mathcal{A}, +, -, 0, 1) \cong (\{r^M : r \in \mathcal{A}\}, \cup, \setminus, \varnothing, 1^M),$$

where, for $r \in \mathcal{A}$, r^M denotes the interpretation of the binary relation symbol r as a binary relation on M (so $r^M \subseteq {}^2M$), and $\setminus$ denotes complement relative to 1^M. Also, 1^M is a reflexive and symmetric relation on M.

M is a complete relativised representation iff for every $x, y \in M$ with $M \models 1(x, y)$, there is an atom (minimal non-zero element) $\alpha \in \mathcal{A}$ such that $M \models \alpha(x, y)$.

2.2 Flat relativised representations

Until the end of this section we fix n with $3 \leq n < \omega$.

Definition 5 Let M be a relativised representation of the relation algebra $\mathcal{A}$. A *clique* in M is a subset $X \subseteq M$ such that $M \models 1(x, y)$ for all $x, y \in X$. We write $C^n(M)$ for the set $\{\bar{a} \in {}^nM : rng(\bar{a}) \text{ is a clique in } M\}$.

Definition 6 We consider the set $\mathcal{L}^n(\mathcal{A})$ of first-order formulas of $\mathcal{L}(\mathcal{A})$ that are written with the variables $x_0, \ldots, x_{n-1}$ only. Let M be a relativised representation of $\mathcal{A}$. We define the *clique-relativised semantics* $M \models_C \varphi(\bar{a})$, for $\varphi \in \mathcal{L}^n(\mathcal{A})$ and $\bar{a} \in C^n(M)$, as follows.

- If φ is $r(x_i, x_j)$ for $r \in \mathcal{A}$ and $i, j < n$, then $M \models_C \varphi(\bar{a})$ iff $M \models r(a_i, a_j)$.
- If φ is $x_i = x_j$ for some $i, j < n$, then $M \models_C \varphi(\bar{a})$ iff $M \models a_i = a_j$.
- $M \models_C \neg\varphi(\bar{a})$ iff $M \not\models_C \varphi(\bar{a})$.
- $M \models_C (\varphi \wedge \psi)(\bar{a})$ iff $M \models_C \varphi(\bar{a})$ and $M \models_C \psi(\bar{a})$, where $\varphi, \psi \in \mathcal{L}^n(\mathcal{A})$.
- For $i < n$, $M \models_C \exists x_i \varphi(\bar{a})$ iff $M \models_C \varphi(\bar{b})$ for some $\bar{b} \in C^n(M)$ with $\bar{b} \equiv_i \bar{a}$.

Notations here are as given in notation 2. We define the abbreviations $\vee, \rightarrow, \forall$ in the usual way.

Definition 7 Let $\mathcal{A}$ be a relation algebra. An *n-flat relativised representation of $\mathcal{A}$* is a relativised representation M of $\mathcal{A}$ with the additional property that for all $\varphi \in \mathcal{L}^n(\mathcal{A})$, all $\bar{a} \in C^n(M)$, and all $i, j < n$, we have

$$M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a}).$$

2.3 Properties of flat relativised representations

We establish two basic properties of any n-flat relativised representation M of a relation algebra $\mathcal{A}$.

LEMMA 8 *Let $i_0, \ldots, i_{k-1} < n$ for some $k < n$, and let $\varphi \in \mathcal{L}^n(\mathcal{A})$ and $\bar{a} \in C^n(M)$. Then $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}} \varphi)(\bar{a})$ iff $M \models_C \varphi(\bar{b})$ for some $\bar{b} \in C^n(M)$ with $\bar{a} \equiv_{i_0,\ldots,i_{k-1}} \bar{b}$.*

PROOF:

'$\Rightarrow$' is clear. We prove '$\Leftarrow$' by induction on k. If $k = 0$, it is trivial, and if $k = 1$, it holds by definition of $\models_C$. Let $k > 1$, and assume the result for smaller k. Since $k < n$, there is $j \in n \setminus \{i_0, \ldots, i_{k-1}\}$. Then $a_j = b_j$. Let $\bar{a}' \in C^n(M)$ be the result of replacing $a_{i_{k-1}}$ by a_j in $\bar{a}$. Define $\bar{b}'$ similarly. Clearly, $\bar{a} \equiv_{i_{k-1}} \bar{a}' \equiv_{i_0,\ldots,i_{k-2}} \bar{b}' \equiv_{i_{k-1}} \bar{b}$. So if $M \models_C \varphi(\bar{b})$, then by three applications of the inductive hypothesis we obtain $M \models_C (\exists x_{i_{k-1}} \varphi)(\bar{b}')$, $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}} \varphi)(\bar{a}')$, and $M \models_C (\exists x_{i_{k-1}} (\exists x_{i_0} \ldots \exists x_{i_{k-1}} \varphi))(\bar{a})$. Now M is n-flat, so by the commutativity of existential quantifiers and a straightforward induction on k, we obtain $M \models_C (\exists x_{i_0} \ldots \exists x_{i_{k-1}} \varphi)(\bar{a})$, as required. $\Box$

Now we prove that free variables of $\mathcal{L}^n(\mathcal{A})$-formulas behave as we would hope. Cf. [31, lemma 20]. Bear in mind that variables can be 're-used' in n-variable formulas, so that x_0 occurs both free and bound in $r(x_0, x_1) \wedge \exists x_0\, s(x_0, x_1)$, for example.

LEMMA 9 *Let $\varphi \in \mathcal{L}^n(\mathcal{A})$ and let x_i (for some $i < n$) be a variable that does not occur free in φ. Then $M \models_C (\varphi \leftrightarrow \exists x_i \varphi)(\bar{a})$ for all $\bar{a} \in C^n(M)$.*

PROOF:

We show by induction on φ that if x_i is not free in φ, $\bar{a}, \bar{b} \in C^n(M)$, and $\bar{a} \equiv_i \bar{b}$, then $M \models_C \varphi(\bar{a})$ iff $M \models_C \varphi(\bar{b})$. If φ is atomic, this is trivial, and the boolean cases are also straightforward. Assume the result for φ and consider $\exists x_j \varphi$, assuming that x_i is not free in $\exists x_j \varphi$. If $j = i$, the result follows from the fact that $\equiv_i$ is an equivalence relation on $C^n(M)$. So assume that $j \neq i$. We let $\bar{a} \equiv_i \bar{b}$ and $M \models_C \exists x_j \varphi(\bar{a})$, and check that $M \models_C \exists x_j \varphi(\bar{b})$, also. (The converse is similar.) Plainly, $M \models_C \exists x_i \exists x_j \varphi(\bar{b})$. By n-flatness, $M \models_C \exists x_j \exists x_i \varphi(\bar{b})$. So there are $\bar{c}, \bar{d} \in C^n(M)$ with $\bar{b} \equiv_j \bar{c} \equiv_i \bar{d}$ and $M \models_C \varphi(\bar{d})$. Now as $i \neq j$, x_i is not free in φ. So by the inductive hypothesis, $M \models_C \varphi(\bar{c})$. Hence, $M \models_C \exists x_j \varphi(\bar{b})$, as required. $\Box$

2.4 From flat representations to RA-reducts

There is a well-known method of obtaining a relation-type algebra $\mathfrak{Ra}(\mathcal{C})$ from an n-dimensional cylindric algebra $\mathcal{C}$ (for any $n \geq 3$): $\mathfrak{Ra}(\mathcal{C})$ is constructed by taking the two-dimensional elements of $\mathcal{C}$ and using the spare dimensions to define converse and composition (see [7, 5.3.7]). $\mathfrak{Ra}(\mathcal{C})$ is called the relation algebra reduct of $\mathcal{C}$.

More formally, this is done as follows. Recall that for $i, j < n$, the substitution operator s^i_j is defined in n-dimensional cylindric algebras by

$$\mathsf{s}^i_j x = \begin{cases} x, & \text{if } i = j; \\ \mathsf{c}_i(\mathsf{d}_{ij} \cdot x), & \text{otherwise.} \end{cases}$$

DEFINITION 10 [7, 5.3.7] Let $\mathcal{C}$ be any n-dimensional cylindric algebra. For $m \leq n$, the *neat m-reduct of* $\mathcal{C}$ (in symbols, $\mathfrak{Nr}_m\mathcal{C}$) is the m-dimensional cylindric algebra with domain $\{a \in \mathcal{C} : \mathsf{c}_j a = a \text{ for all } m \leq j < n\}$ and with operations $+, -, 0, 1, \mathsf{c}_j, \mathsf{d}_{jk}$ for $j, k < m$ induced from $\mathcal{C}$.

The *relation algebra reduct of* $\mathcal{C}$ — in symbols, $\mathfrak{Ra}(\mathcal{C})$ — is the algebra

$$\langle dom(\mathfrak{Nr}_2\mathcal{C}), +, -, 0, 1, 1', \breve{}\ , ; \rangle,$$

where

- $+, -, 0, 1$ are as in $\mathcal{C}$
- $1' = \mathsf{d}_{01}$ $(\in \mathfrak{Nr}_2\mathcal{C})$
- converse is defined by $\breve{r} = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1 r$, for $r \in \mathfrak{Nr}_2\mathcal{C}$.
- composition is defined by $r\,;s = \mathsf{c}_2(\mathsf{s}_2^1 r \cdot \mathsf{s}_2^0 s)$, for $r, s \in \mathfrak{Nr}_2\mathcal{C}$.

$\mathfrak{Ra}(\mathcal{C})$ can be checked to be closed under these operations. For $n = 3$, $\mathfrak{Ra}(\mathcal{C})$ need not be a relation algebra (see [29]), but when $n \geq 4$, it is a relation algebra [7, 5.3.8]. For finite n, if $\mathcal{C}$ is atomic then so are the algebras $\mathfrak{Nr}_m\mathcal{C}$ $(m < n)$, and hence also $\mathfrak{Ra}\mathcal{C}$. The set $At\,\mathfrak{Nr}_m\mathcal{C}$ of atoms of $\mathfrak{Nr}_m\mathcal{C}$ is $\{\mathsf{c}_m\mathsf{c}_{m+1}\ldots\mathsf{c}_{n-1}x : x \text{ an atom of } \mathcal{C}\}$.

We generally identify notationally the algebras $\mathfrak{Nr}_m\mathcal{C}$, $\mathfrak{Ra}\mathcal{C}$ with their domains.

We now prove our first main result.

THEOREM 11 *Let $\mathcal{A}$ be a relation algebra with an n-flat relativised representation. Then $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.*

PROOF:

Let M be an n-flat relativised representation of $\mathcal{A}$. For $\varphi \in \mathcal{L}^n(\mathcal{A})$, write

$$\varphi^C = \{\bar{a} \in C^n(M) : M \models_C \varphi(\bar{a})\}.$$

Let $\mathcal{C}$ be the following CA_n-type algebra (its signature is $\{+, -, 0, 1, \mathsf{d}_{ij}, \mathsf{c}_i : i, j < n\}$).

- The domain of $\mathcal{C}$ is the set $\{\varphi^C : \varphi \in \mathcal{L}^n(\mathcal{A})\}$ of all sets definable by n-variable formulas in the clique-relativised semantics.
- 0 is interpreted in $\mathcal{C}$ as $\emptyset = 0(x_0, x_1)^C$, and 1 as $C^n(M) = 1(x_0, x_1)^C$.
- $+$ and $-$ are interpreted in $\mathcal{C}$ as follows: $\varphi^C + \psi^C = \varphi^C \cup \psi^C = (\varphi \vee \psi)^C$, and $-(\varphi^C) = C^n(M) \setminus \varphi^C = (\neg\varphi)^C$ (this is plainly well-defined).
- d_{ij} is interpreted as $\{\bar{a} \in C^n(M) : a_i = a_j\} = (x_i = x_j)^C$.
- c_i is interpreted by: $\mathsf{c}_i(\varphi^C) = (\exists x_i\varphi)^C$ (this is well-defined).

One may easily check that $\mathcal{C}$ satisfies the axioms for n-dimensional cylindric algebras of [6, §1.1.1]. In particular, n-flatness implies that $\mathcal{C} \models \forall x(\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x)$ for all $i, j < n$. Hence, $\mathcal{C} \in \mathsf{CA}_n$. We show that $\mathcal{A}$ embeds into $\mathfrak{Ra}\mathcal{C}$. For $r \in \mathcal{A}$, let

$$\iota(r) = r(x_0, x_1)^C \in \mathcal{C}.$$

By lemma 9, $\iota(r) \in \mathfrak{Ra}\mathcal{C}$ for all $r \in \mathcal{A}$. We check that ι is an algebra embedding of $\mathcal{A}$ into $\mathfrak{Ra}\mathcal{C}$.

ι plainly preserves the boolean operations, because M is a relativised representation of $\mathcal{A}$. (For negation, note that if $\bar{a} \in C^n(M)$ then $M \models 1(a_0, a_1)$, so $M \models_C (\neg r(x_0, x_1) \leftrightarrow -r(x_0, x_1))(\bar{a})$ by the negation axiom of $S_{\mathcal{A}}$.) Now, to show that ι is one-one, we only need check that if $r \in \mathcal{A} \setminus \{0\}$ then $\iota(r) \neq 0$ in $\mathcal{C}$. But $M \models S_{\mathcal{A}}$, so there are $a, b \in M$ with $M \models r(a, b)$. One may check that $\{a, b\}$ is a clique in M (cf. lemma 4), so that $\bar{a} = (a, b, b, \ldots, b) \in C^n(M)$. Then $\bar{a} \in \iota(r)$, which is therefore non-empty, as required.

We check the relation algebra operations. By the identity axiom of $S_{\mathcal{A}}$, $M \models \forall xy(1\text{'}(x, y) \leftrightarrow x = y)$. So $\iota(1\text{'}) = (x_0 = x_1)^C = \mathsf{d}_{01}$. For converse, let $r \in \mathcal{A}$: we require $\iota(\breve{r}) = \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r)$. By definition of $\mathcal{C}$,

$$\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r) = \big(\exists x_2(x_2 = x_0 \wedge \exists x_0(x_0 = x_1 \wedge \exists x_1(x_1 = x_2 \wedge r(x_0, x_1))))\big)^C.$$

It follows that for any $\bar{a} \in C^n(M)$, $\bar{a} \in \mathsf{s}_0^2\mathsf{s}_1^0\mathsf{s}_2^1\iota(r)$ iff $M \models r(a_1, a_0)$. By the converse axiom of $S_{\mathcal{A}}$, this is iff $M \models \breve{r}(a_0, a_1)$ — i.e., iff $\bar{a} \in \iota(\breve{r})$, as required.

Finally, we check composition. We require that $\iota(r\,;s) = \mathsf{c}_2(\mathsf{s}_2^1\iota(r) \cdot \mathsf{s}_2^0\iota(s))$ for all $r, s \in \mathcal{A}$. Fix such r, s, and let

$$\chi = \exists x_1(x_1 = x_2 \wedge r(x_0, x_1)) \wedge \exists x_0(x_0 = x_2 \wedge s(x_0, x_1)).$$

We require $[r\,;s](x_0, x_1)^C = (\exists x_2\chi)^C$.

First, let $\bar{a} \in (\exists x_2\chi)^C$. So there is $\bar{b} \in C^n(M)$ with $\bar{b} \equiv_2 \bar{a}$ and $M \models r(b_0, b_2) \wedge s(b_2, b_1)$. Because $M \models 1(a_0, a_1)$, it follows by the composition axiom in $S_{\mathcal{A}}$ that $M \models [r\,;s](a_0, a_1)$. So $\bar{a} \in [r\,;s](x_0, x_1)^C$.

Conversely, let $\bar{a} \in [r\,;s](x_0, x_1)^C$. The composition axiom of $S_{\mathcal{A}}$ yields $b \in M$ with $M \models r(a_0, b) \wedge s(b, a_1)$. The $S_{\mathcal{A}}$ axioms now yield $M \models 1(a_0, a_1) \wedge 1(a_0, b) \wedge 1(b, a_1)$. By reflexivity and symmetry of 1^M (lemma 4), $\{a_0, a_1, b\}$ is a clique in M, so that $\bar{b} = (a_0, a_1, b, b, \ldots, b) \in C^n(M)$. Now, $\bar{b} \equiv_{23\cdots(n-1)} \bar{a}$ and $\bar{b} \in \chi^C$. By lemma 8, $\bar{a} \in (\exists x_3 \ldots \exists x_{n-1}\exists x_2\chi)^C$. By repeated use of lemma 9, $\bar{a} \in (\exists x_2\chi)^C$, as required. □

This proves (4) ⇒ (1) of theorem 1.

Remark 12 Crs_n denotes the class of cylindric relativised set algebras (see [6]), and G_n denotes the class of algebras in Crs_n whose unit 1 is closed under permutations and substitutions. In the proof of theorem 11, $\mathcal{C}$ is a subalgebra of the cylindric relativised set algebra with domain $\wp(C^n(M))$, and the unit $C^n(M)$ of $\mathcal{C}$ is clearly closed under permutations and substitutions. It follows that any relation algebra $\mathcal{A}$ with an n-flat relativised representation belongs to $\mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{Crs}_n)$ and to $\mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$.

3 Hypernetworks and hyperbases

In this section, we will define the terms of part 2 of theorem 1, and prove (1) ⇒ (2) of the theorem. This is a more substantial matter than before. We fix finite $n \geq 4$, an atomic relation algebra $\mathcal{A}$, and a non-empty set Λ (of 'labels') disjoint from $At(\mathcal{A})$, the set of atoms of $\mathcal{A}$.

DEFINITION 13 An *n-dimensional Λ-hypernetwork over $\mathcal{A}$* is a map $N : {}^{\leq n}n \to \Lambda \cup At(\mathcal{A})$, such that $N(\bar{a}) \in At\mathcal{A}$ if and only if $|\bar{a}| = 2$, with the following properties.

1. $N(i,i) \leq 1'$ for each $i < n$.
2. For all $i,j,k < n$, $N(i,j) ; N(j,k) \geq N(i,k)$.
3. For every $\bar{a}, \bar{b} \in {}^{\leq n}n$ of equal length, if $N(a_i, b_i) \leq 1'$ for each $i < |\bar{a}|$, then $N(\bar{a}) = N(\bar{b})$.

Property 3 in the case $|\bar{a}| = 2$ is implied by properties 1 and 2. As notation, for Λ-hypernetworks N, M and $i, j < n$, we write $N \equiv_i M$ iff $N(\bar{a}) = M(\bar{a})$ for all $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$, and similarly, $N \equiv_{ij} M$ iff $N(\bar{a}) = M(\bar{a})$ for all $\bar{a} \in {}^{\leq n}(n \setminus \{i,j\})$.

The important labels in hypernetworks are the labels of $(n-2)$-tuples: see in particular lemma 34. We let hypernetworks have labels on longer and shorter sequences in our definition purely for convenience; these labels have less significance. See, for example, definition 30, where sequences of length $n-1$ and n are labelled by a constant — such labelling carries no information.

The following is a basic property of hypernetworks.

LEMMA 14 *Let N be an n-dimensional Λ-hypernetwork over $\mathcal{A}$, and let $i, j < n$. Then $N(i,j) = N(j,i)^\smile$.*

PROOF:

By property 2 of definition 13, $N(i,j) ; N(j,i) \geq N(i,i)$, so by property 1, $(N(i,j) ; N(j,i)) \cdot 1' \neq 0$. By the Peircean law in $\mathcal{A}$, $N(i,j) \cdot (1' ; N(j,i)^\smile) = N(i,j) \cdot N(j,i)^\smile \neq 0$. As $N(i,j)$ and $N(j,i)^\smile$ are atoms, we obtain $N(i,j) = N(j,i)^\smile$, as required. □

Now we can define the terms used in part 2 of theorem 1.

DEFINITION 15 An *n-dimensional Λ-hyperbasis for $\mathcal{A}$* is a set $\mathcal{H}$ of n-dimensional Λ-hypernetworks over $\mathcal{A}$ satisfying:

1. If $r \in \mathcal{A}$ is non-zero, then there is $N \in \mathcal{H}$ with $N(0,1) \leq r$.
2. If $N \in \mathcal{H}$, $i, j < n$, $k \in n \setminus \{i,j\}$, and $r, s \in \mathcal{A}$ satisfy $N(i,j) \leq r ; s$, then there is $M \in \mathcal{H}$ with $M \equiv_k N$, $M(i,k) \leq r$, and $M(k,j) \leq s$.
3. If $N, M \in \mathcal{H}$, $i, j < n$, and $N \equiv_{ij} M$, then there is $P \in \mathcal{H}$ with $N \equiv_i P \equiv_j M$.

Property 3 holds trivially when $i = j$, so we generally assume that $i \neq j$. We drop terms such as 'n-dimensional', 'Λ', and 'over $\mathcal{A}$' when they are clear from the context. Also, 'an n-dimensional hyperbasis' will sometimes mean 'an n-dimensional Λ-hyperbasis for some Λ'.

In the case where $|\Lambda| = 1$, a Λ-hyperbasis is essentially a 'cylindric basis' in the sense of Maddux. The conditions above amount to those of a relational basis plus the 'amalgamation' condition of cylindric bases. See [28, 31]. We will discuss this further in section 8.

3.1 Substitutions in cylindric algebras

We aim to prove that the canonical embedding algebra of any subalgebra of the relation algebra reduct of an n-dimensional cylindric algebra has an n-dimensional hyperbasis. In this section, we prove some necessary preliminary results about substitutions in cylindric algebras. $n \geq 4$ remains fixed. Recall again that the substitution operator s^i_j is defined by

$$\mathsf{s}^i_j x = \begin{cases} x, & \text{if } i = j; \\ \mathsf{c}_i(\mathsf{d}_{ij} \cdot x), & \text{otherwise,} \end{cases}$$

for $i, j < n$. As is standard, e.g., in [6], the map $[i/j] : n \to n$ is given by:

$$[i/j](k) = \begin{cases} j, & \text{if } k = i; \\ k, & \text{otherwise.} \end{cases}$$

Of course, this definition depends implicitly on n. We write maps on the left, and $\circ$ denotes map composition, so that for example, $([1/2] \circ [2/3])(1) = [1/2]([2/3](1)) = 2$.

3.1.1 s-c-words

Definition 16

1. An *s-word* is a finite string of substitutions (s^i_j), a *c-word* is a finite string of cylindrifications (c_k), and an *s-c-word* is a finite string of substitutions and cylindrifications, all of the signature of CA_n.

2. If u, w are s-c-words, we write simply uw for their concatenation. The length (number of symbols) of w is written $|w|$.

3. With each s-c-word w, we associate a partial map $\widehat{w} : n \to n$ by induction on $|w|$:

   - If w is the empty string then $\widehat{w} = Id_n$, the identity map on n.
   - $\widehat{w\mathsf{s}^i_j} = \widehat{w} \circ [i/j]$, for $i, j < n$.
   - $\widehat{w\mathsf{c}_i} = \widehat{w}\restriction_{n \setminus \{i\}} = \widehat{w} \circ Id_{n \setminus \{i\}}$, for $i < n$.

Cf. [36]. Clearly, for any variable x and s-c-word w, wx is a term of the signature of n-dimensional cylindric algebras. And if $x \in \mathcal{C} \in \mathsf{CA}_n$, then $wx \in \mathcal{C}$.

For familiarisation, the following useful lemma may help.

Lemma 17 *Let u, w be s-c-words.*

1. $\widehat{uw} = \widehat{u} \circ \widehat{w}$.

2. $\mathsf{CA}_n \models w\mathsf{d}_{ij} = \mathsf{d}_{\widehat{w}(i)\widehat{w}(j)}$, *for all* $i, j \in dom(\widehat{w})$.

Proof:

1. We prove $\widehat{uw} = \widehat{u} \circ \widehat{w}$ for all u, by a trivial induction on the length of w. If this is zero, then $\widehat{w} = Id_n$ and we are done. Assume the result for w, and let $i, j < n$. Then $\widehat{uw\mathsf{s}^i_j} = \widehat{uw} \circ [i/j] = (\widehat{u} \circ \widehat{w}) \circ [i/j]$ (by the inductive hypothesis), $= \widehat{u} \circ (\widehat{w} \circ [i/j]) = \widehat{u} \circ \widehat{w\mathsf{s}^i_j}$, as required. The proof for $w\mathsf{c}_i$ is similar.
2. The proof is by induction on the length of w. If this is zero, there is nothing to prove. Assume the result for w. We first prove it for $u = w\mathsf{s}^k_l$. Let $i, j \in dom(\widehat{u})$. If $i = j$, then in any n-dimensional cylindric algebra, $\mathsf{d}_{ij} = 1 = \mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$. So suppose that $i \neq j$. Now, by [6, theorem 1.5.4(i)] we have $w\mathsf{s}^k_l\mathsf{d}_{ij} = w\mathsf{d}_{[k/l](i),[k/l](j)}$; by assumption, $[k/l](i), [k/l](j) \in dom(\widehat{w})$, so by the induction hypothesis this is $\mathsf{d}_{\widehat{w}([k/l](i)),\widehat{w}([k/l](j))} = \mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$.

   Next let $u = w\mathsf{c}_k$. Assume that $i, j \in dom(\widehat{u})$ — that is, $i, j \in dom(\widehat{w})$ and $i, j \neq k$. Then $u\mathsf{d}_{ij} = w\mathsf{d}_{ij}$, which by the inductive hypothesis is $\mathsf{d}_{\widehat{w}(i),\widehat{w}(j)}$; and this is clearly equal to $\mathsf{d}_{\widehat{u}(i),\widehat{u}(j)}$, as required.

$\square$

3.1.2 Adapting known results on substitutions

We will need some corollaries of the following known results.

Fact 18 *Let $\alpha \geq 2$ be an ordinal.*

1. *If $\theta : \alpha \to \alpha$ is not a permutation of α, and $\{i < \alpha : \theta(i) \neq i\}$ is finite, then θ is either of the form $[i/j]$ (for some $i, j < \alpha$) or a composition of maps of this form.*

   This is a not-too-difficult exercise, or it can be derived from [19]. [18, theorem 1] proves it for finite α, and [38, corollary 1.2] for arbitrary α.

2. [38, theorem 3.6] *Let $q, r < \omega$. Assume that $i_1, j_1, \ldots, i_q, j_q, k_1, m_1, \ldots, k_r, m_r < \alpha$ are such that $[i_1/j_1] \circ \cdots \circ [i_q/j_q] = [k_1/m_1] \circ \cdots \circ [k_r/m_r] = f \in {}^{\alpha}\alpha$ and $|\alpha \setminus rng(f)| \geq 2$. Then*
$$\mathsf{CA}_\alpha \models \forall x(\mathsf{s}^{i_1}_{j_1} \ldots \mathsf{s}^{i_q}_{j_q}(x) = \mathsf{s}^{k_1}_{m_1} \ldots \mathsf{s}^{k_r}_{m_r}(x)).$$

We need to generalise these facts to partial maps and s-c-words. We are only interested in the case $\alpha = n$. See [7, theorem 3.2.52] for a similar result.

Definition 19 Two s-c-words u, w are said to be *congruent* if $\widehat{u} = \widehat{w}$ and $\mathsf{CA}_n \models \forall x(ux = wx)$. We write $u \simeq w$ in this case.

Lemma 20 *If v, v' are s-c-words and $v \simeq v'$ then $uvw \simeq uv'w$ for all s-c-words u, w.*

Proof:

Assume that $v \simeq v'$. Then $\mathsf{CA}_n \models \forall x(v(wx) = v'(wx))$, so $\mathsf{CA}_n \models \forall x(uvwx = uv'wx)$. By lemma 17.1 and associativity of composition of partial maps, $\widehat{uvw} = \widehat{u} \circ \widehat{v} \circ \widehat{w} = \widehat{u} \circ \widehat{v'} \circ \widehat{w} = \widehat{uv'w}$. So $uvw \simeq uv'w$. $\square$

Definition 21 An s-c-word w is said to be *modest* if whenever u, v are s-c-words, $i < n$, and $w = u\mathsf{c}_i v$, then $|\widehat{v}^{-1}(i)| \leq 1$. Here, $\widehat{v}^{-1}(i) = \{j \in dom(\widehat{v}) : \widehat{v}(j) = i\}$.

Examples 22 The following are easily checked.

1. $\mathsf{c}_0\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_0\mathsf{s}_2^1\mathsf{c}_2$ is a modest s-c-word.

2. The word $\mathsf{c}_0\mathsf{s}_0^1$ is not modest, as $\widehat{\mathsf{s}_0^1}^{-1}(0) = \{0,1\}$.

3. If u is an s-word and v a c-word, then uv is modest.

4. More generally, if u, v are modest s-c-words and $\widehat{v}$ is one-one, then uv is also modest.

5. If u is a modest s-c-word, $i < n$, and $i \notin rng(\widehat{u})$, then $\mathsf{c}_i u$ is modest.

6. More generally, if u, v are s-c-words, v is modest, and $rng(\widehat{v}) \subseteq dom(\widehat{u})$, then uv is modest.

We will prove:

Theorem 23

1. *For any partial map $\theta : n \to n$ with $|rng(\theta)| \leq n-1$, there exists a modest s-c-word w with $\widehat{w} = \theta$.*

2. *Let u, w be modest s-c-words with $|rng(\widehat{u})|, |rng(\widehat{w})| \leq n-2$. If $\widehat{u} = \widehat{w}$, then $\mathsf{CA}_n \models \forall x(ux = wx)$, so that $u \simeq w$.*

Remark 24 Part 2 can fail if the words are not modest. For example, $\widehat{\mathsf{c}_0\mathsf{s}_0^1} = \widehat{\mathsf{c}_0\mathsf{c}_1}$, yet $\mathsf{c}_0\mathsf{s}_0^1(-\mathsf{d}_{01}) = 0$ and $\mathsf{c}_0\mathsf{c}_1(-\mathsf{d}_{01}) \neq 0$, in general. In [8], it will be shown that part 2 can be extended to arbitrary s-c-words by additionally requiring that $u^* = w^*$, where $u^* = \{(i,j) : i,j < n,\ \exists t, v(u = tv \wedge i,j \in dom(\widehat{v}) \wedge \widehat{v}(i) = \widehat{v}(j))\}$, and similarly for w. This is more general because if u, w are modest then $u^* = w^* = \{(i,i) : i < n\}$; we will not need this generality here.

Proof:

The proof of the first part is straightforward. If θ is the empty map, it is trivial — let $w = \mathsf{c}_0\mathsf{c}_1 \ldots \mathsf{c}_{n-1}$. Assume not; let $n \setminus dom(\theta) = \{i_0, \ldots, i_{k-1}\}$, take $j \in rng(\theta)$, and consider the total map $\theta^+ = \theta \cup \{(i_l, j) : l < k\} : n \to n$. Now $rng(\theta^+) = rng(\theta)$, so by fact 18.1, there is an s-word u with $\widehat{u} = \theta^+$. Then $w = u\mathsf{c}_{i_0} \ldots \mathsf{c}_{i_{k-1}}$ is modest (example 22.3) and $\widehat{w} = \theta^+\restriction_{dom(\theta)} = \theta$.

To prove the second part, we need to be able to move cylindrifications rightwards within s-c-words, preserving modesty and $\simeq$-equivalence. Lemma 25 below shows how to do this. Cf. [7, theorem 3.2.51(vi,vii)].

Lemma 25 *Let $\mathsf{c}_i w$ be a modest s-c-word, for some s-c-word w and some $i < n$. Then*

$$\mathsf{c}_i w \simeq \begin{cases} w, & \text{if } i \notin rng(\widehat{w}); \\ w\mathsf{c}_l, & \text{if } l < n \text{ and } \widehat{w}(l) = i. \end{cases}$$

(The 'l' in the second case is unique, as $\mathsf{c}_i w$ is modest. Note that w and $w\mathsf{c}_l$ above are modest.)

PROOF:

The proof is by induction on $|w|$. If this is zero, there is nothing to prove. Assume the result for u. We prove it for words $w = \mathsf{c}_j u$ and $w = \mathsf{s}^j_k u$. We use lemma 20 freely in the proof.

First, consider the case of $w = \mathsf{c}_j u$. Assume that $\mathsf{c}_i\mathsf{c}_j u$ is modest. If $i = j$, then $i \notin rng(\widehat{\mathsf{c}_i u})$ and plainly, $\mathsf{c}_i\mathsf{c}_i u \simeq \mathsf{c}_i u$, which is as required. Assume that $i \neq j$. Then $\mathsf{c}_i\mathsf{c}_j u \simeq \mathsf{c}_j\mathsf{c}_i u$. Clearly, $\mathsf{c}_i u$ is modest, so by the inductive hypothesis, $\mathsf{c}_i u \simeq u$ if $i \notin rng(\widehat{u})$, while $\mathsf{c}_i u \simeq u\mathsf{c}_l$ if $\widehat{u}(l) = i$. But as $j \neq i$, for any $l < n$ we have $\widehat{u}(l) = i$ iff $\widehat{\mathsf{c}_j u}(l) = i$. So

$$\mathsf{c}_i w = \mathsf{c}_i\mathsf{c}_j u \simeq \mathsf{c}_j\mathsf{c}_i u \simeq \begin{cases} \mathsf{c}_j u = w, & \text{if } i \notin rng(\widehat{\mathsf{c}_j u}); \\ \mathsf{c}_j u\mathsf{c}_l = w\mathsf{c}_l, & \text{if } \widehat{\mathsf{c}_j u}(l) = i. \end{cases}$$

We pass to the case $w = \mathsf{s}^j_k u$; we can assume that $j \neq k$. Consider $\mathsf{c}_i\mathsf{s}^j_k u$ (assumed modest). Here, there are three cases.

Case $i \neq j, k$. By [6, theorem 1.5.8(ii)], $\mathsf{c}_i\mathsf{s}^j_k x = \mathsf{s}^j_k\mathsf{c}_i x$ holds in CA_n; and clearly, $\widehat{\mathsf{c}_i\mathsf{s}^j_k} = \widehat{\mathsf{s}^j_k\mathsf{c}_i}$. So $\mathsf{c}_i\mathsf{s}^j_k \simeq \mathsf{s}^j_k\mathsf{c}_i$. Now $\mathsf{c}_i u$ is modest, and by the case assumption, for any $l < n$ we have $i = \widehat{\mathsf{s}^j_k u}(l)$ iff $i = \widehat{u}(l)$. So by the inductive hypothesis,

$$\mathsf{c}_i w = \mathsf{c}_i\mathsf{s}^j_k u \simeq \mathsf{s}^j_k\mathsf{c}_i u \simeq \begin{cases} \mathsf{s}^j_k u = w, & \text{if } i \notin rng(\widehat{\mathsf{s}^j_k u}); \\ \mathsf{s}^j_k u\mathsf{c}_l = w\mathsf{c}_l, & \text{if } \widehat{\mathsf{s}^j_k u}(l) = i. \end{cases}$$

Case $i = j$. By [6, theorem 1.5.9(ii)], $\mathsf{c}_j\mathsf{s}^j_k x = \mathsf{s}^j_k x$ holds in CA_n; also, $\widehat{\mathsf{c}_j\mathsf{s}^j_k} = \widehat{\mathsf{s}^j_k}$. So $\mathsf{c}_j\mathsf{s}^j_k \simeq \mathsf{s}^j_k$, whence $\mathsf{c}_j\mathsf{s}^j_k u \simeq \mathsf{s}^j_k u$. Also, $i \notin rng(\widehat{\mathsf{s}^j_k u})$. We are done.

Case $i = k$. So we are considering $\mathsf{c}_k\mathsf{s}^j_k u$. Since this is assumed modest, either j or k (or both) is not in $rng(\widehat{u})$. Inductively, therefore, $u \simeq \mathsf{c}_j u$ or $u \simeq \mathsf{c}_k u$, so that $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_j u$ or $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_k u$. Now by [6, theorems 1.5.8(i), 1.5.9(i)], $\mathsf{s}^j_k\mathsf{c}_j x = \mathsf{c}_j x$ and $\mathsf{c}_k\mathsf{s}^j_k x = \mathsf{c}_j\mathsf{s}^k_j x$ are valid in CA_n. Of course, $\widehat{\mathsf{s}^j_k\mathsf{c}_j} = \widehat{\mathsf{c}_j}$ and $\widehat{\mathsf{c}_k\mathsf{s}^j_k} = \widehat{\mathsf{c}_j\mathsf{s}^k_j}$. So $\mathsf{s}^j_k\mathsf{c}_j \simeq \mathsf{c}_j$ and $\mathsf{c}_k\mathsf{s}^j_k \simeq \mathsf{c}_j\mathsf{s}^k_j$. Hence, $\mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_j \simeq \mathsf{c}_k\mathsf{c}_j \simeq \mathsf{c}_j\mathsf{c}_k \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k$ and $\mathsf{c}_k\mathsf{s}^j_k\mathsf{c}_k \simeq \mathsf{c}_j\mathsf{s}^k_j\mathsf{c}_k \simeq \mathsf{c}_j\mathsf{c}_k \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k$. We conclude that our $\mathsf{c}_k\mathsf{s}^j_k u$ is congruent to $\mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k u$.

Now $k \notin rng(\widehat{\mathsf{s}^j_k u})$ iff $j, k \notin rng(\widehat{u})$, and $k = \widehat{\mathsf{s}^j_k u}(l)$ iff $\widehat{u}(l) \in \{j, k\}$. The above and two applications of the inductive hypothesis now yield $\mathsf{c}_j\mathsf{c}_k u \simeq u$ in the first case, and $\mathsf{c}_j\mathsf{c}_k u \simeq u\mathsf{c}_l$, in the second. Since $\mathsf{c}_k\mathsf{s}^j_k u \simeq \mathsf{s}^j_k\mathsf{c}_j\mathsf{c}_k u$, the result follows.

$\square$

DEFINITION 26 An s-c-word w is said to be in *normal form* if it has the form uv for some s-word u and some c-word v.

By example 22.3, any s-c-word in normal form is modest. Conversely, repeated use of the preceding lemma allows us to push all cylindrifications in a modest s-c-word rightwards until they either disappear or emerge on the right-hand side, all the time preserving modesty. Thus, any modest s-c-word is congruent to one in normal form. Note that if $w = uv$ is in normal form, with u an s-word and v a c-word, then $dom(\widehat{w}) = dom(\widehat{v}) = \{i < n : \mathsf{c}_i \text{ does not occur in } v\}$.

We may now prove the second part of theorem 23. We have to show that if u, v are modest s-c-words with $\widehat{u} = \widehat{v} = \theta$, say, and $|rng(\theta)| \leq n-2$, then u and v are congruent. As outlined above, we may suppose that they are in normal form; and since $\widehat{u}, \widehat{v}$ have the same domain, we can suppose $u = u'w$ and $v = v'w$ for some s-words u', v' and some fixed c-word w (with $dom(\widehat{w}) = dom(\theta)$).

If θ is empty then every c_i $(i < n)$ occurs in w, and since $\mathsf{c}_i\mathsf{c}_j \simeq \mathsf{c}_j\mathsf{c}_i$ and $\mathsf{s}^i_j\mathsf{c}_i \simeq \mathsf{c}_i$ for all $i \neq j$ (cf. [6, theorem 1.5.8(i)]), it is easily seen that $u'w \simeq w \simeq v'w$. We are done. Assume otherwise. Let $n \setminus dom(\theta) = \{i_0, \ldots, i_{k-1}\}$, choose $j \in dom(\theta)$, and let $t = \mathsf{s}^{i_0}_j \ldots \mathsf{s}^{i_{k-1}}_j$. Then clearly, $\widehat{u't} = \widehat{v't} = \theta^+$, say. θ^+ is a total map : $n \to n$, and $|rng(\theta^+)| \leq n-2$. So by fact 18.1, the s-words $u't$ and $v't$ are congruent. Using $\mathsf{s}^i_j\mathsf{c}_i \simeq \mathsf{c}_i$ and induction on k, we see that tw and w are also congruent. So

$$u = u'w \simeq u'tw \simeq v'tw \simeq v'w = v.$$

This completes the proof of theorem 23. □

We will need the following corollary.

COROLLARY 27 *Let w be a modest s-c-word, let $m \leq n$, and suppose that $m \subseteq dom(\widehat{w})$. Let $\mathcal{C} \in \mathsf{CA}_n$. Then the map $x \mapsto wx$, for $x \in \mathfrak{Nr}_m\mathcal{C}$, is a homomorphism from the boolean reduct of $\mathfrak{Nr}_m\mathcal{C}$ into the boolean reduct of $\mathcal{C}$.*

PROOF:

Let $x, y \in \mathfrak{Nr}_m\mathcal{C}$. Given that substitutions and cylindrifications are additive operators on cylindric algebras and preserve 0 and 1 (for substitutions, see [6, theorem 1.5.3]), a trivial induction on the length of w shows that $w0 = 0$, $w1 = 1$, and $w(x+y) = wx + wy$ for all $x, y \in \mathcal{C}$. We check that $w(-x) = -wx$, for $x \in \mathfrak{Nr}_m\mathcal{C}$. We may suppose that w is in normal form, so has the form uv for an s-word u and a c-word v. If c_i occurs in v, then $i \notin dom(\widehat{w})$, so $i \geq m$. As $x, -x \in \mathfrak{Nr}_m\mathcal{C}$, we have $\mathsf{c}_i x = x$ and similarly for $-x$. Thus, we obtain $wx = ux$ and $w(-x) = u(-x)$. But by [6, theorem 1.5.3(ii)], $\mathsf{s}^i_j(-x) = -\mathsf{s}^i_j x$ for any $i, j < n$ and any $x \in \mathcal{C}$. So by induction on $|u|$, we obtain

$$w(-x) = u(-x) = -ux = -wx,$$

as required. □

3.2 A hyperbasis from a cylindric algebra

Now fix finite $n \geq 5$. (All but one of our results go through unchanged if $n = 4$; and even the one (lemma 34) that requires $n \geq 5$ can be generalised to cover the case $n = 4$ at the cost of complicating the definition of hyperbasis.) We will now prove (1) $\Rightarrow$ (2) of theorem 1: that the canonical embedding algebra $\mathcal{A}^+$ of any $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ has an n-dimensional hyperbasis. We assume familiarity with canonical embedding algebras of relation algebras and cylindric algebras; cf. [21]. In particular, we identify elements of $\mathcal{A}$ with elements of $\mathcal{A}^+$ in the natural way, so that $\mathcal{A} \subseteq \mathcal{A}^+$, and similarly for cylindric algebras.

So suppose that $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. So there is $\mathcal{B} \in \mathsf{CA}_n$ with $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$. Let $\mathcal{C} = \mathcal{B}^+$, the canonical embedding algebra of $\mathcal{B}$. Then $\mathcal{C}$ is atomic, and $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$. Most of the time we will work in $\mathcal{C}$. We generally write $r, s, \ldots$ for elements of $\mathcal{A}$, $f, g, \ldots$ for filters on $\mathcal{A}$, and $x, y, \ldots$ for atoms of $\mathcal{C}$.

Notation 28 If $\bar{a} \in {}^{<n-1}n$, we write $\mathsf{s}_{\bar{a}}$ for an arbitrarily-chosen modest s-c-word w with $\widehat{w} = \bar{a}$. Such a word exists by theorem 23.1; as clearly $|rng(\bar{a})| \leq n-2$, any two such are congruent (theorem 23.2), so $\mathsf{s}_{\bar{a}}x$ for $x \in \mathcal{C}$ is independent of the choice of $\mathsf{s}_{\bar{a}}$. As a slight abbreviation of this, we may write $\mathsf{s}_{a_0 a_1 \ldots a_{l-1}}$ for $\mathsf{s}_{\bar{a}}$, where $\bar{a} = (a_0, \ldots, a_{l-1})$. In this way, we can write s_{ijk} instead of, say, $\mathsf{s}_{(i,j,k)}$. It will help to remember that $\widehat{\mathsf{s}_{ijk}}$ is the partial map from n to n that takes 0 to i, 1 to j, 2 to k, and is undefined on larger numbers.

The definition of relation algebra composition in $\mathfrak{Ra}\mathcal{C}$ (definition 10) is in terms of the indices 0, 1, and 2. These indices can be 'moved', using substitutions.

Lemma 29 *Let $i, j, k < n$ with $k \neq i, j$. Then $\mathsf{s}_{ij}(r\,;s) = \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s)$ for all $r, s \in \mathcal{A}$.*

Proof:

First, as $k \neq i, j$, the s-c-word $\mathsf{c}_k\mathsf{s}_{ijk}$ is modest, $\widehat{\mathsf{c}_k\mathsf{s}_{ijk}} = \widehat{\mathsf{s}_{ij}\mathsf{c}_2}$, and clearly this map has range of size at most $n-2$. As $\mathsf{s}_{ij}\mathsf{c}_2$ is also modest (example 22(4)), we obtain $\mathsf{c}_k\mathsf{s}_{ijk} \simeq \mathsf{s}_{ij}\mathsf{c}_2$ by theorem 23.2. Similarly, $\mathsf{s}_{ik} \simeq \mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2$ and $\mathsf{s}_{kj} \simeq \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2$.

Second, if $3 \leq i < n$ then by [6, theorem 1.5.8(ii)], $\mathsf{c}_i\mathsf{s}^1_2 r = \mathsf{s}^1_2\mathsf{c}_i r = \mathsf{s}^1_2 r$ (as $r \in \mathfrak{Nr}_2\mathcal{C}$), and similarly, $\mathsf{c}_i\mathsf{s}^0_2 s = \mathsf{s}^0_2\mathsf{c}_i s = \mathsf{s}^0_2 s$. So $\mathsf{s}^1_2 r, \mathsf{s}^0_2 s \in \mathfrak{Nr}_3\mathcal{C}$, and $3 \subseteq dom(\widehat{\mathsf{s}_{ijk}})$.

Now,

$$\begin{array}{rcll}
\mathsf{s}_{ij}(r\,;s) & = & \mathsf{s}_{ij}\mathsf{c}_2(\mathsf{s}^1_2 r \cdot \mathsf{s}^0_2 s) & \text{by definition of composition} \\
 & = & \mathsf{c}_k\mathsf{s}_{ijk}(\mathsf{s}^1_2 r \cdot \mathsf{s}^0_2 s) & \text{as } \mathsf{c}_k\mathsf{s}_{ijk} \simeq \mathsf{s}_{ij}\mathsf{c}_2 \\
 & = & \mathsf{c}_k(\mathsf{s}_{ijk}\mathsf{s}^1_2 r \cdot \mathsf{s}_{ijk}\mathsf{s}^0_2 s) & \text{by corollary 27, as } \mathsf{s}^1_2 r, \mathsf{s}^0_2 s \in \mathfrak{Nr}_3\mathcal{C} \\
 & = & \mathsf{c}_k(\mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2 r \cdot \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2 s) & \text{as } r, s \in \mathfrak{Nr}_2\mathcal{C} \\
 & = & \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s) & \text{as } \mathsf{s}_{ijk}\mathsf{s}^1_2\mathsf{c}_2 \simeq \mathsf{s}_{ik} \text{ and } \mathsf{s}_{ijk}\mathsf{s}^0_2\mathsf{c}_2 \simeq \mathsf{s}_{kj},
\end{array}$$

as required. $\Box$

For $m < n-1$ let Λ_m be the set of ultrafilters of $\mathfrak{Nr}_m\mathcal{C}$, or up to a natural identification the set of atoms of its canonical embedding algebra $(\mathfrak{Nr}_m\mathcal{C})^+$, and let λ be any fixed object not in $\bigcup_{m<n-1}\Lambda_m$ or in $At\mathcal{A}^+$. Let $\Lambda = \{\lambda\} \cup \bigcup_{m<n-1}\Lambda_m$.

Definition 30 For each atom $x \in \mathcal{C}$, we define an n-dimensional Λ-hypernetwork N_x over $\mathcal{A}^+$, as follows. For $\bar{a} \in {}^{\leq n}n$, we let:

$$N_x(\bar{a}) = \begin{cases} \{r \in \mathcal{A} : x \leq \mathsf{s}_{\bar{a}}r\}, & \text{if } |\bar{a}| = 2; \\ \lambda, & \text{if } |\bar{a}| \in \{n-1, n\}; \\ \{r \in \mathfrak{Nr}_{|\bar{a}|}\mathcal{C} : x \leq \mathsf{s}_{\bar{a}}r\}, & \text{otherwise (i.e., } 2 \neq |\bar{a}| < n-1). \end{cases}$$

LEMMA 31 *Let* $x \in At\mathcal{C}$ *and* $\bar{a} \in {}^{<n-1}n$.

1. *If* $|\bar{a}| = 2$, $N_x(\bar{a})$ *is an ultrafilter of* $\mathcal{A}$, *and so an atom of* $\mathcal{A}^+$.
2. *If* $|\bar{a}| \neq 2$, $N_x(\bar{a})$ *is an ultrafilter of* $\mathfrak{Nr}_{|\bar{a}|}\mathcal{C}$, *and so an atom of* $(\mathfrak{Nr}_{|\bar{a}|}\mathcal{C})^+$ *and in* Λ.

In the second case, it can be shown that $N_x(\bar{a})$ is actually a principal ultrafilter.

PROOF:

Assume that $|\bar{a}| = m \neq 2$. Trivially, $m \subseteq dom(\widehat{\mathsf{s}_{\bar{a}}})$. So by corollary 27, the map $\psi : \mathfrak{Nr}_m\mathcal{C} \to \mathcal{C}$ given by $\psi(r) = \mathsf{s}_{\bar{a}}r$ is a boolean homomorphism. Let $f = \{y \in \mathcal{C} : y \geq x\}$, an ultrafilter of $\mathcal{C}$. Then $N_x(\bar{a})$ is the inverse image under ψ of f, and hence is an ultrafilter of $\mathfrak{Nr}_m(\mathcal{C})$. This proves the second part. The proof of the first part is similar, using $\psi : \mathcal{A} \to \mathcal{C}$ given by $\psi(r) = \mathsf{s}_{\bar{a}}r$; this is a boolean homomorphism as $\mathcal{A} \subseteq \mathfrak{Nr}_2\mathcal{C}$. $\square$

LEMMA 32 *Let* $x \in At\mathcal{C}$. *Then* N_x *is an* n*-dimensional* Λ*-hypernetwork over* $\mathcal{A}^+$.

PROOF:

Bear in mind that $n \geq 4$. We first check that $\mathcal{A}^+ \models N_x(i,i) \leq 1'$ for each $i < n$. That is, we must show that $1'^{\mathcal{A}} \in N_x(i,i)$, or, that $\mathcal{C} \models x \leq \mathsf{s}_{ii}\mathsf{d}_{01}$. By lemma 17.2, $\mathsf{s}_{ii}\mathsf{d}_{01} = \mathsf{d}_{ii} = 1$ in $\mathcal{C}$, and we are done.

Next, we let $i,j,k < n$ and check that $N_x(i,k)\,;N_x(k,j) \geq N_x(i,j)$ in $\mathcal{A}^+$. We require $r\,;s \in N_x(i,j)$ whenever $r \in N_x(i,k)$ and $s \in N_x(k,j)$. So we let $r,s \in \mathcal{A}$, assume that $x \leq \mathsf{s}_{ik}r$ and $x \leq \mathsf{s}_{kj}s$, and prove that $x \leq \mathsf{s}_{ij}(r\,;s)$. But by lemma 29, $x \leq \mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s \leq \mathsf{c}_k(\mathsf{s}_{ik}r \cdot \mathsf{s}_{kj}s) = \mathsf{s}_{ij}(r\,;s)$, and we are done.

Finally, we have to check that if $\bar{a}, \bar{b} \in {}^{\leq n}n$ are of equal length $l \leq n$ and satisfy $N_x(a_i,b_i) \leq 1'$ for each $i < l$, then $N_x(\bar{a}) = N_x(\bar{b})$. If $|\bar{a}| = |\bar{b}| \in \{n-1, n\}$ then $N_x(\bar{a}) = N_x(\bar{b}) = \lambda$, so suppose $|\bar{a}| = |\bar{b}| = l$ (say) $< n-1$. By lemma 17.2 again, the condition is equivalent to $x \leq \mathsf{d}_{a_ib_i}$ for each $i < l$. Let $d(\bar{a},\bar{b}) = |\{i < l : a_i \neq b_i\}|$. The proof is by induction on $d(\bar{a},\bar{b})$.

If $d(\bar{a},\bar{b}) = 0$, then $\bar{a} = \bar{b}$ and there is nothing to prove. Assume that $d(\bar{a},\bar{b}) = 1$. Let $i < l$ be the index with $a_i \neq b_i$. Now $|rng(\bar{a}) \cup rng(\bar{b})| \leq n-1$, so we may choose $j < n$ with $j \notin rng(\bar{a}) \cup rng(\bar{b})$. Let $\bar{c} \in {}^ln$ be given by $\bar{c} \equiv_i \bar{a}$, $c_i = j$. Then $\widehat{\mathsf{s}_{\bar{a}}} = (\mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}})^\frown$ and $\widehat{\mathsf{s}_{\bar{b}}} = (\mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}})^\frown$; and clearly, the words $\mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}$ and $\mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}$ are modest. So by theorem 23.2, $\mathsf{s}_{\bar{a}} \simeq \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}$ and $\mathsf{s}_{\bar{b}} \simeq \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}$.

Now we show $N_x(\bar{a}) \subseteq N_x(\bar{b})$; the converse is similar (and indeed is not needed as $N_x(\bar{a}), N_x(\bar{b})$ are ultrafilters). Let $r \in N_x(\bar{a})$. Then $x \leq \mathsf{s}_{\bar{a}}r$. Also, by assumption, $x \leq \mathsf{d}_{a_ib_i}$. So

$$\begin{array}{rcll} x & \leq & \mathsf{d}_{a_ib_i} \cdot \mathsf{s}_{\bar{a}}r & \text{by assumption} \\ & = & \mathsf{d}_{a_ib_i} \cdot \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}}r & \text{as } \mathsf{s}_{\bar{a}} \simeq \mathsf{s}^j_{a_i}\mathsf{s}_{\bar{c}} \\ & = & \mathsf{d}_{a_ib_i} \cdot \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}r & \text{by [6, theorem 1.5.6]} \\ & \leq & \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}}r & \\ & = & \mathsf{s}_{\bar{b}}r & \text{as } \mathsf{s}^j_{b_i}\mathsf{s}_{\bar{c}} \simeq \mathsf{s}_{\bar{b}}. \end{array}$$

Hence, $r \in N_x(\bar{b})$, as required.

Now let $d(\bar{a}, \bar{b}) \geq 2$ and assume the result for smaller d. Choose $i < l$ with $a_i \neq b_i$, and let $\bar{c} \in {}^l n$ be given by $\bar{c} \equiv_i \bar{a}$, $c_i = b_i$. Then clearly, $d(\bar{a}, \bar{c}), d(\bar{c}, \bar{b}) < d(\bar{a}, \bar{b})$, and $x \leq \mathsf{d}_{a_j c_j}$, $x \leq \mathsf{d}_{c_j b_j}$ for all $j < l$. By the inductive hypothesis, $N_x(\bar{a}) = N_x(\bar{c}) = N_x(\bar{b})$, which completes the induction and the proof. □

The next two lemmas relate atoms to hypernetworks. The second is in some way the converse of the first.

LEMMA 33 *Let $x, y \in At\mathcal{C}$, $i < n$, and suppose that $\mathsf{c}_i x = \mathsf{c}_i y$. Then $N_x \equiv_i N_y$.*

PROOF:

Let $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$. We show $N_x(\bar{a}) = N_y(\bar{a})$. If $|\bar{a}| \in \{n-1, n\}$ then $N_x(\bar{a}) = N_y(\bar{a}) = \lambda$, so we assume $|\bar{a}| < n-1$. As $N_x(\bar{a}), N_y(\bar{a})$ are ultrafilters, it suffices to show $N_x(\bar{a}) \subseteq N_y(\bar{a})$. Let $r \in N_x(\bar{a})$, so that $x \leq \mathsf{s}_{\bar{a}} r$. Then $y \leq \mathsf{c}_i x \leq \mathsf{c}_i \mathsf{s}_{\bar{a}} r$. But $i \notin rng(\bar{a})$, so $\mathsf{c}_i \mathsf{s}_{\bar{a}}$ is modest, and $\mathsf{c}_i \mathsf{s}_{\bar{a}} \simeq \mathsf{s}_{\bar{a}}$ by lemma 25. Hence, $\mathsf{c}_i \mathsf{s}_{\bar{a}} r = \mathsf{s}_{\bar{a}} r$, so that $y \leq \mathsf{s}_{\bar{a}} r$ and $r \in N_y(\bar{a})$. Hence, $N_x(\bar{a}) \subseteq N_y(\bar{a})$, as required. □

LEMMA 34 *Let $x, y \in At\mathcal{C}$, let $i, j < n$ be distinct, and suppose that $N_x \equiv_{ij} N_y$. Then $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j y$.*

PROOF:

Pick $\bar{a} \in {}^{n-2}n$ with $rng(\bar{a}) = n \setminus \{i, j\}$. Then $N_x(\bar{a}) = N_y(\bar{a})$.

Let w be a modest s-c-word with $\widehat{w} = (\bar{a})^{-1}$, so that $rng(\widehat{w}) = n-2$ and $\bar{a} \circ \widehat{w} = Id_{n \setminus \{i,j\}}$. Such a w exists by theorem 23.1 and because $\bar{a}$ is one-one. By lemma 17.1, $\widehat{\mathsf{s}_{\bar{a}} w} = \bar{a} \circ \widehat{w} = \widehat{\mathsf{c}_i \mathsf{c}_j}$. The word $\mathsf{s}_{\bar{a}} w$ is modest, because $\mathsf{s}_{\bar{a}}$ and w are, and $\widehat{w}$ is one-one (see example 22.4). As $\mathsf{c}_i \mathsf{c}_j$ is certainly modest, and $|rng(\widehat{\mathsf{c}_i \mathsf{c}_j})| = n-2$, we obtain $\mathsf{s}_{\bar{a}} w \simeq \mathsf{c}_i \mathsf{c}_j$ by theorem 23.2.

Thus, $x \leq \mathsf{c}_i \mathsf{c}_j x = \mathsf{s}_{\bar{a}} w x$.

Now $n-2, n-1 \notin rng(\widehat{w})$, so $\mathsf{c}_{n-2} w$ is modest and (by lemma 25) $\mathsf{c}_{n-2} w \simeq w$, and similarly for c_{n-1}. Hence, $wx \in \mathfrak{Nr}_{n-2}\mathcal{C}$. As $n - 2 \neq 2$ (here, we use our assumption that $n \geq 5$ for the only time) we obtain $wx \in N_x(\bar{a})$. But $N_x(\bar{a}) = N_y(\bar{a})$, so $y \leq \mathsf{s}_{\bar{a}} w x = \mathsf{c}_i \mathsf{c}_j x$. Because y is an atom of $\mathcal{C}$, we obtain $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j y$. □

Now let $\mathcal{H} = \{N_x : x \in At\mathcal{C}\}$.

PROPOSITION 35 *$\mathcal{H}$ is an n-dimensional Λ-hyperbasis for $\mathcal{A}^+$.*

PROOF:

Recall that we assumed $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$ for some $\mathcal{B} \in \mathsf{CA}_n$, and $\mathcal{B}^+ = \mathcal{C}$. We will use the following properties of $\mathcal{C}$:

FACT 36

1. If $S \subseteq \mathcal{B}$ is a filter base (that is, for every finite $S' \subseteq S$ there is $s \in S$ with $0 < s \leq s'$ for every $s' \in S'$), then there is an atom $x \in \mathcal{C}$ with $\mathcal{C} \models x \leq s$ for every $s \in S$.

   For, the upward closure $\{b \in \mathcal{B} : \exists s \in S(s \leq b)\}$ of S in $\mathcal{B}$ is a proper filter in $\mathcal{B}$ and so extends to an ultrafilter of $\mathcal{C}$. This ultrafilter corresponds to an atom x of $\mathcal{C}$ which lies beneath every element of S.
2. If $x \in At\mathcal{C}$ and $k < n$, then in $\mathcal{C}$ we have $\mathsf{c}_k x = \prod_{b \in \mathcal{B},\, b \geq x} \mathsf{c}_k b$. See [21, definition 2.14, theorem 2.15].

We now check that $\mathcal{H}$ has the properties of the definition of hyperbasis. First, let $f \in \mathcal{A}^+$ be non-zero. We must find $N \in \mathcal{H}$ with $\mathcal{A}^+ \models N(0,1) \leq f$. Since $\mathcal{A}^+$ is atomic, we may suppose that f is an atom of $\mathcal{A}^+$. Regarding f as an ultrafilter of $\mathcal{A}$, it is a filter base in $\mathcal{B}$, so fact 36.1 gives an atom x of $\mathcal{C}$ with $\mathcal{C} \models x \leq r$ for every $r \in f$. Let $r \in f$. As $\mathcal{C} \models x \leq \mathsf{s}_{01} r$ — for example, by taking $\mathsf{s}_{01} = \mathsf{c}_2 \mathsf{c}_3 \ldots \mathsf{c}_{n-1}$ — we have $r \in N_x(0,1)$. In terms of $\mathcal{A}^+$, this says that $\mathcal{A}^+ \models N_x(0,1) \leq f$, as required.

Second, we let $N_x \in \mathcal{H}$, $i, j < n$, $k \in n \setminus \{i, j\}$, and $f, g \in \mathcal{A}^+$, and assume that $\mathcal{A}^+ \models N_x(i,j) \leq f\,;g$. We must find $N_y \in \mathcal{H}$ with $N_y \equiv_k N_x$, $N_y(i,k) \leq f$, and $N_y(k,j) \leq g$. Since $N_x(i,j)$ is an atom of $\mathcal{A}^+$, by complete additivity of ';' in $\mathcal{A}^+$ we can suppose that $f, g \in At\mathcal{A}^+$, also. That is, f and g are ultrafilters of $\mathcal{A}$.

We know that $\mathcal{A}^+ \models N_x(i,j) \leq f\,;g$. So for every $r \in f$ and $s \in g$, we have $r\,;s \in N_x(i,j)$, whence $x \leq \mathsf{s}_{ij}(r\,;s)$. By lemma 29, $x \leq \mathsf{c}_k(\mathsf{s}_{ik} r \cdot \mathsf{s}_{kj} s)$. Define:

$$\begin{array}{rcl} S & = & \{\mathsf{s}_{ik} r \cdot \mathsf{s}_{kj} s : r \in f,\ s \in g\} \\ X & = & \{b \in \mathcal{B} : \mathcal{C} \models x \leq b\} \\ Y & = & \{\mathsf{c}_k b \cdot s : b \in X, s \in S\}. \end{array}$$

Note that each is a subset of $\mathcal{B}$. Then $x \leq \mathsf{c}_k s$ and so $\mathsf{c}_k x \cdot s \neq 0$ for all $s \in S$, so that every element of Y is non-zero. Since both S and X are filter bases in $\mathcal{B}$ (e.g., use corollary 27 for S), it follows that Y is also a filter base in $\mathcal{B}$, so by fact 36.1, there is an ultrafilter y of $\mathcal{B}$ containing Y. Then $y \in At\mathcal{C}$, so $N_y \in \mathcal{H}$. Also, $\mathcal{C} \models y \leq \mathsf{c}_k b$ for all $b \in X$. By fact 36.2, in $\mathcal{C}$ we have $y \leq \prod_{b \in X} \mathsf{c}_k b = \mathsf{c}_k x$, so $\mathsf{c}_k x = \mathsf{c}_k y$. By lemma 33, $N_x \equiv_k N_y$. Finally, $\mathcal{C} \models y \leq s$ for every $s \in S$, so that $r \in N_y(i,k)$ and $s \in N_y(k,j)$ for every $r \in f$, $s \in g$. In terms of $\mathcal{A}^+$, this says that $N_y(i,k) = f$ and $N_y(k,j) = g$, as we wanted.

Finally, let $N_x, N_y \in \mathcal{H}$, let $i, j < n$ be distinct, and suppose that $N_x \equiv_{ij} N_y$. By lemma 34, $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_i \mathsf{c}_j y$. Hence, by additivity of cylindrifications, there is an atom $z \in \mathcal{C}$ with $\mathsf{c}_i x = \mathsf{c}_i z$ and $\mathsf{c}_j z = \mathsf{c}_j y$. Then $N_z \in \mathcal{H}$, and by lemma 33, $N_x \equiv_i N_z \equiv_j N_y$. □

This completes the proof of (1) ⇒ (2) of theorem 1. The same argument shows that the atomic relation algebra $\mathfrak{Ra}\mathcal{C}$ also has an n-dimensional Λ-hyperbasis.

4 From hyperbasis to relativised representation

Since $\mathcal{A} \subseteq \mathcal{A}^+$, the implication (2) ⇒ (3) of theorem 1 is trivial. In this section, we show that (3) ⇒ (4) in theorem 1. First, we need to make our hyperbases more symmetrical.

4.1 Symmetric hyperbases

Let $\mathcal{A}$ be an atomic relation algebra.

Definition 37 If N is a hypernetwork over $\mathcal{A}$ and $\sigma : n \to n$ is any map, we write $N\sigma$ for the hypernetwork defined by $(N\sigma)(\bar{a}) = N(\sigma(\bar{a}))$, for all $\bar{a} \in {}^{\leq n}n$. (See notation 2.)

It is easy to check that $N\sigma$ is indeed a hypernetwork. In particular, suppose that $\bar{a}, \bar{b} \in {}^{\leq n}n$ have equal length l, and $(N\sigma)(a_i, b_i) \leq 1'$ for each $i < l$. We require $(N\sigma)(\bar{a}) = (N\sigma)(\bar{b})$. Write $\bar{a}'$ for the l-tuple $\sigma(\bar{a}) = (\sigma(a_0), \ldots, \sigma(a_{l-1}))$, and define $\bar{b}'$ similarly. Then $N(a'_i, b'_i) = (N\sigma)(a_i, b_i) \leq 1'$ for each i, so $(N\sigma)(\bar{a}) = N(\bar{a}') = N(\bar{b}') = (N\sigma)(\bar{b})$, as required.

Lemma 38 *Let M, N be hypernetworks over $\mathcal{A}$, and let $i, j < n$ be distinct. Then $M = N[i/j]$ iff $M \equiv_i N$ and $M(i,j) \leq 1'$.*

Let $\mathcal{H}$ be an n-dimensional hyperbasis for $\mathcal{A}$, let $N \in \mathcal{H}$, and let $i, j < n$. Then $N[i/j] \in \mathcal{H}$.

Proof:

Left-to-right of the first part is clear. For the converse, suppose that $M \equiv_i N$ and $M(i,j) \leq 1'$. For each $k < n$, let $k' = [i/j](k)$. Let $\bar{a} \in {}^{\leq n}n$, and define $\bar{a}' = [i/j](\bar{a})$. Then $N[i/j](\bar{a}) = N(\bar{a}') = M(\bar{a}')$, since $\bar{a}' \in {}^{\leq n}(n \setminus \{i\})$. Now because $M(i,j) \leq 1'$, it follows that $M(a'_m, a_m) \leq 1'$ for each $m < |\bar{a}|$. So $M(\bar{a}') = M(\bar{a})$, and hence $N[i/j] = M$.

If $i = j$, the second part is trivial. Assume otherwise. Let $N(j,j) = r$. Then $r \leq 1'$, and it follows in relation algebras that $r = r\,;r$. By the second defining property of hyperbases, there is $M \in \mathcal{H}$ with $N \equiv_i M$ and $M(j,i) = M(i,j) = r$. By the first part, $M = N[i/j]$. □

Definition 39 A hyperbasis $\mathcal{H}$ for $\mathcal{A}$ is said to be *symmetric* if whenever $N \in \mathcal{H}$ and $\sigma : n \to n$ then $N\sigma \in \mathcal{H}$.

Lemma 40 *If an atomic relation algebra has an n-dimensional hyperbasis, then it has a symmetric n-dimensional hyperbasis.*

Proof:

Let $\mathcal{H}$ be an n-dimensional hyperbasis for the atomic relation algebra $\mathcal{A}$. Let $\mathcal{H}^+ = \{N\sigma : N \in \mathcal{H}, \sigma : n \to n\}$. We show that $\mathcal{H}^+$ is a symmetric n-dimensional hyperbasis for $\mathcal{A}$.

The elements of $\mathcal{H}^+$ are certainly hypernetworks, and $\mathcal{H}^+$ is certainly symmetric. For any non-zero $r \in \mathcal{A}$, there is $N \in \mathcal{H}^+$ with $N(0,1) \leq r$, because $\mathcal{H}^+ \supseteq \mathcal{H}$.

We check the second defining property of hyperbases. Let $N \in \mathcal{H}$ and $\sigma : n \to n$, let $i, j, k < n$ with $k \neq i, j$, and let $r, s \in \mathcal{A}$ with $N\sigma(i,j) \leq r\,;s$. We seek $P \in \mathcal{H}^+$ with $P \equiv_k N\sigma$, $P(i,k) \leq r$, and $P(k,j) \leq s$.

Well, $N(\sigma(i), \sigma(j)) \leq r\,;s$. Pick $l \in n \setminus \{\sigma(m) : m < n, m \neq k\}$. As $N \in \mathcal{H}$, there is $M \in \mathcal{H}$ with $M \equiv_l N$, $M(\sigma(i), l) \leq r$, and $M(l, \sigma(j)) \leq s$. Define $\tau : n \to n$ by $\tau(k) = l$ and $\tau(m) = \sigma(m)$ for $m \neq k$. Then $M\tau \in \mathcal{H}^+$. If $\bar{a} \in {}^{\leq n}(n \setminus \{k\})$, then $M\tau(\bar{a}) = M(\tau(\bar{a})) = M(\sigma(\bar{a}))$ (because $k \notin rng(\bar{a})$), $= N(\sigma(\bar{a}))$ (because $l \notin rng(\sigma(\bar{a}))$ and $M \equiv_l N$), $= N\sigma(\bar{a})$. So $M\tau \equiv_k N\sigma$. Furthermore,

$M\tau(i,k) = M(\tau(i),\tau(k)) = M(\sigma(i),l) \leq r$, and (similarly) $M\tau(k,j) \leq s$. This is as required.

Lastly, we check that if $N, M \in \mathcal{H}$, $\sigma, \tau : n \to n$, and $N\sigma \equiv_{ij} M\tau$ for some distinct $i, j < n$, then $N\sigma \equiv_i P \equiv_j M\tau$ for some $P \in \mathcal{H}^+$. Clearly,

$$(N\sigma)[i/j] \equiv_i N\sigma \equiv_{ij} M\tau \equiv_j (M\tau)[j/i], \tag{1}$$

so that $(N\sigma)[i/j] \equiv_{ij} (M\tau)[j/i]$. Now $(N\sigma)[i/j] = N(\sigma \circ [i/j])$. By fact 18.1, $\sigma \circ [i/j]$ is a product of substitutions, so by lemma 38, $(N\sigma)[i/j] \in \mathcal{H}$. Similarly, $(M\tau)[j/i] \in \mathcal{H}$. As $\mathcal{H}$ is a hyperbasis, there is $P \in \mathcal{H}$ with $(N\sigma)[i/j] \equiv_i P \equiv_j (M\tau)[j/i]$. So $P \in \mathcal{H}^+$, and by (1), $N\sigma \equiv_i P \equiv_j M\tau$, as required. □

4.2 Constructing a relativised representation from a hyperbasis

THEOREM 41 *Let $\mathcal{A}$ be an atomic relation algebra with an n-dimensional hyperbasis, where $n \geq 4$. Then $\mathcal{A}$ has a (complete) n-flat relativised representation.*

PROOF:

Let $\mathcal{H}$ be an n-dimensional Λ-hyperbasis for $\mathcal{A}$, for some set Λ. By lemma 40, we may suppose that $\mathcal{H}$ is symmetric. We will extract a (complete) n-flat relativised representation of $\mathcal{A}$ directly from $\mathcal{H}$. We will build a chain of possibly uncountable labelled, directed hypergraphs M_t ($t < \omega$); they will not be complete hypergraphs. Their union, M_ω, will essentially be the representation we seek. Each M_t will have edges (ordered pairs) labelled by atoms of $\mathcal{A}$, and hyperedges (l-tuples for $l \leq n$, $l \neq 2$) labelled by elements of Λ. No non-edges or non-hyperedges are labelled. We will require (inductively) that for each $t < \omega$, M_t satisfies conditions 1–4 below:

1. The set of edges forms a reflexive and symmetric binary relation on M_t.
2. Each directed edge (x, y) of M_t is labelled by an atom of $\mathcal{A}$, written $M_t(x, y)$.
3. If (x, y) is an edge of M_t, then $M_t(x, y) \leq 1$' iff $x = y$.

For such a graph M_t, and a hypernetwork $N \in \mathcal{H}$, a map $\nu : N \to M_t$ (formally a map from n into $dom(M_t)$) is said to be an *embedding* if whenever $i, j < n$ then $(\nu(i), \nu(j))$ is an edge of M_t and $M_t(\nu(i), \nu(j)) = N(i, j)$; and whenever $\bar{a} \in {}^{\leq n}n$ with $|\bar{a}| \neq 2$, then $\nu(\bar{a})$ is a hyperedge of M_t and is labelled with $N(\bar{a})$. Note that despite their name, embeddings need not be one-to-one, but they do preserve atoms under 1'. Say (as usual in graph theory) that a subset C of M_t is a *clique* if (x, y) is an edge of M_t for all $x, y \in C$. We further require of M_t:

4. Any clique in M_t is contained in $rng(\nu)$ for some $N \in \mathcal{H}$ and some embedding $\nu : N \to M_t$.

Let N be a hypernetwork and let $S \subseteq n$. We say that the labelled hypergraph $N{\restriction}_S$ induced by N on S is *strict* if for all distinct $i, j \in S$ we have $N(i, j) \cdot 1' = 0$. $N{\restriction}_S$ is *maximal strict* if it is strict and for all $S \subset T \subseteq n$, $N{\restriction}_T$ is not strict. Let M_0 be the disjoint union of all maximal strict labelled hypergraphs $N{\restriction}_S$, where $N \in \mathcal{H}$ and $S \subseteq n$. Thus, M_0 satisfies requirements 1–3 above. If $N{\restriction}_S$ is maximal strict, then for all $i < n$ there is unique $s_i \in S$ such that $N(i, s_i) \leq 1$'.

The map $\nu = \{(i, s_i) : i < n\}$ is an embedding of N onto $N{\restriction}_S$. So M_0 satisfies requirement 4, too.

Assume inductively that M_t is defined for some $t < \omega$. Then we define the extension M_{t+1} of M_t so that for every quadruple (N, ν, k, N'), where $N, N' \in \mathcal{H}$, $k < n$, $N \equiv_k N'$, and $\nu : N \to M_t$ is an embedding, the restriction $\nu{\restriction}_{n\setminus\{k\}}$ of ν extends to an embedding $\nu' : N' \to M_{t+1}$. We do this as follows.

- If $N'(k,i) \leq 1$' for some $i \neq k$, then we may (must) set $\nu'(k) = \nu(i)$. This is well-defined if there are several such i, and is an embedding. No change to M_t is made for these (N, ν, k, N').
- For each other (N, ν, k, N'), we adjoin a new point $\pi = \pi_{(t,N,\nu,k,N')}$ to M_t. We add just the following new edges: (π, π), and $(\pi, \nu(i))$, $(\nu(i), \pi)$ for each $i \in n \setminus \{k\}$.
- The new edges are labelled by atoms of $\mathcal{A}$ as follows: (π, π) is labelled by $N'(k,k)$, $(\pi, \nu(i))$ by $N'(k,i)$, and $(\nu(i), \pi)$ by $N'(i,k)$.
- We may extend $\nu{\restriction}_{n\setminus\{k\}}$ to ν' defined on k, by setting $\nu'(k) = \pi$.
- We also add a new hyperedge $\nu'(\bar{a})$ for every $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$ with $k \in rng(\bar{a})$. We label it by $N'(\bar{a})$. This is well-defined.

  Because $N' \equiv_k N$, ν' is an embedding $: N' \to M_{t+1}$.
- For distinct (N, ν, k, N'), the new points $\pi_{(t,N,\nu,k,N')}$ are distinct.
- M_{t+1} will consist of M_t, with its old labels and edges, together with all these new points, edges, and labels.

It is easy to check that the properties 1–4 above are preserved by these actions. For 4, note that any clique in M_{t+1} is either a clique in M_t, for which we have the result inductively, or else it contains a new point $\pi_{(t,N,\nu,k,N')}$, in which case it is contained in $rng(\nu')$, since the only edges involving $\pi_{(t,N,\nu,k,N')}$ lie in this set.

Now define a labelled hypergraph M in the obvious way by $M = M_\omega = \bigcup_{t<\omega} M_t$. Clearly, M satisfies properties 1–4 (for the last, observe that any clique C in M_ω is finite of size at most n, hence $C \subseteq M_t$ for some $t < \omega$). It also has a further property:

5. If $k < n$, $N \equiv_k N'$ in $\mathcal{H}$, and $\nu : N \to M$ is an embedding, then $\nu{\restriction}_{n\setminus\{k\}}$ extends to an embedding $\nu' : N' \to M$.

Now define M as an $\mathcal{L}(\mathcal{A})$-structure by:

$$\begin{aligned} M \models r(x,y) &\iff (x,y) \text{ is an edge of } M \text{ and } M(x,y) \leq r \\ &\iff \exists t < \omega\ ((x,y) \text{ is an edge of } M_t \wedge M_t(x,y) \leq r), \end{aligned}$$

for each $r \in \mathcal{A}$ and $x, y \in M$.

Lemma 42 *M is an n-flat relativised representation of $\mathcal{A}$.*

Proof:

First, we show that $M \models S_{\mathcal{A}}$ (see definition 3). To see that $M \models \forall xy(1'(x,y) \leftrightarrow x = y)$, use properties 1 and 3 above. The boolean clauses are easy to check, since edges of M are labelled by atoms of $\mathcal{A}$.

We check the axiom for converse, $\forall xy(r(x,y) \leftrightarrow \breve{r}(y,x))$. Suppose that $M \models r(x,y)$. By property 1, $\{x,y\}$ is a clique in M. By property 4, there are $N \in \mathcal{H}$, an embedding $\nu : N \to M$, and $i,j < n$ with $\nu(i) = x$ and $\nu(j) = y$. Then $N(i,j) = M(x,y) \leq r$, and by lemma 14 and as the map $r \mapsto \breve{r}$ preserves $\leq$, we have $M(y,x) = N(j,i) = N(i,j)^{\smile} \leq \breve{r}$. So $M \models \breve{r}(y,x)$, as required. The other direction is similar.

Now consider the composition axiom. Let $x,y \in M$ with $M \models 1(x,y)$. First, suppose that $M \models r(x,z) \wedge s(z,y)$ for some $z \in M$. We require $M \models [r\,;s](x,y)$. By property 1, $\{x,y,z\}$ is a clique in M, so by property 4 there are $N \in \mathcal{H}$, an embedding $\nu : N \to M$, and $i,j,k < n$ with $\nu(i) = x$, $\nu(j) = y$, $\nu(k) = z$. Then $M(x,y) = N(i,j) \leq N(i,k)\,;N(k,j) = M(x,z)\,;M(z,y) \leq r\,;s$. So $M \models [r\,;s](x,y)$.

Conversely, suppose that $M \models [r\,;s](x,y)$. Since $\{x,y\}$ is a clique in M, there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $x,y \in rng(\nu)$ — say, $x = \nu(i), y = \nu(j)$. Let $k < n$ with $k \neq i,j$. Clearly, $N(i,j) \leq r\,;s$, so as $\mathcal{H}$ is a hyperbasis, there is $P \in \mathcal{H}$ with $N \equiv_k P$, $P(i,k) \leq r$, and $P(k,j) \leq s$. By property 5, there is an embedding $\pi : P \to M$ extending $\nu{\restriction}_{n\setminus\{k\}}$. Let $z = \pi(k)$. Then $M(x,z) = P(i,k) \leq r$ and $M(z,y) = P(k,j) \leq s$, giving the result.

The axiom $\exists xy\, r(x,y)$, for $r \neq 0$ in $\mathcal{A}$, holds because for any atom $a \leq r$ there is $N \in \mathcal{H}$ with $N(0,1) = a$ (as $\mathcal{H}$ is an n-dimensional hyperbasis for $\mathcal{A}$). Note that if $a \leq 1'$ then $a\,;a = \breve{a} = a$, so $N(0,0) = a$, too. Choose $S \subseteq n$ containing 0 and, if possible, 1, such that $N{\restriction}_S$ is maximal strict. Since $N{\restriction}_S \subseteq M_0$, there is evidently an edge (x,y) of M_0 with $M_0(x,y) = a$ (using the above if $a \leq 1'$). Hence, $M \models r(x,y)$.

So M is a relativised representation of $\mathcal{A}$. It remains to show that M is n-flat. Recall (definition 5) that $C^n(M) = \{\bar{a} \in {}^nM : rng(\bar{a}) \text{ is a clique in } M\}$. Let $\bar{a} \in C^n(M)$, $\varphi \in \mathcal{L}^n(\mathcal{A})$, and $i,j < n$. We have to show that $M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a})$ (see definitions 6, 7). We may assume that $i \neq j$. We begin with two claims.

Claim 1. Let $\bar{a} \in C^n(M)$. Then there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $\nu(i) = a_i$ for each $i < n$.

Proof of claim. By property 4, there is $N \in \mathcal{H}$ and an embedding $\nu : N \to M$ with $rng(\bar{a}) \subseteq rng(\nu)$. For each $i < n$, choose $i' < n$ with $\nu(i') = a_i$, and let σ be the map $\{(i,i') : i < n\} : n \to n$. As $\mathcal{H}$ is symmetric, $N\sigma \in \mathcal{H}$; and $\nu \circ \sigma : N\sigma \to M$ is an embedding with $\nu \circ \sigma(i) = a_i$ for each $i < n$.

Claim 2. Let $N \in \mathcal{H}$ and let $\mu, \nu : N \to M$ be embeddings. Let $\bar{a}, \bar{b} \in C^n(M)$ be given by $a_i = \mu(i)$, $b_i = \nu(i)$, for each $i < n$. Then $M \models_C \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$, for all $\varphi \in \mathcal{L}^n(\mathcal{A})$.

Proof of claim. By induction on φ. If φ is atomic, of the form $r(x_i, x_j)$, the result follows because embeddings preserve labels on edges of hypernetworks; and if φ is $x_i = x_j$ then φ is equivalent in M to $1'(x_i, x_j)$, so the preceding case gives the result here. The boolean cases are easy and we omit them. Consider $\exists x_i\varphi$. If $M \models_C \exists x_i\varphi(\bar{a})$, then there is $\bar{a}' \in C^n(M)$ with $\bar{a}' \equiv_i \bar{a}$ and $M \models_C \varphi(\bar{a}')$. By claim 1, there is $N' \in \mathcal{H}$ and an embedding $\mu' : N' \to M$ with $\mu'(l) = a'_l$ for all $l < n$. It follows that $N' \equiv_i N$. By property 5 of M, $\nu\restriction_{n\setminus\{i\}}$ extends to an embedding $\nu' : N' \to M$. Let $\bar{b}' \in C^n(M)$ be given by $b'_l = \nu'(l)$, for $l < n$. By the inductive hypothesis, $M \models_C \varphi(\bar{b}')$. But $\bar{b}' \equiv_i \bar{b}$, so $M \models_C \exists x_i\varphi(\bar{b})$. The converse is similar. This proves the claim.

Now assume that $M \models_C \exists x_i\exists x_j\varphi(\bar{a})$ for some $\bar{a} \in C^n(M)$, $\varphi \in \mathcal{L}^n(\mathcal{A})$, and distinct $i, j < n$. So there are $\bar{b}, \bar{c} \in C^n(M)$ with $\bar{a} \equiv_i \bar{b} \equiv_j \bar{c}$ and $M \models_C \varphi(\bar{c})$. As $\bar{a}, \bar{c} \in C^n(M)$, claim 1 provides $P, Q \in \mathcal{H}$ and embeddings $\pi : P \to M$, $\psi : Q \to M$ with $\pi(l) = a_l$ and $\psi(l) = c_l$ for all $l < n$. Thus, $\pi(l) = \psi(l)$ if $l \neq i, j$. As π, ψ are embeddings, $P \equiv_{ij} Q$. As $\mathcal{H}$ is a hyperbasis, there is $R \in \mathcal{H}$ with $P \equiv_j R \equiv_i Q$. By property 5 of M, $\pi\restriction_{n\setminus\{j\}}$ extends to an embedding $\rho : R \to M$, and $\rho\restriction_{n\setminus\{i\}}$ extends to an embedding $\psi' : Q \to M$. Let $d_l = \rho(l)$ and $e_l = \psi'(l)$, for $l < n$. Then $\bar{a} \equiv_j \bar{d} \equiv_i \bar{e}$ in $C^n(M)$; and by claim 2, $M \models_C \varphi(\bar{e})$. Hence, $M \models_C \exists x_i\varphi(\bar{d})$ and $M \models_C \exists x_j\exists x_i\varphi(\bar{a})$, as required.

Thus, M is n-flat. This proves lemma 42. $\square$

Thus M is an n-flat relativised representation of $\mathcal{A}$. By lemma 4, since every edge of M is labelled by an atom, it has the further property of being *complete* — respecting all meets and joins that exist in $\mathcal{A}$. $\square$

Now suppose that $\mathcal{A}$ is a subalgebra of an atomic relation algebra $\mathcal{B}$ with an n-dimensional hyperbasis, as in theorem 1(3). By theorem 41, $\mathcal{B}$ has an n-flat relativised representation, M^+, say. M^+ is a $\mathcal{L}(\mathcal{B})$-structure. It follows from the definitions that the reduct of M^+ to the language $\mathcal{L}(\mathcal{A})$ is an n-flat relativised representation of $\mathcal{A}$. We obtain theorem 1, (3) $\Rightarrow$ (4).

So the equivalence of parts 1–4 of theorem 1 has now been shown. The following corollary is also obtained. Recall that Crs_n is the class of all cylindric relativised set algebras, and G_n the subclass consisting of the algebras with unit closed under permutations and substitutions.

Corollary 43 *For all finite $n \geq 5$, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{Crs}_n) = \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$.*

Proof:

Since $\mathsf{CA}_n \supseteq \mathsf{CA}_n \cap \mathsf{Crs}_n \supseteq \mathsf{CA}_n \cap \mathsf{G}_n$, it suffices to prove that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$. If $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, then by theorem 1, $\mathcal{A}$ has an n-flat relativised representation. Now, by remark 12, $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}(\mathsf{CA}_n \cap \mathsf{G}_n)$. $\square$

5 Smooth relativised representations

There is an alternative approach to representations of algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ — what we call *n-smooth relativised representations.* Essentially, we make an n-flat relativised representation n-smooth by dropping explicit mention of the formulas φ of $\mathcal{L}^n(\mathcal{A})$, and stating instead by means of equivalence relations which n-tuples of a relativised representation agree on all these formulas with respect to $\models_C$. We can axiomatise the properties required for quantifiers to commute, by stating that the equivalence relations should have certain 'n-back-and-forth' properties. The reader may consult [4] for similar work in finite model theory, showing that the equivalence relations can be taken to be definable in fixed-point logic. n-smooth representations have the disadvantage (over n-flat ones) that one must expand a relativised representation M by adding further relations, but the advantage that the infinitely many conditions $M \models_C (\exists x_i \exists x_j \varphi \leftrightarrow \exists x_j \exists x_i \varphi)(\bar{a})$, for all formulas φ, reduce to a single one. In section 6 we will use them to obtain an equational axiomatisation of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We fix $n \geq 3$ here. Recall from notation 2 that if $\bar{x}, \bar{y}$ are m-tuples, we write $(\bar{x} \mapsto \bar{y})$ for $\{(x_i, y_i) : i < |\bar{x}|\}$; this may or may not be a well-defined map. The concatenation of tuples $\bar{x}, \bar{y}$ is denoted $\bar{x}\bar{y}$.

DEFINITION 44 Let M be a relativised representation of the relation algebra $\mathcal{A}$. Recall that a clique in M is a subset $C \subseteq M$ with $M \models 1(x, y)$ for all $x, y \in C$. For $m \leq n$, $C^m(M) = \{\bar{x} \in {}^m M : rng(\bar{x}) \text{ is a clique in } M\}$.

1. M is said to be *n-square* if for any clique C in M with $|C| < n$, if $r, s \in \mathcal{A}$, $x, y \in C$, and $M \models (r\,;s)(x, y)$, then there is a point $z \in M$ such that $C \cup \{z\}$ is a clique in M and $M \models r(x, z) \wedge s(z, y)$.

2. A *local isomorphism* of M is a partial, finite map $\iota : M \to M$ such that $dom(\iota), rng(\iota)$ are cliques, and satisfying $M \models a(x, y)$ iff $M \models a(\iota(x), \iota(y))$, for all $x, y \in dom(\iota)$ and $a \in \mathcal{A}$.

3. M is said to be an *n-smooth relativised representation* if it is n-square,[2] and also, for each $0 < m \leq n$, there is an equivalence relation E^m on $C^m(M)$ such that:
   - if $0 < l, m \leq n$, $(\bar{x}, \bar{y}) \in E^m$, and $\theta : l \to m$, then $(\bar{x} \circ \theta, \bar{y} \circ \theta) = ((x_{\theta(0)}, \ldots, x_{\theta(l-1)}), (y_{\theta(0)}, \ldots, y_{\theta(l-1)})) \in E^l$,
   - for any $(\bar{x}, \bar{y}) \in E^m$, $r \in \mathcal{A}$, and $i, j < m$, if $M \models r(x_i, x_j)$ then $M \models r(y_i, y_j)$ (i.e., $(\bar{x} \mapsto \bar{y})$ is a well-defined local isomorphism of M), and
   - if $(\bar{x}, \bar{y}) \in E^{n-2}$ and $rng(\bar{x}x)$ and $rng(\bar{y}y)$ are cliques in M, then there exists a point $z \in M$ such that $(\bar{x}x, \bar{y}z) \in E^{n-1}$ and $rng(\bar{y}yz)$ is a clique.

4. We can extend the theory $S_{\mathcal{A}}$ to an $\mathcal{L}(\mathcal{A})$-theory $Sq^n(\mathcal{A})$ whose models (if any) are precisely the n-square relativised representations of $\mathcal{A}$; and we can extend the language $\mathcal{L}(\mathcal{A})$ to the language $\mathcal{L}^+(\mathcal{A})$ by adding n extra symbols E^m, for $0 < m \leq n$, where E^m is a $2m$-ary predicate symbol, and extend the theory $Sq^n(\mathcal{A})$ to a $\mathcal{L}^+(\mathcal{A})$-theory $Sm^n(\mathcal{A})$ whose models are precisely the n-smooth relativised representations of $\mathcal{A}$. So, for example, the statement 'E^m is an equivalence relation on $C^m(M)$' translates to the $\mathcal{L}^+(\mathcal{A})$-sentence

$$\forall \bar{x}, \bar{y}, \bar{z}\, [(Clique(\bar{x}) \leftrightarrow E^m(\bar{x}, \bar{x})) \wedge (E^m(\bar{x}, \bar{y}) \wedge E^m(\bar{y}, \bar{z}) \to E^m(\bar{z}, \bar{x}))],$$

   where $\bar{x}, \bar{y}, \bar{z}$ are sequences of m distinct variables and $Clique(\bar{x})$ abbreviates the formula $\bigwedge_{j,k<m} 1(x_j, x_k)$.

[2]This condition is not needed, as it follows from the others, but it is easier to add it explicitly.

Remark 45 For n-smooth M, the set $\Theta = \{(\bar{x} \mapsto \bar{y}) : (\bar{x}, \bar{y}) \in \bigcup_{m \leq n} E^m\}$ is a certain kind of n-back-and-forth system of local isomorphisms of M. Each map $\theta \in \Theta$ with $0 < |dom(\theta)| \leq n - 2$ can be extended within Θ to be defined on a new point a, so long as $dom(\theta) \cup \{a\}$ is a clique. Moreover, the extension can be chosen so that its range extends to a clique containing some other new point b, so long as $rng(\theta) \cup \{b\}$ was already a clique. Such n-back-and-forth systems offer an alternative definition of n-smooth.

Remark 46 In [14], we showed that n-square relativised representations correspond to subalgebras of relation algebras with n-dimensional relational bases: a relation algebra $\mathcal{A}$ has such a representation iff $\mathcal{A} \in \mathsf{RA}_n$, in the notation of [31]. A relativised representation is n-square iff all formulas of the form $1(x_i, x_j) \wedge \exists x_k(r(x_i, x_k) \wedge s(x_k, x_j))$, where $i, j, k < n$, $k \neq i, j$, and $r, s \in \mathcal{A}$, have the same meaning whether evaluated classically or in the relativised semantics $\models_C$. We already gave an argument (in theorem 11) that shows that any n-flat relativised representation is n-square. Theorem 1 will show that n-smooth relativised representations correspond exactly to $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We now show that (4) $\Rightarrow$ (5) in theorem 1.

Proposition 47 *Let $\mathcal{A}$ be a relation algebra with an n-flat relativised representation, for some $n \geq 3$. Then $\mathcal{A}$ has an n-smooth relativised representation.*

Proof [sketch]:

Let M be an n-flat relativised representation of $\mathcal{A}$. By replacing M with a suitable elementary extension, we may assume it is ω-saturated. This preserves n-flatness, as $C^n(M)$ is first-order definable. Using lemmas 8 and 9, it can be checked that M is n-square; this is similar to the last part of the proof of theorem 11, and is left to the reader.

A partial map $\rho : M \to M$ is said to be *n-elementary* if whenever $\bar{a} \in C^n(M)$ and $rng(\bar{a}) \subseteq dom(\rho)$, then $M \models_C \varphi(\bar{a})$ iff $M \models_C \varphi(\rho(\bar{a}))$ for all formulas φ of $\mathcal{L}^n(\mathcal{A})$. Now, for $\bar{x}, \bar{y} \in C^m(M)$ $(m \leq n)$, let $E^m(\bar{x}, \bar{y})$ hold iff $(\bar{x} \mapsto \bar{y})$ is a well-defined n-elementary map.

We check that the E^m meet the n-smoothness conditions. First, an n-elementary map must preserve edge relations, as these are just the atomic formulas of $\mathcal{L}^n(\mathcal{A})$. Second, if $(\bar{x}, \bar{y}) \in E^m$ and $\theta : l \to m$ is any map, we must check that $(\bar{x} \circ \theta, \bar{y} \circ \theta) \in E^l$ — i.e., $(\bar{x} \circ \theta \mapsto \bar{y} \circ \theta)$ is a well-defined n-elementary map. But this is clear, since it is a restriction of $(\bar{x} \mapsto \bar{y})$.

Finally, suppose that $(\bar{x}, \bar{y}) \in E^{n-2}$ and $rng(\bar{x}x)$, $rng(\bar{y}y)$ are cliques in M. So the map $(\bar{x} \mapsto \bar{y})$ is well-defined and n-elementary. Let

$$\begin{aligned} \bar{a} &= (x, x_0, x_1, \ldots, x_{n-3}, x_0) \\ \bar{a}' &= (x_0, x_0, x_1, \ldots, x_{n-3}, x_0) \\ \bar{b}' &= (y_0, y_0, y_1, \ldots, y_{n-3}, y_0) \\ \bar{b} &= (y_0, y_0, y_1, \ldots, y_{n-3}, y). \end{aligned}$$

These lie in $C^n(M)$. For each $\varphi \in \mathcal{L}^n(\mathcal{A})$ such that $M \models_C \varphi(\bar{a})$, we have $M \models_C \exists x_0 \varphi(\bar{a}')$, so (by n-elementarity) $M \models_C \exists x_0 \varphi(\bar{b}')$ and $M \models_C \exists x_{n-1} \exists x_0 \varphi(\bar{b})$. By

n-flatness, $M \models_C \exists x_0 \exists x_{n-1} \varphi(\bar{b})$, for all such φ. So using ω-saturation twice, we can find $\bar{b} \equiv_0 \bar{c} \equiv_{n-1} \bar{d}$ in $C^n(M)$ with $M \models_C \varphi(\bar{d})$ for all these φ. So

$$\begin{array}{rcl} \bar{c} & = & (z, y_0, y_1, \ldots, y_{n-3}, y) \\ \bar{d} & = & (z, y_0, y_1, \ldots, y_{n-3}, y_0), \end{array}$$

for some $z \in M$. (That $d_{n-1} = y_0$ is seen by taking φ to be $x_{n-1} = x_1$.) Then $rng(\bar{y}yz) = rng(\bar{c})$ is a clique. Further, $(\bar{a} \mapsto \bar{d}) = (\bar{x}x \mapsto \bar{y}z)$ is a well-defined n-elementary map. So $(\bar{x}x, \bar{y}z) \in E^{n-1}$, as required. □

The following two theorems show (5) ⇒ (2) of theorem 1, by applying theorem 49 to $\mathcal{A}^+$.

THEOREM 48 *If $\mathcal{A}$ has an n-smooth relativised representation then $\mathcal{A}^+$ has a complete n-smooth relativised representation.*

Cf. Monk's theorem (reported in [32], theorem 2.12) that if a relation algebra $\mathcal{A}$ is representable then its canonical extension $\mathcal{A}^+$ has a complete representation.

PROOF [SKETCH]:

To say that $\mathcal{A}$ has an n-smooth relativised representation is to say that the first-order theory Sm^n, defined in definition 44 part 4, is consistent. Let M be an ω-saturated model of Sm^n. The proof of [11, theorem 22] or [14, lemma 26] shows how to check that M is a complete, n-square relativised representation of $\mathcal{A}^+$. Since $M \models Sm^n$, it follows that M is n-smooth. □

THEOREM 49 *If the atomic relation algebra $\mathcal{A}$ has a complete n-smooth relativised representation then $\mathcal{A}$ has an n-dimensional hyperbasis.*

PROOF:

Let M be a complete n-smooth relativised representation of $\mathcal{A}$. For $m \leq n$, E^m is an equivalence relation on $C^m(M)$. So the union $E = \bigcup_{0<m\leq n} E^m$ is an equivalence relation on $\bigcup_{0<m\leq n} C^m(M)$. Let Λ be the set of E-equivalence classes. For any n-dimensional Λ-hypernetwork N over $\mathcal{A}$ and map $\nu : n \to M$, we say that ν is an *embedding* of N into M if (i) for $r \in At(\mathcal{A})$ and $i, j < n$, we have $N(i,j) = r$ iff $M \models r(\nu(i), \nu(j))$, and (ii) for any $\bar{a} \in {}^{\leq n}n$ with $|\bar{a}| \neq 2$, $\nu(\bar{a})$ is a member of the equivalence class $N(\bar{a}) \in \Lambda$.

We let $\mathcal{H}$ be the set of all n-dimensional Λ-hypernetworks over $\mathcal{A}$ that embed into M, and check that $\mathcal{H}$ is a hyperbasis. The first two properties of hyperbases are easy to verify, using n-squareness. We check the 'amalgamation' condition for $\mathcal{H}$. So take $P, Q \in \mathcal{H}$ and distinct $i, j < n$ with $P \equiv_{ij} Q$. We seek a hypernetwork $R \in \mathcal{H}$ with $P \equiv_i R \equiv_j Q$.

Let $\bar{b}$ be any $(n-2)$-tuple enumerating $n \setminus \{i, j\}$. Since $P, Q \in \mathcal{H}$, there are embeddings $\pi : P \to M$ and $\psi : Q \to M$. Since $P \equiv_{ij} Q$, we know that $P(\bar{b}) = Q(\bar{b}) \in \Lambda$, so that $(\pi(\bar{b}), \psi(\bar{b})) \in E$. Now $rng(\pi\restriction_{n\setminus\{i\}})$ and $rng(\psi\restriction_{n\setminus\{j\}})$ are both cliques, so since M is n-smooth, there is a point $z \in M$ such that $rng(\psi\restriction_{n\setminus\{j\}}) \cup \{z\}$ is a clique and $((\pi(j), \pi(b_0), \ldots, \pi(b_{n-3})), (z, \psi(b_0), \ldots, \psi(b_{n-3}))) \in E$. Let $\rho : n \to M$ be defined by: $\rho(k) = \psi(k)$ for $k \neq j$, and $\rho(j) = z$. By the first part of the definition of n-smooth, for any $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$ we have $(\pi(\bar{a}), \rho(\bar{a})) \in E$.

We use ρ to define the required hypernetwork $R \in \mathcal{H}$ in the obvious way: for $k, l < n$, $R(k,l)$ is the atom α of $\mathcal{A}$ satisfying $M \models \alpha(\rho(k), \rho(l))$, and for $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$, $R(\bar{a})$ is the E-class of $\rho(\bar{a})$. Since ψ and ρ agree on all points except perhaps j, it follows that $R \equiv_j Q$. For any $\bar{a} \in {}^{\leq n}(n \setminus \{i\})$, because $E(\pi(\bar{a}), \rho(\bar{a}))$, we have $P(\bar{a}) = R(\bar{a})$. Hence $R \equiv_i P$, as required. □

Now we have proved the equivalence of parts (1) to (5) of theorem 1.

Remark 50 Adjustments to the equivalence relations E^m yield rather different classes of algebras. So at one extreme, if for each $m \leq n$ we make E^m as small as possible (subject to being an equivalence relation) we get the identity relation $(\bar{x}, \bar{y}) \in E^m \Leftrightarrow \bar{x} = \bar{y}$. From this, if M expanded by the E^m is n-smooth, we can show that $\{(x,y) : M \models 1(x,y)\}$ is transitive. So this gives a classical representation and thus we obtain the class RRA. This would be too strong a requirement to characterise $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$.

At the other extreme, suppose that each E^m is as big as possible (subject to preserving all edge relations). That is, $E^m(\bar{x}, \bar{y})$ holds iff $(\bar{x} \mapsto \bar{y})$ is a well-defined local isomorphism of M. Under this condition, an n-smooth relativised representation becomes *n-homogenous* — that is, for all local isomorphisms ι of M with $0 < |\iota| \leq n-2$ and for all $x, y \in M$ with $\{x\} \cup dom(\iota)$ and $\{y\} \cup rng(\iota)$ both cliques, there is $z \in M$ such that $\iota \cup \{(x,z)\}$ is a local isomorphism and $\{y, z\} \cup rng(\iota)$ is a clique (cf. remark 45). Then we can show, in much the same way as earlier, that for any relation algebra $\mathcal{A}$:

1. if $\mathcal{A}$ has an n-flat relativised representation with quantifier elimination with respect to the semantics $\models_C$, then it has an n-homogeneous relativised representation;

2. if $\mathcal{A}$ has an n-homogeneous relativised representation M, then M 'is' a complete n-homogeneous relativised representation of the atomic relation algebra $\mathcal{B} \supseteq \mathcal{A}$ consisting of unions of isomorphism types of elements of 1^M;

3. if $\mathcal{A}$ is atomic, then it has a complete n-homogeneous relativised representation iff it has an n-dimensional cylindric basis (also cf. theorem 63 below).

Again, these conditions are too strong for $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$, but in a different direction (see theorem 60).

6 Axiomatising $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$

We can use theorem 1 to construct a recursive axiomatisation of $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ by defining a game to determine whether a relation algebra has an n-smooth relativised representation. This method was used in [11, section 9] to axiomatise the representable relation algebras and cylindric algebras following earlier axiomatisations in [24, 35, 7]. A similar game-theoretic method was outlined in [14, section 4.3], to axiomatise the class RA_n. A summary of various game-theoretic axiomatisations appeared in [9]. In [17] it is shown how to axiomatise the class of complex algebras over any given variety; and in [8] it will be shown how to obtain an explicit universal axiomatisation of any pseudo-elementary class of algebras that is closed under subalgebras. As these methods have been used before and will be made available in a general form soon, it does not seem necessary to go through the axiomatisation in detail here. Instead, we only sketch an outline of the method.

We can define a two-player game $G^n_\omega(\mathcal{A})$ over a relation algebra $\mathcal{A}$, and show, for countable $\mathcal{A}$, that a winning strategy for the second player is equivalent to the existence of an n-smooth relativised representation of $\mathcal{A}$. This game has countably many rounds played over finite structures that are intended to provide better and better approximations to a genuine n-smooth relativized representation. The first player, '$\forall$', picks defects in the current approximation, and the second player, '$\exists$', tries to repair the defect by refining the approximation.

These approximations are finite structures (X, h, E) where X is a finite set, h is a partial labelling $h : {}^{\leq n}X \to \mathcal{A}$ such that $dom(h) \cap (X \times X)$ is reflexive and symmetric, and E is an equivalence relation over $\{\bar{x} \in {}^{\leq n}X : (x_i, x_j) \in dom(h),\ i, j < |\bar{x}|\}$ satisfying, if we may speak loosely, all the universal conditions for an n-smooth relativised representation. So, (i) for $x \in X$ we have $h(x, x) \leq 1'$, (ii) if $(x, y), (y, z), (x, z) \in dom(h)$ then $h(x, y)\,; h(y, z) \cdot h(x, z) \neq 0$, (iii) for $(\bar{x}, \bar{y}) \in E$ if $|\bar{x}| = |\bar{y}| = l$, say, and $\theta : l \to m$ (some $m \leq n$) then $(\bar{x} \circ \theta, \bar{y} \circ \theta) \in E$, and (iv), for $(\bar{x}, \bar{y}) \in E$ and $i, j < l$ we have $h(x_i, x_j) \cdot h(y_i, y_j) \neq 0$.

An approximation (X, h, E) determines an $\mathcal{L}(\mathcal{A})$-structure $M = M(X, h, E)$ with domain X and defined by $M \models a(x, y)$ iff $h(x, y) \leq a$, for $x, y \in X$ and $a \in \mathcal{A}$. M may fail to be an n-smooth relativised representation for four reasons.

1. There could be a non-zero $a \in A$ but no edge $(x, y) \in dom(h)$ with $h(x, y) \leq a$.

2. The approximation might not carry enough information: there could be an edge $(x, y) \in dom(h)$ and an element $a \in \mathcal{A}$ such that $h(x, y) \not\leq a$ and $h(x, y) \not\leq (-a)$. In a genuine n-smooth relativised representation, for any $a \in \mathcal{A}$ and for any labelled edge in the relativised representation, we know that either a or $-a$ holds on that edge.

3. There might be an edge $(x, y) \in dom(h)$ with $x, y \in C$ for some clique C of the approximation with $|C| < n$ and $r, s \in \mathcal{A}$ such that $h(x, y) \leq r; s$. This is a defect if there is no witness z in the approximation such that $\{z\} \cup C$ is a clique, $(x, z), (z, y) \in dom(h)$ and $h(x, z) \leq r,\ h(z, y) \leq s$.

4. Finally there are 'hyperdefects'. A hyperdefect is a pair $\bar{x}, \bar{y} \in {}^{\leq n-2}X$ with $(\bar{x}, \bar{y}) \in E$ and two points $x, y \in X$ such that $rng(\bar{x}x), rng(\bar{y}y)$ are cliques but there is no witness $z \in X$ such that $rng(\bar{y}yz)$ is a clique of h and $(\bar{x}x, \bar{y}z) \in E$.

We can define a game played on approximations in which player $\forall$ will pick inaccuracies of these four types in the current approximation, and player $\exists$ will try to repair them by refining it to a better one. If she suceeds in finding the required approximation in every round of the game, $\exists$ wins; if not, $\forall$ wins.

Crucially, for each type of defect, $\exists$ can narrow her set of possible responses to a finite number and still win the game, if she has a winning strategy at all.

The first kind of defect, 'lack of faithfulness', is dealt with in the initial round of the game. In this round, $\forall$ is allowed to pick any non-zero $a \in \mathcal{A}$, and $\exists$ must respond with an approximation (X, h, E) containing a labelled edge (x, y) such that $h(x, y) \leq a$. If such an approximation exists then there is a unique approximation that she can play which gives the least possible information and we assume that she plays this.

For case 2, when presented with a labelled edge $(x, y) \in dom(h)$ and $a \in \mathcal{A}$, $\exists$ either 'accepts' by resetting the label on (x, y) to $h(x, y) \cdot a$ in the new approximation, or she 'rejects' by resetting the label on (x, y) to $h(x, y) \cdot (-a)$; in either case other labels and the relation E are unaltered. These two responses repair the inaccuracy in the approximation but give no more information than is required.

Given a defect $((x,y),C,r,s)$ of type 3, $\exists$ chooses a set $C' \subseteq X$ containing C, adds a new point z to X, labels all edges $(z,c),(c,z) : c \in C'$ with 1 except that $(x,z),(z,y)$ are labelled by r,s respectively, labels (z,z) with 1', and leaves other labels unaltered. She also chooses one of the finitely many ways of extending E to an equivalence relation over ${}^{\leq n}(X \cup \{z\})$.

Finally, if presented with a hyperdefect $(\bar{x},\bar{y},x,y)$, $\exists$ adds a new point z to X, extends the domain of h so that $rng(\bar{y}yz)$ is a clique, leaving old labels unchanged and labelling new edges with 1, and chooses an extension of E containing the tuple $(\bar{x}x,\bar{y}z)$.

These games, which have countably many rounds, are designed to test membership of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ in that the algebra belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if it has an n-smooth relativised representation which is equivalent, at least for countable algebras, to the existence of a winning strategy in the game for $\exists$. We can define approximations to the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ by curtailing the games to finitely many rounds. We write $G^n_k(\mathcal{A})$ for the game with k rounds. As before, if $\exists$ successfully plays the required refinement in each of the k rounds she wins, otherwise $\forall$ wins.

We now summarise the main results on these games. Recall that a relation algebra is *simple* iff $1\,;r\,;1 = 1$ for all non-zero elements r of the algebra.

Proposition 51

1. *For any relation algebra $\mathcal{A}$, $\exists$ has a winning strategy in $G^n_k(\mathcal{A})$ (for all $k < \omega$) if and only if she has a winning strategy in $G^n_\omega(\mathcal{A})$.*
2. *There is a universal first-order sentence σ^n_k, effectively constructible from k, n, such that for any relation algebra $\mathcal{A}$, $\mathcal{A} \models \sigma^n_k$ if and only if $\exists$ has a winning strategy in $G^n_k(\mathcal{A})$.*
3. *A countable simple relation algebra $\mathcal{A}$ has an n-smooth relativised representation if and only if $\exists$ has a winning strategy in $G^n_\omega(\mathcal{A})$.*

The first part, which uses a version of König's tree lemma, depends on the fact that $\exists$ has only finitely many choices for her moves. The sentences in the second part are universal because $\exists$ is never required to choose an element of the algebra. In the third part, if $\exists$ has a winning strategy then, as $\mathcal{A}$ is countable, there is a play of the game in which $\forall$ picks all possible defects. The 'limit' of such a play will determine an n-smooth relativised representation of $\mathcal{A}$, if $\mathcal{A}$ is simple.

Thus, for countable simple $\mathcal{A}$, we have $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \iff \mathcal{A} \models \{\sigma^n_k : k < \omega\}$.

The final step is to replace the universal axioms σ^n_k by equations and to remove the assumptions of countability and simplicity. This is a fairly standard argument. The analogue of the following lemma for neat reducts of cylindric algebras was proved by Monk [33].

Proposition 52 *For $n \geq 4$, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety contained in* RA.

Proof:

We show that $\mathbf{HSP}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Evidently, if $\mathcal{B}_i \in \mathsf{CA}_n$, $i \in I$, then $\prod_{i\in I}\mathfrak{Ra}\mathcal{B}_i = \mathfrak{Ra}\prod_{i\in I}\mathcal{B}_i \in \mathfrak{Ra}\mathsf{CA}_n$. So $\mathbf{P}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, and it suffices to check that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is closed under homomorphic images. Let $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{B}$ for $\mathcal{B} \in \mathsf{CA}_n$, and let I be an ideal of $\mathcal{A}$. Plainly, I is a subset of $\mathcal{B}$. Let J be the ideal of $\mathcal{B}$ generated by I. By [6, theorem 2.3.8], $J = \{b \in \mathcal{B} : b \leq \mathsf{c}_{i_0}\ldots\mathsf{c}_{i_l}(x_0+\cdots+x_{k-1})$ for some $i_0,\ldots,i_l < n$ and $x_0,\ldots,x_{k-1} \in I\}$. Now since I is an ideal of $\mathcal{A}$, it is closed under $+$. Further, if $x \in I$ then $1\,;x \in I$. By [6, 1.5.9(i),1.5.8(i)] and the fact that x is 2-dimensional,

$$1\,;x = \mathsf{c}_2(\mathsf{s}_2^1 1 \cdot \mathsf{s}_2^0 x) = \mathsf{c}_2(1 \cdot \mathsf{s}_2^0 \mathsf{c}_2 x) = \mathsf{c}_2 \mathsf{s}_2^0 \mathsf{c}_2 x = \mathsf{c}_0 \mathsf{s}_0^2 \mathsf{c}_2 x = \mathsf{c}_0 \mathsf{c}_2 x = \mathsf{c}_0 x.$$

So $\mathsf{c}_0 x \in I$, and similarly, $\mathsf{c}_1 x \in I$. So the above expression simplifies to $J = \{b \in \mathcal{B} : b \leq x$ for some $x \in I\}$. It follows that $J \cap \mathcal{A} = I$.

Now define a homomorphism from $\mathcal{A}/I$ into $\mathfrak{Ra}(\mathcal{B}/J)$ by $a/I \mapsto a/J$ (for $a \in \mathcal{A}$). As $J \cap \mathcal{A} = I$, this map is one-one. Since $\mathcal{B}/J \in \mathsf{CA}_n$, we have $\mathcal{A}/I \in \mathbf{S}\mathfrak{Ra}\{\mathcal{B}/J\} \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

So $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is closed under $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$. By Birkhoff's theorem, this shows that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety and can be equationally axiomatised. It is known (see proposition 61) that $\mathsf{RA} = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4$, and clearly, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supseteq \cdots$, which completes the proof. □

LEMMA 53 *Let $\psi(\bar{x})$ be any quantifier-free formula of the language of relation algebras. Then there is an equation, of the form $s = 0$ for some relation algebra term $s(\bar{x})$, that is equivalent in any simple relation algebra to ψ and which can be obtained effectively from ψ.*

PROOF:

By induction on ψ. The equation $t = u$ is equivalent in any relation algebra to $(t \cdot -u) + (u \cdot -t) = 0$. Assume inductively that ψ is equivalent to $t = 0$, and χ to $u = 0$. Then $\neg\psi$ is equivalent to $\neg(t = 0)$ and so (in simple relation algebras) to $1\,;t\,;1 = 1$, and so to $-(1\,;t\,;1) = 0$. Clearly, $\psi \wedge \chi$ is equivalent in any relation algebra to $t + u = 0$. □

Each universal sentence σ_k^n $(k < \omega)$ from proposition 51 can be put in prenex form in an effective manner. It then follows from lemma 53 that each σ_k^n is equivalent in simple relation algebras to an equation ε_k^n which can be obtained effectively from k.

We can now prove the final part of theorem 1, which provides a recursive axiomatisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

THEOREM 54 *For $n \geq 5$, the variety $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is axiomatised by the equations defining RA together with the equations ε_k^n for $k < \omega$.*

PROOF:

It suffices to show that a relation algebra $\mathcal{A}$ is in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ iff $\mathcal{A} \models \varepsilon_k^n$ for each k. As $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is elementary (by proposition 52), we may suppose that $\mathcal{A}$ is countable. Now by [22, theorem 4.15], $\mathcal{A}$ is embeddable in a direct product of simple algebras $\mathcal{A}_i$ $(i \in I)$, where each $\mathcal{A}_i$ is a homomorphic image of $\mathcal{A}$. The $\mathcal{A}_i$ are clearly countable also. Since $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety (proposition 52), $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ iff $\mathcal{A}_i \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for all $i \in I$; and since the ε_k^n are equations, $\mathcal{A} \models \varepsilon_k^n$ (all $k < \omega$) iff $\mathcal{A}_i \models \varepsilon_k^n$ (all k) for each $i \in I$. So we may further assume that $\mathcal{A}$ is a simple relation algebra. But now, $\mathcal{A} \models \{\varepsilon_k^n : k < \omega\}$ iff $\mathcal{A} \models \{\sigma_k^n : k < \omega\}$, iff $\mathcal{A}$ has an n-smooth relativised representation (by proposition 51), iff $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ (by the equivalence of parts (1) and (5) of theorem 1). □

The proof of theorem 1 is now complete.

7 Remarks on the theorem

Here, we present some observations and questions.

REMARK 55 Theorem 1 also holds degenerately for $n = 4$ — all five parts are true for any relation algebra in this case. For part 1, it is well-known that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}$ [7, theorems 5.3.8, 5.3.17]. Maddux shows in [28] that the set of all 4-dimensional atomic networks, for any atomic relation algebra, is a 4-dimensional 'relational basis', and in dimension 4, such a basis is essentially a Λ-hyperbasis for $|\Lambda| = 1$ (the 'hyperlabels' carry no information). Since RA is a canonical variety, it follows that if $\mathcal{A} \in \mathsf{RA}$ then $\mathcal{A}^+ \in \mathsf{RA}$ and hence that $\mathcal{A}^+$ has a 4-dimensional hyperbasis. As shown in [14], any relation algebra has a '4-square' relativised representation, and the notions of 4-square and 4-flat coincide; or one may simply apply theorem 41. So parts 2 and 4 also hold for any relation algebra. Part 5 holds, by proposition 47. Since $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}$, by proposition 51 we obtain that every countable relation algebra satisfies $\{\sigma_k^4 : k < \omega\}$ and hence an arbitrary relation algebra satisfies $\{\varepsilon_k^4 : k < \omega\}$, proving part 6. Of course, we are not very interested in obtaining a recursive axiomatisation of RA.

REMARK 56 We saw in proposition 52 that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is a variety, for $n \geq 4$. We remark at this point that it is a canonical variety (closed under the map $\mathcal{A} \mapsto \mathcal{A}^+$). For, let $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. By theorem 1, $\mathcal{A}^+$ has an n-dimensional hyperbasis, and by (3) $\Rightarrow$ (1) of the theorem, $\mathcal{A}^+ \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

REMARK 57 In [13] it was shown that the problem of whether a finite relation algebra is representable is undecidable. The same techniques will show that for finite $n \geq 5$, it is undecidable whether a finite relation algebra is in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. The set of isomorphism types of finite algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is co-r.e., since we have a recursive axiomatisation of it. So if $\mathcal{A}$ is a finite relation algebra in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, it does not follow that $\mathcal{A}$ has a *finite* n-dimensional hyperbasis or a finite n-flat relativised representation, nor that $\mathcal{A} \subseteq \mathfrak{Ra}\mathcal{C}$ for a finite $\mathcal{C} \in \mathsf{CA}_n$, as if any of these implications held, it would mean that the set of isomorphism types of finite algebras in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ would be r.e. and hence recursive, a contradiction.

By modifying the definition of n-flat relativised representation so that the unit $C^n(M)$ is replaced by a n-ary relation contained in $C^n(M)$, it can be shown that a finite relation algebra is a subalgebra of $\mathfrak{Ra}\mathcal{C}$ for a finite n-dimensional cylindric algebra $\mathcal{C}$ iff it has a finite (modified) n-flat relativised representation. The proof uses techniques of [1]. As far as we know, whether this holds for 'unmodified' n-flat relativised representations is an open question.

REMARK 58 In [31, §6], a discussion of n-variable proof theory is given. Using n-dimensional cylindric bases, an algebraic semantics for n-variable formulas is introduced; it is shown that each axiom of a Hilbert system for n-variable first-order logic taken from [37] is valid in this semantics, and that the proof rules preserve validity. We note that n-flat relativised representations form an alternative to this algebraic semantics, with the advantage of being quite close to classical Tarskian semantics for first-order logic. One may hope to prove a soundness and completeness theorem for n-flat relativised representations with respect to n-variable proof theory, but a full treatment would involve cylindric algebras and we do not pursue it further here. The connection of neat reducts with proof theory is discussed further in Part II of this paper.

8 Hyperbases, cylindric bases, and relational bases

Hyperbases are related to the relational and cylindric bases of Maddux. In this section we investigate the connections.

Let $\mathcal{A}$ be an atomic relation algebra, and fix $n \geq 4$. An *n-dimensional (atomic) network* over $\mathcal{A}$ is similar to a n-dimensional hypernetwork N over $\mathcal{A}$, except that $N(\bar{a})$ is only defined if $|\bar{a}| = 2$. The same information is carried by an Λ-hypernetwork N over $\mathcal{A}$ if we insist that $|\Lambda| = 1$.

- An *n-dimensional relational basis* is then a set of n-dimensional networks over $\mathcal{A}$, satisfying conditions 1 and 2 of definition 15.
- An *n-dimensional cylindric basis* is a set of n-dimensional networks over $\mathcal{A}$, satisfying all three conditions of definition 15.

The first definition is the same as that of Maddux [28], and the second is equivalent to Maddux's definition [31].

8.1 Direct connections between bases

Proposition 59 *Let $\mathcal{A}$ be an atomic relation algebra with an n-dimensional Λ-hyperbasis for some set Λ and some $n \geq 4$. Then $\mathcal{A}$ has a n-dimensional relational basis.*

Proof:

Let $\mathcal{H}$ be an n-dimensional Λ-hyperbasis for $\mathcal{A}$. If $N \in \mathcal{H}$, write $N^\dagger$ for the n-dimensional network over $\mathcal{A}$ obtained by throwing away the labels $N(\bar{a})$ for any $\bar{a}$ of length not equal to 2. We may regard $N^\dagger$ as a reduct of N. Write $\mathcal{H}^\dagger$ for $\{N^\dagger : N \in \mathcal{H}\}$.

We claim that $\mathcal{H}^\dagger$ is an n-dimensional relational basis for $\mathcal{A}$. We check the two properties required.

1. Let $r \in \mathcal{A}$ be non-zero. As $\mathcal{H}$ is a hyperbasis, we may choose $N \in \mathcal{H}$ with $N(0,1) \leq r$. Then certainly, $N^\dagger(0,1) \leq r$, also.
2. Let $N^\dagger \in \mathcal{H}^\dagger$, let $i, j, k < n$ with $k \neq i, j$, let $r, s \in \mathcal{A}$ with $N^\dagger(i,j) \leq r\,;s$. Then $N(i,j) \leq r\,;s$, so there is $M \in \mathcal{H}$ with $N \equiv_k M$, $M(i,k) \leq r$, and $M(k,j) \leq s$. Clearly, $M^\dagger \in \mathcal{H}^\dagger$, $N^\dagger \equiv_k M^\dagger$, $M^\dagger(i,k) \leq r$, and $M^\dagger(k,j) \leq s$, as required.

□

$\mathcal{H}^\dagger$ in the proof need not be an n-dimensional *cylindric* basis for $\mathcal{A}$. Indeed, we will now see that $\mathcal{A}$ may not have such a basis.

Take p to be a whole number, at least 2. The *Lyndon algebra* $\mathcal{A}_p$ is a finite relation algebra with $p+2$ atoms, say 1', $a_0, \ldots, a_p$. It is defined by:

- $a_i\,; a_i = a_i + 1'$ if $p \geq 3$, and $a_i; a_i = 1'$ if $p = 2$.
- $a_i\,; a_j = \sum_{k \neq i,j} a_k$ if $i \neq j$,
- (necessarily) $\breve{a}_i = a_i$,

where $i, j, k \leq p$. On arbitrary elements of $\mathcal{A}_p$, ';' can be calculated from this using distributivity over $+$. So can '$\smile$': we have $\breve{r} = r$ for all $r \in \mathcal{A}_p$.

It can be shown that any given Lyndon algebra $\mathcal{A}_p$ is representable iff there is a projective plane of order p. In particular, infinitely many $\mathcal{A}_p$ are representable [25, theorem 1].

THEOREM 60 *There exists a (finite) relation algebra with an n-dimensional hyperbasis for all finite $n \geq 5$, but with no 5-dimensional cylindric basis.*

PROOF:

Choose $p \geq 4$ such that $\mathcal{A}_p$ is representable. By [7, theorem 5.3.16], $\mathcal{A}_p \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ for all finite n. By theorem 1, $\mathcal{A}_p^+$ has an n-dimensional hyperbasis for all finite $n \geq 5$. Since $\mathcal{A}_p$ is finite, $\mathcal{A}_p = \mathcal{A}_p^+$, so the same holds for $\mathcal{A}_p$.

It remains to show that $\mathcal{A}_p$ does not have a 5-dimensional cylindric basis. Assume for contradiction that $\mathcal{M}$ is such a basis. For convenience, let a, b, c, d, e be distinct diversity atoms ($\neq 1'$) of $\mathcal{A}_p$. By property 1 of the definition of cylindric basis, there is a network $N_0 \in \mathcal{M}$ with $N_0(0,1) = a$. Now $N_0(0,1) \leq b\,;c$, so by property 2 there are $N_3, N_4 \in \mathcal{M}$ with

- $N_3 \equiv_3 N_0$, $N_3(0,3) = b$, and $N_3(3,1) = c$, and
- $N_4 \equiv_4 N_0$, $N_4(0,4) = b$, and $N_4(4,1) = c$.

As $c \leq d\,;a$ and $c \leq e\,;a$, there are $N_3', N_4' \in \mathcal{M}$ with

- $N_3' \equiv_2 N_3$, $N_3'(3,2) = d$, and $N_3'(2,1) = a$, and
- $N_4' \equiv_2 N_4$, $N_4'(4,2) = e$, and $N_4'(2,1) = a$.

Notice that $N_3'(0,2) \leq (N_3'(0,1)\,;N_3'(1,2))\cdot(N_3'(0,3)\,;N_3'(3,2)) = (a\,;a)\cdot(b\,;d) = a$, so $N_3'(0,2) = a$. Similarly, $N_4'(0,2) = a$. See figure 2.

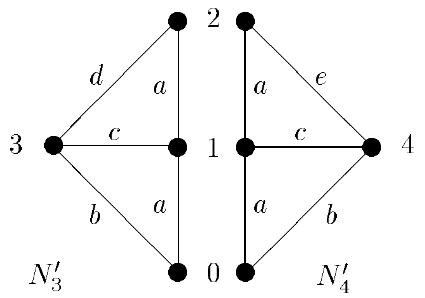

Figure 2: The networks N_3', N_4'

Thus, $N_3' \equiv_{34} N_4'$. By property 3 of the definition of cylindric basis, there is $P \in \mathcal{M}$ with $N_3' \equiv_4 P \equiv_3 N_4'$. But now, $P(3,4) \leq P(3,0)\,;P(0,4) = N_3'(3,0)\,;N_4'(0,4) = b\,;b$. Similarly, using 1 and 2 instead of 0, we obtain $P(3,4) \leq c\,;c$ and $P(3,4) \leq d\,;e$. So $P(3,4) \leq b\,;b \cdot c\,;c \cdot d\,;e = 0$, a contradiction. $\square$

So theorem 1 fails if we replace 'hyperbasis' by 'cylindric basis': for all $n \geq 5$, $\mathcal{A}_p \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ but $\mathcal{A}_p^+ = \mathcal{A}_p$ has no n-dimensional cylindric basis.

8.2 Varieties generated by bases

However, better results may be expected if we close under taking subalgebras. Recall from [31] that RA_n is the class of all relation-type algebras that embed into a (complete) atomic algebra with an n-dimensional relational basis. The paper [14] studied these varieties. Also define CB_n analogously, but using cylindric bases instead of relational bases.

PROPOSITION 61

1. $\mathsf{CB}_4 = \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 = \mathsf{RA}_4 = \mathsf{RA}$.
2. *For* $n \geq 4$, *we have* $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$.
3. $\bigcap_{n<\omega} \mathsf{CB}_n = \bigcap_{n<\omega} \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n = \bigcap_{n<\omega} \mathsf{RA}_n = \mathsf{RRA}$.

PROOF:

1. As Maddux proved that $\mathsf{RA}_4 = \mathsf{RA}$, (1) follows from (2) if we show that $\mathsf{RA}_4 \subseteq \mathsf{CB}_4$. But it is easily verified that any 4-dimensional relational basis is also a cylindric basis.
2. From any cylindric basis $\mathcal{M}$ we may obtain a $\{0\}$-hyperbasis, by letting $N(\bar{a}) = 0$ for all $\bar{a} \in {}^{\leq n}n$ of length $\neq 2$ and all $N \in \mathcal{M}$. So by theorem 1, $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.
   Now let $4 \leq n < \omega$ and $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. Then by theorem 1 and remark 55, $\mathcal{A}^+$ has an n-dimensional hyperbasis, and so, by proposition 59, an n-dimensional relational basis. So $\mathcal{A} \in \mathsf{RA}_n$ by definition of RA_n.
3. By [28, theorems 6, 10], $\bigcap_{n<\omega} \mathsf{RA}_n = \mathsf{RRA}$. So using the second part, we only need verify that $\mathsf{RRA} \subseteq \mathsf{CB}_n$ for all n. Let $\mathcal{A} \in \mathsf{RRA}$, and let $h : \mathcal{A} \to \wp(X \times X)$ be a representation of $\mathcal{A}$. Then h is an embedding of $\mathcal{A}$ into the proper relation algebra $\mathcal{B}$ with domain $\wp(h(1))$. For any n, let Φ be the set of all n-tuples $\bar{a} = (a_0, \ldots, a_{n-1})$ of elements of X such that $(a_i, a_j) \in h(1)$ for each $i, j < n$. For each $\bar{a} \in \Phi$, define a network $N_{\bar{a}}$ by $N_{\bar{a}}(i,j) = \{(a_i, a_j)\} \in At\mathcal{B}$. It can be checked that $\{N_{\bar{a}} : \bar{a} \in \Phi\}$ is an n-dimensional cylindric basis for $\mathcal{B}$.

$\square$

Hence, $\mathcal{A}_p$ as in theorem 60 is a *subalgebra* of a (finite) relation algebra with a 5-dimensional cylindric basis. Indeed, any representable relation algebra has this property. This shows an important distinction between RA_n and CB_n. It is known that $\mathcal{A} \in \mathsf{RA}_n$ iff the canonical extension $\mathcal{A}^+$ has an n-dimensional relational basis. But the algebra $\mathcal{A}_p$ of the theorem belongs to CB_n yet its canonical extension $\mathcal{A}_p^+ \cong \mathcal{A}_p$ has no n-dimensional cylindric basis. We do not know if the inclusion $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is proper. An example in [15] shows that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subset \mathsf{RA}_n$ for all finite $n \geq 5$. In Part 2 of this paper [12], we show that the gap is not finitely axiomatisable.

PROBLEM 1 *For each finite $n \geq 5$, is the inclusion $\mathsf{CB}_n \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ proper? That is, does any algebra $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ embed in a relation algebra with an n-dimensional cylindric basis?*

For each $n < \omega$, is there an $m < \omega$ such that $\mathsf{RA}_m \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$?

For each $n < \omega$, is there an $m < \omega$ such that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_m \subseteq \mathsf{CB}_n$?

8.3 Cylindric bases and homogeneous representations

A (classical) representation of a relation algebra $\mathcal{A}$ can be equivalently regarded as a relativised representation of $\mathcal{A}$ in the sense of definition 3 with the property that 1^M is an equivalence relation on M. Such a representation is said to be *homogeneous* if it is 'ultra-homogeneous' in the model-theoretic sense: every partial (some say 'local') isomorphism of M with finite domain is induced by a full automorphism of M.

For finite relation algebras, an argument in the style of [5] shows that cylindric bases and homogeneous representations 'coincide'. The n-dimensional analogue was discussed in remark 50.

LEMMA 62 *If $\mathcal{M}_n$ is an n-dimensional cylindric basis for $\mathcal{A}$ then the set $\mathcal{M}_n^+ = \{N\sigma : N \in \mathcal{M}_n,\ \sigma : n \to n\}$ is also an n-dimensional cylindric basis.*

PROOF:

Copy the proof of lemma 40. □

THEOREM 63 *Let $\mathcal{A}$ be a finite relation algebra. Then $\mathcal{A}$ has an n-dimensional cylindric basis for all finite $n \geq 4$, iff $\mathcal{A}$ has a homogeneous representation.*

PROOF [SKETCH]:

(Cf. theorem 41.) If M is a homogeneous representation of $\mathcal{A}$, then consider the set of all atomic networks of dimension n that embed in M, as in proposition 61.3. This is an n-dimensional cylindric basis for $\mathcal{A}$.

To prove the converse, suppose that $\mathcal{A}$ has a n-dimensional cylindric basis $\mathcal{M}_n$ for all finite $n \geq 4$. Below, n, m, p, q will satisfy $4 \leq m, n, p, q < \omega$ unless otherwise stated. If $m < n$, write $\mathcal{M}_n\lceil_m$ for the set $\{N\lceil_m : N \in \mathcal{M}_n\}$ of m-dimensional networks, where for a network $N \in \mathcal{M}_n$, $N\lceil_m$ denotes the m-dimensional network obtained by restricting N to indices $< m$. Then it can be checked (cf. [12, lemma 12]) that $\mathcal{M}_n\lceil_m$ is an m-dimensional cylindric basis for $\mathcal{A}$.

Now there are only finitely many m-dimensional networks over $\mathcal{A}$, for any m. So by König's tree lemma, we may suppose without loss of generality that $\mathcal{M}_n\lceil_m = \mathcal{M}_m$ for all $m < n$. By lemma 62 we may suppose that $\mathcal{M}_n$ is symmetric (closed under permutations), for each n.

We are going to build a homogeneous representation of $\mathcal{A}$ by a game. The following lemma will be used to show that it is homogeneous. First, a definition.

DEFINITION 64 Let $q \leq p$, $P \in \mathcal{M}_p$, and $Q \in \mathcal{M}_q$. Let $\theta : q \to p$ be a one-one map. We say that θ is an *embedding* from Q to P if $Q(i,j) = P(\theta(i), \theta(j))$ for all $i, j < q$.

Informally, '$Q = P\theta$'.

LEMMA 65 *Let $p, q < \omega$, $P \in \mathcal{M}_p$, $Q \in \mathcal{M}_{q+1}$, and let $\theta : Q\lceil_q \to P$ be an embedding. Then there is $P^+ \in \mathcal{M}_{p+1}$ with $P^+\lceil_p = P$, and an embedding $\theta^+ : Q \to P^+$ extending θ.*

PROOF:

Take P, Q, θ as in the lemma, so that $\theta : q \to p$ is an embedding $: Q\restriction_q \to P$. Clearly, $p \geq q$; the proof is by induction on $p - q$. If this is zero, extend θ to a permutation θ' of $q+1$ in the unique way. Let $P^+ = Q(\theta')^{-1}$. By our assumption that $\mathcal{M}_{q+1}$ is symmetric, $P^+ \in \mathcal{M}_{q+1}$. If $i, j \leq q$ then $Q(i,j) = Q(\theta')^{-1}(\theta'(i), \theta'(j))$, so $\theta' : Q \to P^+$ is an embedding. Clearly, $P^+\restriction_p = P$.

Let $p - q \geq 1$ and assume the result for smaller $p - q$. By symmetry of $\mathcal{M}_p$, we may suppose that $p - 1 \notin rng(\theta)$, so that θ is an embedding $: Q\restriction_q \to P\restriction_{p-1}$. By the inductive hypothesis, there is $P' \in \mathcal{M}_p$ with $P'\restriction_{p-1} = P\restriction_{p-1}$, and an embedding $\theta' : Q \to P'$ extending θ. By symmetry of $\mathcal{M}_p$, we may suppose that $\theta'(q) = p - 1$. Take any $R' \in \mathcal{M}_{p+1}$ with $R'\restriction_p = P'$, let $\pi : p+1 \to p+1$ be the transposition swapping $p-1$ and p, and define $\rho = \pi \circ \theta' : q+1 \to p+1$ and $R = R'\pi$. Then $\rho : Q \to R$ is an embedding extending θ, and $\rho(q) = p$. Take any $P^* \in \mathcal{M}_{p+1}$ with $P^*\restriction_p = P$. Clearly, $P^* \equiv_{p-1,p} R$ in $\mathcal{M}_{p+1}$. As $\mathcal{M}_{p+1}$ is a cylindric basis, we may find $S \in \mathcal{M}_{p+1}$ with $P^* \equiv_p S \equiv_{p-1} R$. So $S\restriction_p = P$, and $\rho : Q \to S$ is an embedding extending θ. $\square$

Two players, $\forall$ (male) and $\exists$ (female), now play a game to build a representation of $\mathcal{A}$. The game has ω rounds, for convenience numbered $3, 4, \ldots, t, \ldots$ $(3 \leq t < \omega)$. In round 3, $\forall$ picks non-zero $r \in \mathcal{A}$, and $\exists$ responds with some $N_4 \in \mathcal{M}_4$ with $N_4(0,1) \leq r$; she can find such a network because $\mathcal{M}_4$ is a cylindric basis. In each subsequent round $t \geq 4$, if the current network is $N_t \in \mathcal{M}_t$, then $\forall$ can make one of two types of move:

Triangle move: he chooses $i, j < t$ and $r, s \in \mathcal{A}$ with $N_t(i,j) \leq r\,;s$. $\exists$ must respond with some $N_{t+1} \in \mathcal{M}_{t+1}$ such that $N_{t+1}\restriction_t = N_t$, $N_{t+1}(i,t) \leq r$ and $N_{t+1}(t,j) \leq s$. She can do so because there is $N' \in \mathcal{M}_{t+1}$ with $N'\restriction_t = N_t$, so as $\mathcal{M}_{t+1}$ is a cylindric basis, there is $N_{t+1} \in \mathcal{M}_{t+1}$ with $N_{t+1} \equiv_t N'$, $N_{t+1}(i,t) \leq r$ and $N_{t+1}(t,j) \leq s$.

Amalgamation move: he chooses $Q \in \mathcal{M}_{q+1}$ for some $q \leq t$, and an embedding $\theta : Q\restriction_q \to N_t$. $\exists$ must respond with $N_{t+1} \in \mathcal{M}_{t+1}$ such that $N_{t+1}\restriction_t = N_t$ and θ extends to an embedding $\theta^+ : Q \to N_{t+1}$. Lemma 65 shows that she can do this.

Consider a play $N_4, N_5, \ldots$ of the game in which $\forall$ plays $r_0 \in \mathcal{A} \setminus \{0\}$ and then makes every possible move in some round. That is, if $N_t(i,j) \leq r\,;s$ for some t, i, j, r, s, then he plays a triangle move i, j, r, s in some round $u \geq t$; and if $t \geq 4$, $q \leq t$, $Q \in \mathcal{M}_{q+1}$, and $\theta : Q\restriction_q \to N_t$ is an embedding, then $\forall$ plays Q, θ in some round $u \geq t$ (note that $\theta : Q \to N_u$ will also be an embedding). He can do all this because $\mathcal{A}$ and each $\mathcal{M}_n$ are countable, and he makes countably many moves during the game. Since $N_u\restriction_t = N_t$ whenever $4 \leq t \leq u < \omega$, the play has a well-defined limit N_ω, a network of dimension ω, where for each $i, j < \omega$, $N_\omega(i,j) = N_t(i,j)$ for any $t > 4, i, j$. N_ω has the following properties:

1. $N_\omega(0,1) \leq r_0$ (by $\forall$'s first move).
2. N_ω is a network: $N_\omega(i,i) \leq$ 1' and $N(i,k) \leq N(i,j)\,;N(j,k)$ for all $i, j, k < \omega$ (because N_ω is the 'limit' of networks).

3. If $N_\omega(i,j) \leq r\,;s$ for some $i,j < \omega$ and $r,s \in \mathcal{A}$, there is $k < \omega$ with $N_\omega(i,k) \leq r$ and $N_\omega(k,j) \leq s$. (Take $k > i,j$ such that $\forall$ played the triangle move i,j,r,s in round k.)

A *partial isomorphism* of N_ω is a partial one-one map $\phi : \omega \to \omega$ such that $N_\omega(i,j) = N_\omega(\phi(i),\phi(j))$ for all $i,j \in dom(\phi)$.

4. If $\phi : N_\omega \to N_\omega$ is a finite partial isomorphism (i.e., as a subset of $\omega \times \omega$, ϕ is finite), and $i < \omega$, then ϕ extends to a finite partial isomorphism defined on i.

   We may suppose $i \notin dom(\phi)$. Let $t < \omega$ be such that $\{i\} \cup dom(\phi) \cup rng(\phi) \subseteq t$. Let $|\phi| = q$, and choose $\theta : q+1 \to t$ with $rng(\theta\restriction_q) = dom(\phi)$ and $\theta(q) = i$. Let ψ be a permutation of t extending θ, and let $Q = (N_t\psi)\restriction_{q+1}$. Then $Q \in \mathcal{M}_{q+1}$, and $\theta : Q \to N_t$ is an embedding. Clearly, for all $u \geq t$, the map $\phi \circ \theta : Q\restriction_q \to N_u$ is also an embedding. Choose $u \geq t$ such that $\forall$ made the amalgamation move $Q, \phi\circ\theta$ in round u of the game. So there is an embedding $\chi : Q \to N_{u+1}$ extending $\phi\circ\theta$. Then $\chi\circ\theta^{-1}$ is a finite partial isomorphism of N_ω extending ϕ and defined on i.

5. Any partial isomorphism of N_ω is induced by a partial isomorphism that is actually a permutation of ω. (As N_ω is countable, repeated 'back-and-forth' application of property 4 shows this.)

Define a binary relation $\sim$ on ω by $i \sim j$ iff $N_\omega(i,j) \leq 1$'. It is easily checked that $\sim$ is an equivalence relation, and indeed a *congruence,* in that if $i \sim i'$ and $j \sim j'$ then $N_\omega(i,j) = N_\omega(i',j')$. Write $\omega/{\sim}$ for the set of equivalence classes, and $i^\sim$ for the equivalence class of i ($i < \omega$). Let M_{r_0} be the $\mathcal{L}(\mathcal{A})$-structure with domain $\omega/{\sim}$ given by

$$M_{r_0} \models r(i^\sim, j^\sim) \text{ iff } N_\omega(i,j) \leq r,$$

for $r \in \mathcal{A}$. (We make explicit the dependence on $\forall$'s first move r_0 here.) There is such a structure M_{r_0} for each non-zero $r_0 \in \mathcal{A}$. As is easily seen, property 5 ensures that each M_{r_0} is homogeneous.

Let $r \approx s$ in $At\mathcal{A}$ iff $r \leq 1\,;s\,;1$. Then $\approx$ is an equivalence relation on $At\mathcal{A}$, and, as can be checked, $r \approx s$ iff $M_r \models \exists xy\, s(x,y)$, for all atoms r,s. Take a set E of representatives for the $\approx$-classes, and let M be the disjoint union of the structures M_e for $e \in E$. Then $M \models \exists xy\, r(x,y)$ for all non-zero $r \in \mathcal{A}$. It can be checked routinely that M 'is' a representation of $\mathcal{A}$; see [11] for more details. Moreover, if θ is any finite partial isomorphism of M, then take $x \in dom(\theta) \cap M_e$. There is $a \in At\mathcal{A}$ such that $M \models a(x,x)$. If $\theta(x) \in M_{e'}$, then since $M \models a(\theta(x),\theta(x))$, then $e \approx a \approx e'$, so $e = e'$. So θ is a union of partial isomorphisms of the M_e. Each one extends to an automorphism of M_e, and their union is an extension of θ to an automorphism of M. So M is homogeneous. $\square$

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