Status of Problems Listed in the Book

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All problems listed here are from Hirsch and Hodkinson *Relation Algebras by Games*, North Holland, 2002, but some have been slightly paraphrased.

Problem 3.27, Jónsson. This is [Madd94a, problem 2], wrongly quoted in our book. Find all simple relation algebras with no subalgebras other than the whole algebra and the minimal relation subalgebra whose elements are $0, 1', 0', 1$. Maddux stated that 22 simple relation algebras with no non-trivial subalgebras had been found.
Status: open.

Problem 5.18. If $\mathcal{C} \in \mathbf{RCA}_\omega$ does $\mathcal{C}^+$ necessarily have a complete, ω-dimensional representation?
Status: open.

Problem 5.47, Németi, Sayed Ahmed. For finite $n \geq 4$, is $\mathfrak{Ra}\mathbf{CA}_n$ elementary?
Status: open. [Hir07] appeared to show that the classes are not elementary, for $n \geq 5$, but in the light of the erratum [Hir13] we may only deduce that $\mathbf{S}_c\mathfrak{Ra}\mathbf{CA}_n$ is not elementary for $n \geq 5$. For $n = 4$ the problem is also open, although it is known that $\mathbf{RA} = \mathbf{S}\mathfrak{Ra}\mathbf{CA}_4$ and for complete and atomic $\mathcal{A}$ we have $\mathcal{A} \in \mathbf{RA} \iff \mathcal{A} \in \mathfrak{Ra}\mathbf{CA}_4$ [Madd78b, Theorem 21].

Problem 5.55. Is there a good definition of $\mathfrak{Ra}\mathcal{D}$ for $\mathcal{D} \in \mathbf{D}_\alpha$?
Status: open.

Problem 5.56. How are $\mathbf{S}\mathfrak{Ra}\mathbf{D}_3$ and $\mathbf{S}\mathfrak{Ra}(\mathbf{CA}_3 \cap \mathbf{D}_3)$ related to $\mathbf{NA}, \mathbf{WA}, \mathbf{SA}$ and $\mathbf{RA}$?
Status: open.

Problem 9.3. Is the class of relation algebras with a homogeneous representation an elementary class?
Status: open.

Problem 9.4, Németi. Is the class $\mathbf{IG}_\omega$ (the isomorphism-closure of the ω-dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts? See [Ném96, AndGol$^+$98, And01].
Status: open.

Problem 9.16, Venema. If a class of structures is closed under ultraproducts, must it be pseudo-elementary?
Status: it is consistent with ZFC that the answer is 'no'. Under the assumption that there are no measurable cardinals, Keisler [Kei65, pp216–217] gives an example of a (proper) class of pseudo-elementary classes whose intersection is not pseudo-elementary. The intersection is of course closed under ultraproducts.

Problem 11.6. Is there a recursive function $f : \omega \to \omega$ such that for any finite relation algebra $\mathcal{A}$ and and $n < \omega$, if $\exists$ has a winning strategy in $G_{f(n)}(\mathcal{A})$ of definition 7.12 then she has a winning strategy strategy in the game $G^a_n(\mathcal{A})$.
Status: open.

Problem 12.33. Investigate the parallel complexity of determining whether a finite non-associative algebra is in $\mathbf{RA}_n$. Investigate the complexity of determining, for a finite relation algebra atom structure $\mathcal{S}$, whether $\mathfrak{Cm}\,\mathcal{S} \in \mathbf{RA}_n$.
Status: the investigation has not yet been completed.

Problem 12.38. For arbitrary finite $n \geq 5$, does every atomic relation algebra with an n-dimensional hyperbasis embed in an atomic relation algebra with an n-dimensional cylindric basis?
Status: open.

Problem 12.39. Investigate the complexity of deciding, for fixed $n \geq 3$, whether a finite weakly associative algebra has an n-dimensional cylindric basis. [It is clearly in NP; is it NP-complete?]
Status: open.

Problem 13.52. Must any n-smooth relativised representation of an arbitrary non-associative algebra be infinitarily n-flat?
Status: open.

Problem 13.53. Is every n-flat relativised representation of a non-associative algebra also infinitarily n-flat?
Status: open.

Problem 13.54. For finite $n \geq 5$, is there an atomic non-associative algebra with a complete n-flat relativised representation but with no complete infinitarily n-flat representation?
Status: open.

Problem 14.19. If $\mathcal{S}, \mathcal{S}'$ are elementarily equivalent relation algebra atom structures, must the term algebras of $\mathcal{S}$ and $\mathcal{S}'$ be elementarily equivalent?
Status: solved, the answer is no. Proved in [AndNem18].

Problem 14.20. For which $\alpha \geq 3$ is $\mathrm{Str}\,\mathbf{RCA}_\alpha$ elementary?
Status: solved for $3 \leq \alpha < \omega$, none of these classes is elementary [HH09]. Also see [BH13] for corresponding results over other algebras of relations. The case $\alpha \geq \omega$ remains open.

Problem 15.14. For precisely which k, r ($r \geq k > n \geq 4$) does $\mathcal{A}(n, r)$ have a k-dimensional basis?
Status: open.

Problem 15.18. Let $n \geq 3$ be finite. Is there a finite set of n-schemata whose set Σ of n-instances satisfies

$$\Sigma \vdash_{n,n} \phi \iff \vdash_{n,n+1} \phi$$

for all L^n- formulas ϕ?
Status: open.

Problem 17.13. A universal formula is built out of RA equations, using $\wedge, \vee, \neg$ and $\forall$ where each universal quantifier occurs under an even number of negations. Note that variables can be reused. Is there a universal axiomatisation of **RRA** using a finite number of variables?
Status: open, surprisingly.

Problem 17.14. It is known that **RRA** cannot be axiomatised by any set of equations using only finitely many variables [Jón91]. Prove that **RRA** cannot be axiomatised with a set of first-order sentences using only finitely many variables.
Status: `https://arxiv.org/abs/2008.01329` proves that **RRA** cannot be axiomatised by any set of first-order formulas of bounded quantifier-depth. However, this does not cover possible axiomatisation with finitely many variables but unbounded quantifier-depth, so the problem as stated remains open.

Problem 17.30. Is it the case that for any finite relational structures A, B in a given binary signature, if $\mathcal{A}_{A,B} \in \mathbf{RA}_5$ then $\mathcal{A}_{A,B} \in \mathbf{wRRA}$?
Status: open.

Problem 17.39, Venema. Find (or show non-existence of) a set of equations $\{e_i : i < \omega\}$ with the following properties:

- $\mathcal{A} \models \{e_i : i < \omega\} \iff \mathcal{A} \in \mathbf{RRA}$.

- Each e_i is canonical, i.e. $\mathcal{A} \models e_i \Rightarrow \mathcal{A}^+ \models e_i$.

Repeat, replacing **RRA** by $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ for $n \geq 5$. Repeat for $\mathbf{RA}_n$ as well.
Status: partly solved. Such canonical equations do not exist [HV05], for the class **RRA**. Moreover, any first order axiomatisation of **RRA** contains infinitely many non-canonical sentences. See [BH13] for corresponding negative results for the classes $\mathbf{RCA}_n, \mathbf{RPA}_n, \mathbf{RPEA}_n, \mathbf{RDf}_n$ for finite $n \geq 3$. The problem remains open for $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n, \mathbf{RA}_n$ $(n \geq 5)$.

Problem 17.40. Are $\mathbf{RA}_5$ and $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_5$ closed under completions?
Status: open.

Problem 17.41. Is the class **wRRA** of weakly representable relation algebras closed under completions?
Status: solved, negatively [HM12].

Problem 18.17. For fixed finite $n \geq 5$, is it decidable whether an arbitrary finite relation algebra has a finite n-dimensional hyperbasis?
Status: open.

Problem 18.18, Maddux. Is it decidable whether an arbitrary finite relation algebra has a finite representation?
Status: open.

Problem 18.26, partly Maddux. A relation algebra is said to be *weakly representable* if it has a representation respecting the relation algebra operations (1’, $\breve{}$, ;) and $0, 1, \cdot$, but not necessarily $+, -$. The class of weakly representable relation algebras is denoted by **wRRA**. Do we have $\mathbf{wRRA} \subseteq \mathbf{RA}_n$ for some $n \geq 5$? What inclusions hold between **wRRA** and the $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ $(n \geq 5)$?
Status: Partly solved. **wRRA** is not contained in $\mathbf{RA}_5$ (hence it is not contained in $\mathbf{RA}_n$ for any $n \geq 5$) and $\mathbf{RA}_n$ is not contained in **wRRA**, for any finite n [HHM11]. The former property proves that **wRRA** is not contained in $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$, for any $n \geq 5$, but whether $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ is contained in **wRRA** (any finite n) remains open.

Problem 19.17. Let $\mathcal{L}$ be the class of all residuated algebras $(A, ;, \backslash, /, \leq)$, where A is a non-empty set of binary relations on some base set U, $W = \bigcup A$ is transitive and $\{x, y : (x,y) \in W\} = U$, and for $r, s \in A$,

- $r;s = \{(x,y) : \exists z((x,z) \in r \wedge (z,y) \in s)\}$,
- $r \setminus s = \{(x,y) \in W : \forall z((z,x) \in r \Rightarrow (z,y) \in s)\}$,
- $r/s = \{(x,y) \in W : \forall z((y,z) \in s \Rightarrow (x,z) \in r)\}$,
- $r \leq s$ iff $r \subseteq s$.

Is any finite algebra in $\mathcal{L}$ isomorphic to an algebra in $\mathcal{L}$ with finite base?
Status: open.

Problem 19.22. For $n \geq 4$, let $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n^f$ denote the class $\mathbf{S}\mathfrak{Ra}\{\mathcal{C} \in \mathbf{CA}_n : \mathcal{C} \text{ is finite}\}$. Exercise 15.4(1) showed that $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n^f \supset \mathbf{S}\mathfrak{Ra}\mathbf{CA}_{n+1}^f$ for all finite $n \geq 3$. Study the class $\bigcap_{n<\omega} \mathbf{S}\mathfrak{Ra}\mathbf{CA}_n^f$.

This is the class of all finite relation algebras with a finite n-dimensional hyperbasis for every finite $n \geq 3$. Clearly it is a class of finite representable relation algebras. It contains a finite relation algebra with no finite representation — for example, the point algebra. Is it the class of finite relation algebras with ω-categorical representations? (It contains all of these.) Is it the class of finite relation algebras that have a representation M with finitely many $\mathcal{L}(\mathcal{A})_n$-definable n-ary relations for all finite n? Is membership of it decidable? Is it the case that for all finite $n \geq 4$ there is a finite *representable* relation algebra $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathbf{CA}_n^f \setminus \mathbf{S}\mathfrak{Ra}\mathbf{CA}_{n+1}^f$? (This is true if the condition that $\mathcal{A} \in \mathbf{RRA}$ is dropped.)
Status: open.

Problems from chapter 21 other than those already listed above:

Problem 21.7. Old problem, stated in [AndTho88] as well as other sources. For infinite α, is the equational theory of $\mathbf{D}_\alpha$ decidable? For finite α it is know that the equational theory is decidable [Ném86].
Status: open.

Problem 21.9. Is **wRRA** a variety? Is it canonical?
Status: solved, [Péc09] proves that **wRRA** is a variety but [HM12] proves that it is not canonical.

Problem 21.12, Sayed Ahmed. Which $\mathbf{S}\mathfrak{Nr}_m\mathbf{CA}_n$ for $m < n < \omega$ are closed under completions?
Status: it was claimed in [Ahm15, corollary 5.10] for $2 < n < \omega$ and $\omega \geq n \geq m+3$ that $\mathbf{S}\mathfrak{Nr}_m\mathbf{CA}_n$ was not closed under completions, but Sayed Ahmed discovered a mistake in his proof, his revised proof shows only that $\mathbf{S}\mathfrak{Nr}_n\mathbf{CA}_{2n}$ is not closed under completions for $2 < n < \omega$. $\mathbf{S}\mathfrak{Nr}_n\mathbf{CA}_{n+1}$ is Sahlqvist axiomatisable hence closed under completions [And90], other cases remain open.

Problem 21.20, Madd94a, problems 15, 16. Following on from exercise 12.5(13), is it true that *almost all* finite relation algebras are representable? More precisely, if $RA(n), RRA(n)$ are the numbers of isomorphism types of relation algebras and representable relation algebras (respectively) with no more than n elements, is it the case that

$$\lim_{n\to\infty} \frac{RRA(n)}{RA(n)} = 1?$$

Status: open.

Problem 21.21, Madd94a, problem 9. Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Does $\mathcal{A}$ necessarily have a finite representation?
Status: open.

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