

Status of Problems Listed in the Book

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All problems listed here are from Hirsch and Hodkinson *Relation Algebras by Games*, North Holland, 2002, but some have been slightly paraphrased.

Problem 3.27, Jónsson. This is [Madd94a, problem 2], wrongly quoted in our book. Find all simple relation algebras with no subalgebras other than the whole algebra and the minimal relation subalgebra whose elements are $0, 1', 0', 1$. Maddux stated that 22 simple relation algebras with no non-trivial subalgebras had been found.

Status: open.

Problem 5.18. If $C \in \mathbf{RCA}_\omega$ does C^+ necessarily have a complete, ω -dimensional representation?

Status: open.

Problem 5.47, Némethi, Sayed Ahmed. For finite $n \geq 4$, is \mathfrak{RaCA}_n elementary?

Status: open. [Hir07] appeared to show that the classes are not elementary, for $n \geq 5$, but in the light of the erratum [Hir13] we may only deduce that $\mathbf{S}_c\mathfrak{RaCA}_n$ is not elementary for $n \geq 5$. For $n = 4$ the problem is also open, although it is known that $\mathbf{RA} = \mathbf{S}\mathfrak{RaCA}_4$ and for complete and atomic \mathcal{A} we have $\mathcal{A} \in \mathbf{RA} \iff \mathcal{A} \in \mathfrak{RaCA}_4$ [Madd78b, Theorem 21].

Problem 5.55. Is there a good definition of \mathfrak{RaD} for $\mathcal{D} \in \mathbf{D}_\alpha$?

Status: open.

Problem 5.56. How are $\mathbf{S}\mathfrak{RaD}_3$ and $\mathbf{S}\mathfrak{Ra}(\mathbf{CA}_3 \cap \mathbf{D}_3)$ related to $\mathbf{NA}, \mathbf{WA}, \mathbf{SA}$ and \mathbf{RA} ?

Status: open.

Problem 9.3. Is the class of relation algebras with a homogeneous representation an elementary class?

Status: open.

Problem 9.4, Némethi. Is the class \mathbf{IG}_ω (the isomorphism-closure of the ω -dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts? See [Ném96, AndGol⁺98, And01].

Status: open.

Problem 9.16, Venema. If a class of structures is closed under ultraproducts, must it be pseudo-elementary?

Status: it is consistent with ZFC that the answer is ‘no’. Under the assumption that there are no measurable cardinals, Keisler [Kei65, pp216–217] gives an example of a (proper) class of pseudo-elementary classes whose intersection is not pseudo-elementary. The intersection is of course closed under ultraproducts.

Problem 11.6. Is there a recursive function $f : \omega \rightarrow \omega$ such that for any finite relation algebra \mathcal{A} and $n < \omega$, if \exists has a winning strategy in $G_{f(n)}(\mathcal{A})$ of definition 7.12 then she has a winning strategy in the game $G_n^a(\mathcal{A})$.

Status: open.

Problem 12.33. Investigate the parallel complexity of determining whether a finite non-associative algebra is in \mathbf{RA}_n . Investigate the complexity of determining, for a finite relation algebra atom structure \mathcal{S} , whether $\mathfrak{Cm}\mathcal{S} \in \mathbf{RA}_n$.

Status: the investigation has not yet been completed.

Problem 12.38. For arbitrary finite $n \geq 5$, does every atomic relation algebra with an n -dimensional hyperbasis embed in an atomic relation algebra with an n -dimensional cylindric basis?

Status: open.

Problem 12.39. Investigate the complexity of deciding, for fixed $n \geq 3$, whether a finite weakly associative algebra has an n -dimensional cylindric basis. [It is clearly in NP; is it NP-complete?]

Status: open.

Problem 13.52. Must any n -smooth relativised representation of an arbitrary non-associative algebra be infinitarily n -flat?

Status: open.

Problem 13.53. Is every n -flat relativised representation of a non-associative algebra also infinitarily n -flat?

Status: open.

Problem 13.54. For finite $n \geq 5$, is there an atomic non-associative algebra with a complete n -flat relativised representation but with no complete infinitarily n -flat representation?

Status: open.

Problem 14.19. If $\mathcal{S}, \mathcal{S}'$ are elementarily equivalent relation algebra atom structures, must the term algebras of \mathcal{S} and \mathcal{S}' be elementarily equivalent?

Status: solved, the answer is no. Proved in [AndNem18].

Problem 14.20. For which $\alpha \geq 3$ is StrRCA_α elementary?

Status: solved for $3 \leq \alpha < \omega$, none of these classes is elementary [HH09]. Also see [BH13] for corresponding results over other algebras of relations. The case $\alpha \geq \omega$ remains open.

Problem 15.14. For precisely which k, r ($r \geq k > n \geq 4$) does $\mathcal{A}(n, r)$ have a k -dimensional basis?

Status: open.

Problem 15.18. Let $n \geq 3$ be finite. Is there a finite set of n -schemata whose set Σ of n -instances satisfies

$$\Sigma \vdash_{n,n} \phi \iff \vdash_{n,n+1} \phi$$

for all L^n -formulas ϕ ?

Status: open.

Problem 17.13. A universal formula is built out of RA equations, using \wedge, \vee, \neg and \forall where each universal quantifier occurs under an even number of negations. Note that variables can be reused. Is there a universal axiomatisation of **RRA** using a finite number of variables?

Status: open, surprisingly.

Problem 17.14. It is known that **RRA** cannot be axiomatised by any set of equations using only finitely many variables [Jón91]. Prove that **RRA** cannot be axiomatised with a set of first-order sentences using only finitely many variables.

Status: <https://arxiv.org/abs/2008.01329> proves that **RRA** cannot be axiomatised by a set of universal formulas with finitely many variables, the problem as stated is open.

Problem 17.30. Is it the case that for any finite relational structures A, B in a given binary signature, if $\mathcal{A}_{A,B} \in \mathbf{RA}_5$ then $\mathcal{A}_{A,B} \in \mathbf{wRRA}$?

Status: open.

Problem 17.39, Venema. Find (or show non-existence of) a set of equations $\{e_i : i < \omega\}$ with the following properties:

- $\mathcal{A} \models \{e_i : i < \omega\} \iff \mathcal{A} \in \mathbf{RRA}$.
- Each e_i is canonical, i.e. $\mathcal{A} \models e_i \Rightarrow \mathcal{A}^+ \models e_i$.

Repeat, replacing **RRA** by **S \mathfrak{R} aCA $_n$** for $n \geq 5$. Repeat for **RA $_n$** as well.

Status: partly solved. Such canonical equations do not exist [HV05], for the class **RRA**. Moreover, any first order axiomatisation of **RRA** contains infinitely many non-canonical sentences. See [BH13] for corresponding negative results for the classes **RCA $_n$** , **RPA $_n$** , **RPEA $_n$** , **RDF $_n$** for finite $n \geq 3$. The problem remains open for **S \mathfrak{R} aCA $_n$** , **RA $_n$** ($n \geq 5$).

Problem 17.40. Are **RA $_5$** and **S \mathfrak{R} aCA $_5$** closed under completions?

Status: open.

Problem 17.41. Is the class **wRRA** of weakly representable relation algebras closed under completions?

Status: solved, negatively [HM12].

Problem 18.17. For fixed finite $n \geq 5$, is it decidable whether an arbitrary finite relation algebra has a finite n -dimensional hyperbasis?

Status: open.

Problem 18.18, Maddux. Is it decidable whether an arbitrary finite relation algebra has a finite representation?

Status: open.

Problem 18.26, partly Maddux. A relation algebra is said to be *weakly representable* if it has a representation respecting the relation algebra operations $(1', \sim, ;)$ and $0, 1, \cdot, \cdot$, but not necessarily $+, -$. The class of weakly representable relation algebras is denoted by **wRRA**. Do we have **wRRA** \subseteq **RA $_n$** for some $n \geq 5$? What inclusions hold between **wRRA** and the **S \mathfrak{R} aCA $_n$** ($n \geq 5$)?

Status: Partly solved. **wRRA** is not contained in **RA $_5$** (hence it is not contained in **RA $_n$** for any $n \geq 5$) and **RA $_n$** is not contained in **wRRA**, for any finite n [HHM11]. The former property proves that **wRRA** is not contained in **S \mathfrak{R} aCA $_n$** , for any $n \geq 5$, but whether **S \mathfrak{R} aCA $_n$** is contained in **wRRA** (any finite n) remains open.

Problem 19.17. Let \mathcal{L} be the class of all residuated algebras $(A, ;, \backslash, /, \leq)$, where A is a non-empty set of binary relations on some base set U , $W = \bigcup A$ is transitive and $\{x, y : (x, y) \in W\} = U$, and for $r, s \in A$,

- $r; s = \{(x, y) : \exists z((x, z) \in r \wedge (z, y) \in s)\}$,
- $r \backslash s = \{(x, y) \in W : \forall z((z, x) \in r \Rightarrow (z, y) \in s)\}$,
- $r / s = \{(x, y) \in W : \forall z((y, z) \in s \Rightarrow (x, z) \in r)\}$,
- $r \leq s$ iff $r \subseteq s$.

Is any finite algebra in \mathcal{L} isomorphic to an algebra in \mathcal{L} with finite base?

Status: open.

Problem 19.22. For $n \geq 4$, let **S \mathfrak{R} aCA $_n^f$** denote the class **S \mathfrak{R} a** $\{C \in \mathbf{CA}_n : C \text{ is finite}\}$. Exercise 15.4(1) showed that **S \mathfrak{R} aCA $_n^f$** \supset **S \mathfrak{R} aCA $_{n+1}^f$** for all finite $n \geq 3$. Study the class $\bigcap_{n < \omega} \mathbf{S}\mathfrak{R}\mathbf{a}\mathbf{C}\mathbf{A}_n^f$.

This is the class of all finite relation algebras with a finite n -dimensional hyperbasis for every finite $n \geq 3$. Clearly it is a class of finite representable relation algebras. It contains a finite relation algebra with no finite representation — for example, the point algebra. Is it the class of finite relation algebras with ω -categorical representations? (It contains all of these.) Is it the class of finite relation algebras that have a representation M with finitely many $\mathcal{L}(\mathcal{A})_n$ -definable n -ary relations for all finite n ? Is membership of it decidable? Is it the case that for all finite $n \geq 4$ there is a finite *representable* relation algebra $\mathcal{A} \in \mathbf{S}\mathfrak{R}\mathbf{a}\mathbf{C}\mathbf{A}_n^f \setminus \mathbf{S}\mathfrak{R}\mathbf{a}\mathbf{C}\mathbf{A}_{n+1}^f$? (This is true if the condition that $\mathcal{A} \in \mathbf{RRA}$ is dropped.)

Status: open.

Problems from chapter 21 other than those already listed above:

Problem 21.7. Old problem, stated in [AndTho88] as well as other sources. For infinite α , is the equational theory of **D $_\alpha$** decidable? For finite α it is known that the equational theory is decidable [Ném86].

Status: open.

Problem 21.9. Is **wRRA** a variety? Is it canonical?

Status: solved, [Péc09] proves that **wRRA** is a variety but [HM12] proves that it is not canonical.

Problem 21.12, Sayed Ahmed. Which $\mathfrak{S}\mathfrak{R}_m\mathbf{CA}_n$ for $m < n < \omega$ are closed under completions?

Status: it was claimed in [Ahm15, corollary 5.10] for $2 < n < \omega$ and $\omega \geq n \geq m + 3$ that $\mathfrak{S}\mathfrak{R}_m\mathbf{CA}_n$ was not closed under completions, but Sayed Ahmed discovered a mistake in his proof, his revised proof shows only that $\mathfrak{S}\mathfrak{R}_n\mathbf{CA}_{2n}$ is not closed under completions for $2 < n < \omega$. $\mathfrak{S}\mathfrak{R}_n\mathbf{CA}_{n+1}$ is Sahlqvist axiomatisable hence closed under completions [And90], other cases remain open.

Problem 21.20, Madd94a, problems 15, 16. Following on from exercise 12.5(13), is it true that *almost all* finite relation algebras are representable? More precisely, if $RA(n), RRA(n)$ are the numbers of isomorphism types of relation algebras and representable relation algebras (respectively) with no more than n elements, is it the case that

$$\lim_{n \rightarrow \infty} \frac{RRA(n)}{RA(n)} = 1?$$

Status: open.

Problem 21.21, Madd94a, problem 9. Let \mathcal{A} be a finite relation algebra with a flexible atom. Does \mathcal{A} necessarily have a finite representation?

Status: open.

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