Peirce Algebras and Boolean Modules*

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Abstract

A boolean module comprises a relation algebra, a boolean algebra and a Peircean operator which must obey a certain finite set of equational axioms. A Peirce algebra is a boolean module with one more operator, right cylindrification, which must obey another finite set of equational axioms. A Peirce algebra can be interpreted in a two sorted algebra built within its relation algebra part, but no such interpretation can be found for boolean modules.

We consider three different types of representations for relation algebras, boolean modules and Peirce algebras: classical (abbreviated as class.) representations, where all operators must be interpreted according to their natural set-theoretic definitions; non-additive (non-+) representations, where the relation algebra negation and sum need not be interpreted correctly; non-multiplicative (non-$\times$) representations, where the relation algebra negation and intersection need not be interpreted correctly. For $\lambda \in \{\text{class, non-+, non-}\times\}$ the class of boolean modules with λ-representations and the class of Peirce algebras with λ-representations form quasi-varieties of two sorted algebras. The class of classically representable boolean modules (respectively, Peirce algebras) is an equational variety, it is the variety generated by the class of all full boolean modules (Peirce algebras). An λ-representation of a boolean module or of a Peirce algebra is *straight* if the representations of the boolean unit and the relation algebra unit have the same base set. For $\lambda \in \{\text{class, non-+, non-}\times\}$ a λ-representation of a Peirce algebra is necessarily straight, but there are λ-representable boolean modules with no straight λ-representations at all. For each such λ, write (i) $\mathbf{RPA}_\lambda$, (ii) $\mathbf{RBM}_\lambda$, (iii) $\mathbf{RRA}_\lambda$ and (iv) $\mathbf{SRBM}_\lambda$ for the classes consisting of Peirce type algebras where (i) the algebra has an λ-representation, (ii) the reduct to boolean modules has an λ-representation, (iii) the reduct to relation algebras has an λ-representation and (iv) the reduct to boolean modules has a straight λ-representation.

$$\begin{array}{ccccccc}
\mathbf{RRA}_{\text{non-}\times} & \supset & \mathbf{RBM}_{\text{non-}\times} & \Supset & \mathbf{SRBM}_{\text{non-}\times} & \supset & \mathbf{RPA}_{\text{non-}\times} \\
\Cup & & \Cup & & \Cup & & \Cup \\
\mathbf{RRA}_{\text{non-+}} & \Supset & \mathbf{RBM}_{\text{non-+}} & \Supset & \mathbf{SRBM}_{\text{non-+}} & \supset & \mathbf{RPA}_{\text{non-+}} \\
\Cup & & \Cup & & \Cup & & \Cup \\
\mathbf{RRA}_{\text{class}} & \Supset & \mathbf{RBM}_{\text{class}} & \Supset & \mathbf{SRBM}_{\text{class}} & \supset & \mathbf{RPA}_{\text{class}}
\end{array}$$

All inclusions are strict and $K \Supset L$ means that $K \supset L$ and L cannot be defined within K using only finitely many axioms.

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1 Introduction

Boolean algebras correspond to unary relations (sets), relation algebras correspond to binary relations. Boolean algebras and relation algebras have a long history and have been studied extensively, but algebras which can explicitly handle both unary and binary relations have received less attention.

Yet much work in logic uses both types of relations. Description logic, for example, has unary relations called 'concepts' and binary relations called 'roles'. In multimodal logic the modalities have semantics as binary relations and the propositions are interpreted as unary relations. Propositional dynamic logic (PDL) has propositional formulae, which are interpreted as unary relations, and programs, which act as modalities and are interpreted as binary relations. If we are to treat both kinds of elements algebraically we need a two-sorted algebra with one sort for unary relations and one sort for binary relations. The two main kinds of algebras for unary and binary relations are *boolean modules* and *Peirce algebras*.

Boolean modules were defined and studied carefully in [1]. A boolean module comprises a relation algebra, a boolean algebra and an operator (the Peircean operator) which takes a relation algebra element and a boolean algebra element and returns a boolean algebra element. An application of the Peircean operator $R : B$ corresponds to the existential formula $\exists R.B$ in description logic, to the formula $\Diamond_R B$ in modal logic and to the formula $\langle R \rangle B$ of PDL. For a first-order approximation, if we translate R, B as the atomic formulas $R(x, y), B(x)$ respectively then $R : B$ translates to a formula with one free variable, namely $\exists y\ (R(x, y) \wedge B(y))$. This is only an approximation, because boolean modules are defined algebraically, by a finite set of equations, and there is no guarantee that a boolean module is representable. We will say more about representations later.

A Peirce algebra (see [2]) is a boolean module with an additional operator, *right cylindrification* c, which takes a boolean element and returns a relation algebra element. Right cylindrification does not correspond to a description logic formula. A formula corresponding to $c(B)$ in PDL is $(B?); 1$, where 1 is the program that can take you from a state to any other state. An approximate first-order translation of $c(B)$ is the two variable formula $B(x) \wedge \top(x, y)$, where $\top$ is a binary predicate symbol that always evaluates to true. Again, the class of Peirce algebras is defined by a finite set of algebraic equations.

There are some obvious connections between relation algebras, boolean modules and Peirce algebras. A relation algebra $\mathcal{R} = (R, 0, 1, +, -, 1', \breve{}\ , ;)$ already contains a boolean algebra $\mathcal{B}(\mathcal{R}) = (\{r; 1 \mid r \in R\}, 0, 1, +, -)$. It is possible to take an arbitrary relation algebra $\mathcal{R}$ together with this boolean subalgebra $\mathcal{B}(\mathcal{R})$ and an appropriate relation algebraic definition of the Peircean operator to obtain a boolean module $\mathcal{M}(\mathcal{R})$ (see definition 18). And we can extend $\mathcal{M}(\mathcal{R})$ to a Peirce algebra $\mathcal{P}(\mathcal{R})$ by giving a relation algebraic definition for right cylindrification. We will see that an arbitrary Peirce algebra whose relation algebra part is $\mathcal{R}$ must be isomorphic to $\mathcal{P}(\mathcal{R})$ but a boolean module whose relation algebra part is $\mathcal{R}$ need not be isomorphic to $\mathcal{M}(\mathcal{R})$. This amounts to saying that there is a uniform *interpretation* of Pierce algebras in two sorted algebras of their relation algebra parts, but no such interpretation for boolean modules. The interpretation of a Peirce algebra in its relation algebra part preserves representability and non-representability. It follows that a Peirce algebra is representable (in any of the senses we discuss below) if and only if its relation algebra part is representable. The majority of our investigation, then, concerns representations of boolean modules.

There is more than one relevant notion of representation here. Brink gives two different definitions of representability for boolean modules. The first definition, the main focus of his paper, is for a type of 'weak representation' — where the relation algebra part is only weakly represented, in the sense of [11]. It is unfortunate that there are at least two different notions of weak representation for relation algebras in the literature. Brink is referring to a representation where all the operators of a boolean module must be interpreted in the natural, set-theoretic way except perhaps the relation algebra $\cdot$ and $-$. The advantage of weak representations is that every relation algebra has a weak representation [11, theorem 4.22]. We will also consider the weak representations given in [10], where all operators are interpreted normally except perhaps the relation algebra $+$ and $-$. To distinguish these two we will refer to the former as a *non-multiplicative representation* and the latter as an *non-additive representation*. However, we consider both of these kinds of weak representations as unnatural for the signatures under consideration. We will also consider *classical representations* where all the operators must be interpreted in the natural set-theoretic way. We consider all three kinds of representations for relation algebras, boolean modules and Peirce algebras. Very little has been written about the representability of boolean modules and Peirce algebras (apart from [1], and [2, 18, 19] are also relevant).

Lets consider Peirce algebras first. Whichever type of representation we consider, it is part of the definition of a Peirce algebra representation that it induces a relation algebra representation (of the same type) of the relation algebra part of the Peirce algebra. But it turns out that representability of the relation algebra part is sufficient (given that the finite set of defining equations for Peirce algebras are satisfied) for representability of the Peirce algebra. It follows that the class of representable Peirce algebras is defined by an axiomatisation of the class of relation algebras with the relevant kind of representation together with a small (finite) number of axioms for the two extra Peirce algebra operators.

Now consider boolean modules. A complication arises not visible in the case of representations of Peirce algebras. In any type of representation of a boolean module we interpret both the relation algebra part and the boolean part of the boolean module. In a representation of a Peirce algebra the domain of the interpretation of the boolean part necessarily coincides with the base of the interpretation of the relation algebra part, but for boolean modules although it is easy to show that the domain of the interpretation of the boolean part is a subset of the base of the interpretation of the relation algebra part, the two base sets need not be identical. A representation where the two base sets are equal will be called a *straight representation*. As we just mentioned, a representation of a Peirce algebra is necessarily straight, but some representable boolean modules do not have any straight representations at all (see remark 13).

In his conclusion [1, page 312] Brink asks for a general representation result for classical representations. Now of course, with the classical definition of a representation, a boolean module might not be representable simply because the relation algebra part is not a representable relation algebra. So an initial question is: must a boolean module whose relation algebra part is representable be a representable boolean module? We show that the answer is no. We give examples of boolean modules, where the relation algebra part is classically representable but the boolean module is not classically representable. These boolean modules do not possess straight non-multiplicative representations either.

Then we show that an ultraproduct of these unrepresentable boolean modules has a classical straight representation. It follows that infinitely many axioms are needed, in addition to an axiomatisation of the representable relation algebras, to define the class of classically representable boolean modules. And any axiomatisation of the class of boolean modules with straight non-multiplicative representations must include infinitely many axioms not derivable from an axiomatisation of the representable relation algebras.

Brink asks whether every boolean module must possess a non-multiplicative representation [1, page 312]. We show that this is so, if we allow representations not to be straight. We have already mentioned that a boolean module could fail to have a straight non-multiplicative representation.

We also consider non-additive representations and show that infinitely many axioms are needed, in addition to an axiomatisation of the relation algebras with non-additive representations, to define the class of boolean modules with non-additive representations.

Peirce algebras therefore appear to have an advantage over boolean modules. Their representability essentially boils down to the representability of their relation algebra part. This is not the case for boolean modules.

In the next section we define boolean modules and Peirce algebras. The following section defines the various kinds of representations of these algebras. Section 4 shows that the representability of a Peirce algebra reduces to the representability of its relation algebra part alone. In section 5 we show that the classically representable boolean modules form a variety — the variety generated by the boolean modules with straight classical representations. Section 6 answers a question of Brink by showing that every boolean module has a non-multiplicative representation, though not necessarily a straight one. The next three sections prove non-finite axiomatisability for various classes defined by different types of representations, relative to certain containing classes.

Those who like to read backwards should start with theorem 47 in the final section, where many of the key results are summarised.

REMARK 1 *In his study of the connections between Peirce algebras and modal logic [18, 19], De Rijke gives a complete characterisation of the* Peirce frames *as the frames that validate a finite set of formulas in a modal language corresponding to Peirce algebras. Peirce frames are quite close to classically representable Peirce algebras (they correspond to the atom structures of full Peirce algebras, while the variety generated by full Peirce algebras is identical to the class of classically representable Peirce algebras). An example of one of these formulas is*

$$\langle r\rangle\langle s\rangle p \rightarrow \langle r;s\rangle p$$

where $\langle r\rangle$ is a modality corresponding to the binary relation r, p is a proposition and $r;s$ denotes the composition of the binary relations r and s. Note, however, that these axioms are not pure algebraic axioms in the language of Peirce algebras, but modal formulas involving propositions.

2 Notation and preliminaries

The colon symbol : is used to denote the Peircean operator, so we will write $\{x \mid P(x)\}$ rather than $\{x : P(x)\}$. The cardinality of a set S is denoted $||S||$. If f, g are functions then $f \circ g$ means 'first g then f' and is defined by $(f \circ g)(x) = f(g(x))$. We assume familiarity with relation algebra. References include [15, 7]. We identify a structure with its domain. Thus, for a structure $\mathcal{W}$, we write $x \in \mathcal{W}$ if x belongs to the domain of $\mathcal{W}$, and we write $\mathcal{W} \times \mathcal{W}$ for the (unstructured) Cartesian product of the domain of $\mathcal{W}$ with itself.

In a boolean algebra $(B, 0, 1, \cdot, +, -)$ we write $b \leq c$ if $b \cdot c = b$ or, equivalently, if $b + c = c$. If S is a subset of the domain of a boolean algebra $\mathcal{B}$ then we call S a *filter base* if for all finite subsets $\{s_0, \ldots, s_{n-1}\}$ of S we have $s_0 \cdot s_1 \cdot \ldots \cdot s_{n-1} \neq 0$. S is a *filter* if it satisfies $s \in S \wedge s \leq t \Rightarrow t \in S$ and $s, t \in S \Rightarrow s \cdot t \in S$ and it is *proper* if $0 \notin S$. A filter S is an *ultrafilter* if it is a maximal proper filter, or equivalently a filter base where for all $b \in \mathcal{B}$ either $b \in S$ or $-b \in S$.

An algebra is said to be *degenerate* if it contains only a single element. If $(\mathcal{A}_i \mid i \in I)$ is a family of structures and D is an ultrafilter over the boolean algebra $\wp(I)$ of all subsets of I, then $\Pi_D \mathcal{A}_i$ denotes the *ultraproduct* over the $\mathcal{A}_i$s. It is obtained from the direct product of the $\mathcal{A}_i$s by taking the quotient by the congruence relation $\sim$ defined by $(a_i \mid i \in I) \sim (b_i \mid i \in I) \iff \{i \in I \mid a_i = b_i\} \in D$.

Let κ be any cardinal and let L be a language. An L-structure $\mathcal{M}$ is said to be *κ-saturated* if every consistent set of formulas of the form $\phi(\bar{x})$, for some fixed tuple of variables $\bar{x}$ in a language obtained by adding less than κ parameters $m \in \mathcal{M}$ to the language L, is realised in $\mathcal{M}$.

LEMMA 2 *If Σ is a consistent L-theory and κ is any cardinal then Σ has a κ-saturated model.*

See [4, lemma 5.1.4] for a proof. If $L \subseteq K$ are classes of algebras and S is a set of formulas in the signature of these algebras then we say that S defines L within K if for all $\mathcal{A} \in K$, $(\mathcal{A} \models S \iff \mathcal{A} \in L)$.

DEFINITION 3 (Boolean Module, Brink 1981) *A (left)* boolean module $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ *consists of a relation algebra $\mathcal{R}$, a boolean algebra $\mathcal{B}$ and a* Peircean operator *: from $\mathcal{R} \times \mathcal{B}$ to $\mathcal{B}$ satisfying,*

1. $r : (a + b) = r : a + r : b$
2. $(r + s) : a = r : a + s : a$
3. $(r; s) : a = r : (s : a)$
4. $1' : a = a$
5. $0 : a = 0$

6. $r^{\smile} : (-(r : a)) \leq -a$

for all $r, s \in \mathcal{R}$ and all $a, b \in \mathcal{B}$. We may refer to $\mathcal{R}$ as the relation algebra part *and $\mathcal{B}$ as the* boolean algebra part *of $\mathcal{M}$ and an element of $\mathcal{R}$ is called a* relation algebra element *while an element of $\mathcal{B}$ is called a* boolean element. *Here $1', {}^{\smile}, ;$ are the relation algebra operators identity, converse and composition. The symbols $0, 1, +, -$ are used to denote either operators on $\mathcal{R}$ or $\mathcal{B}$, the cases distinguished by use. Where necessary we may write $0_{\mathcal{B}}, 1_{\mathcal{B}}, \cdot_{\mathcal{B}}, +_{\mathcal{B}}, -_{\mathcal{B}}$ for the zero, unit, multiplication, addition and negation of the boolean part and $0_{\mathcal{R}}, 1_{\mathcal{R}}, \cdot_{\mathcal{R}}, +_{\mathcal{R}}, -_{\mathcal{R}}$ as the corresponding operators in the relation algebra part of a boolean module, to avoid confusion.*

Right boolean modules are defined similarly, but we will not refer to them further. Henceforth, instead of a "left boolean module" we'll simply say a "boolean module".

It will be useful to recall a lemma about boolean modules here.

LEMMA 4 *Let $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ be a boolean module.*

1. *The Peircean operator $:$ is completely additive in both arguments — if the supremum $\sum S$ of $S \subseteq \mathcal{R}$ exists then the supremum $\sum\{s : b \mid s \in S\}$ also exists and equals $(\sum S) : b$, for $b \in \mathcal{B}$, and if the supremum $\sum C$ of $C \subseteq \mathcal{B}$ exists then the supremum $\sum\{r : c \mid c \in C\}$ also exists and equals $r : \sum C$, for $r \in \mathcal{R}$.*

2. *For any $r \in \mathcal{R}$ and any $a, b \in \mathcal{B}$, we have $(r : b) \cdot a \neq 0 \Rightarrow (r^{\smile} : a) \cdot b \neq 0$.*

PROOF:

See [1, T6] and [1, T5], respectively. $\square$

DEFINITION 5 *For boolean modules $\mathcal{M} = (\mathcal{R}, \mathcal{A}, :_1)$ and $\mathcal{N} = (\mathcal{S}, \mathcal{B}, :_2)$, a two sorted map, (i.e. a map from the domain of $\mathcal{R}$ to the domain of $\mathcal{S}$ and another map from the domain of $\mathcal{A}$ to the domain of $\mathcal{B}$) preserving all the operators is called a* homomorphism *from $\mathcal{M}$ into $\mathcal{N}$. If both parts of this homomorphism are injective then we say that $\mathcal{M}$ is a* subalgebra *of $\mathcal{N}$. If both parts of this homomorphism are surjective then $\mathcal{N}$ is a* homomorphic image *of $\mathcal{M}$. If both parts of this homomorphism are bijective then we say that $\mathcal{M}$ and $\mathcal{N}$ are* isomorphic. *The* direct product *$\mathcal{M} \times \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ is defined to be $(\mathcal{R} \times \mathcal{S}, \mathcal{A} \times \mathcal{B}, :)$, where $:$ is defined by $(r, s) : (a, b) = (r :_1 a, s :_2 b)$ for $r \in \mathcal{R}$, $s \in \mathcal{S}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. More generally, if $(\mathcal{M}_i \mid i \in I)$ is a sequence of boolean modules, say $\mathcal{M}_i = (\mathcal{R}_i, \mathcal{B}_i, :_i)$ for each $i \in I$, then the* direct product *$\Pi_{i \in I}\mathcal{M}_i$ is defined to be $(\Pi_i \mathcal{R}_i, \Pi_i \mathcal{B}_i, :)$ where $(r_i \in \mathcal{R}_i \mid i \in I) : (b_i \in \mathcal{B}_i \mid i \in I) = (r_i :_i b_i \mid i \in I)$.*

$\mathbf{0}$ denotes the trivial, one element boolean algebra. For any boolean algebra $\mathcal{B}$ we have $\mathbf{0} \times \mathcal{B} \cong \mathcal{B}$. The map from $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ to $(\mathcal{R}, \mathbf{0}, :)$ which is the identity over $\mathcal{R}$ and maps all boolean elements to $0_{\mathcal{B}}$ is a homomorphism.

Henceforth, we promise to avoid notations like $:_1, :_2$ and will let the reader determine which Peircean operator we mean, by the context.

DEFINITION 6 (Peirce Algebra) *A* Peirce algebra *$\mathcal{P} = (\mathcal{R}, \mathcal{B}, :, c)$ consists of a boolean module $(\mathcal{R}, \mathcal{B}, :)$ and a* right cylindrifier *c mapping $\mathcal{B}$ to $\mathcal{R}$ such that*

1. *c is a boolean homomorphism, $c(0_{\mathcal{B}}) = 0_{\mathcal{R}}$, $c(1_{\mathcal{B}}) = 1_{\mathcal{R}}$, $c(a+b) = c(a)+c(b)$ and $c(-a) = -c(a)$,*

2. *$c(b); 1_{\mathcal{R}} = c(b)$,*

3. *$c(b) : 1_{\mathcal{B}} = b$,*

4. *$c(r : b) = r; c(b)$,*

for all $a, b \in \mathcal{B}$ and all $r \in \mathcal{R}$.

As with boolean modules, we can define homomorphism, subalgebra and direct product of Peirce algebras.

REMARK 7 *Other operators involving the unary and binary relations have also been considered. For example, we can define the range and domain of a binary relation. Or, given two unary relations, we can define their Cartesian product. However, these new operators can all be defined using the operators of Peirce algebra. If r is a relation algebra element then $r : 1_{\mathcal{B}}$ defines the domain and $r^{\smile} : 1_{\mathcal{B}}$ defines the range of r. For boolean elements a, b, their Cartesian product $a \times b$ is defined by $c(a) \cdot c(b)^{\smile}$.*

REMARK 8 *We defined a homomorphism from $(\mathcal{R}, \mathcal{B}, :)$ to $(\mathcal{R}, \mathbf{0}, :)$ for boolean modules (see definition 5). But the corresponding map from $(\mathcal{R}, \mathcal{B}, :, c)$ to $(\mathcal{R}, \mathbf{0}, :, c)$ will not be a homomorphism of Peirce algebras (except in the degenerate case) because $1_{\mathcal{B}}$ maps to $1_{\mathcal{B}} = 0_{\mathcal{B}}$ but $c(1_{\mathcal{B}}) = 1_{\mathcal{R}}$ maps to $1_{\mathcal{R}}$, yet $1_{\mathcal{R}} \neq 0_{\mathcal{R}} = c(0_{\mathcal{B}})$.*

3 Representations

We shall now define various kinds of representations of boolean algebras, relation algebras, boolean modules and Peirce algebras. In each case, a representation will be defined to be a structure for a defining theory whose signature depends on the algebra under consideration. For example, a representation of a boolean algebra $\mathcal{B}$ will be defined in a language with a unary predicate symbol for each element of $\mathcal{B}$, while a representation of a relation algebra $\mathcal{R}$ will be defined in a language with a binary predicate symbol for each element of $\mathcal{R}$. Thus we can apply the terminology and results from the general theory of first-order structures.

DEFINITION 9 (Representations of boolean and relation algebras) *Let $\mathcal{B} = (B, 0, 1, \cdot, +, -)$ be a boolean algebra.*

- *A $\mathcal{B}$-*structure *$\mathcal{X}$ interprets each element of b as a subset $b^{\mathcal{X}}$ of the domain of $\mathcal{X}$. $\mathcal{X}$ is* faithful *if $b \neq b'$ implies $b^{\mathcal{X}} \neq {b'}^{\mathcal{X}}$.*

- *A* faithful *$\mathcal{B}$-structure $\mathcal{X}$ is called a* representation *if for all $a, b \in \mathcal{B}$,*

$$\begin{aligned} 0^{\mathcal{X}} &= \emptyset \\ (a \cdot b)^{\mathcal{X}} &= a^{\mathcal{X}} \cap b^{\mathcal{X}} \\ (a+b)^{\mathcal{X}} &= a^{\mathcal{X}} \cup b^{\mathcal{X}} \\ (-a)^{\mathcal{X}} &= 1^{\mathcal{X}} \setminus a^{\mathcal{X}} \end{aligned}$$

- *Let* $\mathcal{R} = (R, 0, 1, \cdot, +, -, 1', \smile, ;)$ *be any relation algebra. An* $\mathcal{R}$-structure $\mathcal{W}$ *interprets each element* $r \in \mathcal{R}$ *as a binary relation* $r^{\mathcal{W}}$ *over the domain of* $\mathcal{W}$. *If* $r \neq s \in \mathcal{R}$ *implies* $r^{\mathcal{W}} \neq s^{\mathcal{W}}$ *then* $\mathcal{W}$ *is faithful.*

- *A* pseudo-representation $\mathcal{W}$ *of* $\mathcal{R}$ *is a faithful* $\mathcal{R}$*-structure satisfying*

$$\begin{array}{rcl} 0^{\mathcal{W}} & = & \emptyset \\ r \leq s & \Rightarrow & r^{\mathcal{W}} \subseteq s^{\mathcal{W}} \\ 1'^{\mathcal{W}} & = & \{(w, w) \mid w \in \mathcal{W}\} \\ (r^{\smile})^{\mathcal{W}} & = & \{(v, w) \mid (w, v) \in r^{\mathcal{W}}\} \\ (r; s)^{\mathcal{W}} & = & \{(u, w) \mid \exists v\, ((u, v) \in r^{\mathcal{W}} \wedge (v, w) \in s^{\mathcal{W}})\} \end{array}$$

  for all $r, s \in \mathcal{R}$. *Note, for any pseudo-representation* $\mathcal{W}$ *of* $\mathcal{R}$, *that* $1^{\mathcal{W}}$ *is an equivalence relation over* $\mathcal{W}$, *since* $1' \leq 1$, $1^{\smile} = 1$ *and* $1; 1 = 1$ *hold in all relation algebras.*

- *In a pseudo-representation* $\mathcal{W}$, *if* $1^{\mathcal{W}} = \mathcal{W} \times \mathcal{W}$ *(recall that we identify* $\mathcal{W}$ *with its domain) then* $\mathcal{W}$ *is called* square.

- *If*

$$(r \cdot s)^{\mathcal{W}} = r^{\mathcal{W}} \cap s^{\mathcal{W}}$$

  holds (for all $r, s \in \mathcal{R}$*) then the pseudo representation* $\mathcal{W}$ *is called a* non-additive representation. *The class of relation algebras with non-additive representations is usually called* **wRRA**, *in the literature.*

- *If (instead)*

$$(r + s)^{\mathcal{W}} = r^{\mathcal{W}} \cup s^{\mathcal{W}}$$

  holds in a pseudo-representation then it is called a non-multiplicative *representation.*

- *If* $\mathcal{W}$ *is a non-additive representation and*

$$(-r)^{\mathcal{W}} = 1^{\mathcal{W}} \setminus r^{\mathcal{W}}$$

  then $\mathcal{W}$ *is called a* classical representation.

Of course, by the De Morgan laws, if $\mathcal{W}$ *is a non-multiplicative representation in which the negation axiom holds then it must be a classical representation.*

LEMMA 10 (Stone, [20]) *Let* κ *be a cardinal and let* $\mathcal{B}$ *be a boolean algebra.* $\mathcal{B}$ *has a classical representation* $\mathcal{X}$ *such that (i)* $1_{\mathcal{B}}^{\mathcal{X}} = \mathcal{X}$, *(ii) for all* $x, y \in \mathcal{X}$ *if* $\forall b \in \mathcal{B}\ (x \in b^{\mathcal{X}} \leftrightarrow y \in b^{\mathcal{X}})$ *then* $x = y$ *and (iii)* $\mathcal{X}$ *is* κ*-saturated.*

The main use of κ-saturated representations that we will use is the following.

LEMMA 11 *Let* $\mathcal{B}$ *be a boolean algebra and let* $\mathcal{X}$ *be a* $||\mathcal{B}||$*-saturated representation of it. For any set* $S \subseteq \mathcal{B}$ *if* S *has the* finite intersection property *(i.e.* $s_0, s_1, \ldots, s_k \in S \rightarrow s_0 \cdot s_1 \cdot \ldots \cdot s_k \neq 0$*) then* $\bigcap_{s \in S} s^{\mathcal{X}} \neq \emptyset$.

The following paragraph summarises a few of the known results concerning these representations of relation algebras. The class of all classically representable relation algebras is not finitely axiomatisable [17] but forms an equationally definable variety [21]. The class of relation algebras with non-additive representations is not finitely axiomatisable [9]. By contrast, every relation algebra possesses a non-multiplicative representation [11, theorem 4.22].

DEFINITION 12 (Representations of Boolean Modules and Peirce Algebras)

*Let $\lambda \in \{$pseudo, non-multiplicative, non-additive, classical$\}$. An λ-*representation *$\mathcal{W}$ of a boolean module $(\mathcal{R}, \mathcal{B}, :)$ interprets each element $r \in \mathcal{R}$ as a binary relation $r^{\mathcal{W}}$ over the domain of $\mathcal{W}$ and interprets each element $b \in \mathcal{B}$ as a subset $b^{\mathcal{W}}$ of the domain of $\mathcal{W}$ such that the reduct of $\mathcal{W}$ to the boolean signature gives a boolean representation of $\mathcal{B}$, the reduct to the signature of relation algebras gives a λ-representation of the relation algebra $\mathcal{R}$ and the Peircean operator $:$ satisfies*

$$(r : a)^{\mathcal{W}} = \{v \in \mathcal{W} \mid \exists w \in a^{\mathcal{W}} \ ((v, w) \in r^{\mathcal{W}})\} \tag{1}$$

If $\mathcal{W}$ is a λ-representation and for all $v \in \mathcal{W}$ we have $(v, v) \in 1_{\mathcal{R}}^{\mathcal{W}} \iff v \in 1_{\mathcal{B}}^{\mathcal{W}}$ then we call $\mathcal{W}$ a straight *λ-representation.*

A λ-representation $\mathcal{W}$ of a Peirce algebra $(\mathcal{R}, \mathcal{B}, :, c)$ is a λ-representation of the boolean module $(\mathcal{R}, \mathcal{B}, :)$ satisfying

$$(c(b))^{\mathcal{W}} = \{(x, y) \in 1_{\mathcal{R}}^{\mathcal{W}} \mid x \in b^{\mathcal{W}}\}$$

REMARK 13 *Let $\lambda \in \{$pseudo, non-additive, non-multiplicative, classical$\}$ and let $\mathcal{W}$ be a λ-representation of the non-degenerate relation algebra $\mathcal{R}$. In definition 5 we defined a boolean module $\mathcal{M} = (\mathcal{R}, \mathbf{0}, :)$. Trivially, we can extend $\mathcal{W}$ to a λ-representation of $\mathcal{M}$ by letting $\mathbf{0}^{\mathcal{W}} = \emptyset$. But in $\mathcal{M}$ we have $1_{\mathcal{B}} = \mathbf{0}$ since the boolean part is degenerate, hence $1_{\mathcal{B}}^{\mathcal{W}} = \emptyset$. It follows that although $\mathcal{M}$ has λ-representations it has no straight λ-representations.*

We have just defined a classical representation of a boolean module or Peirce algebra to be an isomorphism to a concrete two sorted algebra of binary and unary relations. Another way of defining representable algebras is as the variety generated by the *full algebras* (for example, the class of representable relation algebras is defined as the variety generated by the full relation algebras in [6, definition 5.3.2]).

DEFINITION 14 *The* full boolean module *$\mathcal{M}(U) = (\Re(U), \wp(U), :)$ consists of the full relation algebra $\Re(U)$ of all binary relations over some set U, the full boolean algebra $\wp(U)$ of all subsets of U, and the Peircean operator $:$ defined by*

$$R : B = \{u \in U \mid \exists v \ (v \in B \wedge (u, v) \in R)\}$$

for any $B \subseteq U$, $R \subseteq U \times U$. The full Peirce algebra *$\mathcal{P}(U) = (\Re(U), \wp(U), :, c)$ is defined similarly, with*

$$c(B) = \{(u, v) \in U \times U \mid u \in B\}$$

It will turn out that the variety generated by the full boolean modules coincides with the class of classically representable boolean modules (theorem 27).

LEMMA 15 *Let $\mathcal{W}$ be a pseudo-representation of a boolean module $\mathcal{M}$. For all $x \in \mathcal{W}$ we have*

$$x \in 1_{\mathcal{B}}^{\mathcal{W}} \Rightarrow (x, x) \in 1_{\mathcal{R}}^{\mathcal{W}}$$

If $\mathcal{W}$ is a square, pseudo-representation of $\mathcal{M}$ then $\mathcal{W}$ is straight.

PROOF:

The first part is noted in [1]. If $x \in 1_{\mathcal{B}}^{\mathcal{W}}$ then, since $1_{\mathcal{B}} = 1' : 1_{\mathcal{B}}$, we have $x \in (1' : 1_{\mathcal{B}})^{\mathcal{W}}$ so there is y such that $(x, y) \in 1'^{\mathcal{W}}_{\mathcal{R}}$, hence $y = x$ and $(x, x) \in 1'^{\mathcal{W}}_{\mathcal{R}} \subseteq 1_{\mathcal{R}}^{\mathcal{W}}$.

For the second part, suppose $\mathcal{W}$ is a square representation of $\mathcal{M}$, so $1_{\mathcal{R}}^{\mathcal{W}} = \mathcal{W} \times \mathcal{W}$ and suppose $(x, x) \in 1_{\mathcal{R}}^{\mathcal{W}}$. Pick any $y \in 1_{\mathcal{B}}^{\mathcal{W}}$. Then $(x, y) \in 1_{\mathcal{R}}^{\mathcal{W}}$ so $x \in (1_{\mathcal{R}} : 1_{\mathcal{B}})^{\mathcal{W}} = 1_{\mathcal{B}}^{\mathcal{W}}$. Hence $(x, x) \in 1_{\mathcal{R}}^{\mathcal{W}} \Rightarrow x \in 1_{\mathcal{B}}^{\mathcal{W}}$, so $\mathcal{W}$ is straight. $\Box$

4 Representations of Peirce algebras

LEMMA 16 *Let $\lambda \in \{$classical, non-additive, non-multiplicative$\}$. Every λ-representation of a Peirce algebra is a straight λ-representation.*

PROOF:

Let $\mathcal{W}$ be a λ-representation of $\mathcal{P}$. By lemma 15, $w \in 1_{\mathcal{B}}^{\mathcal{W}} \Rightarrow (w, w) \in 1_{\mathcal{R}}^{\mathcal{W}}$. Conversely suppose $(w, w) \in 1_{\mathcal{R}}^{\mathcal{W}}$. Then since $1_{\mathcal{R}} = c(1_{\mathcal{B}})$ we have $(w, w) \in (c(1_{\mathcal{B}}))^{\mathcal{W}}$ and so $w \in 1_{\mathcal{B}}^{\mathcal{W}}$. Hence $\mathcal{W}$ is a straight λ-representation. $\Box$

LEMMA 17 *Let $\mathcal{P} = (\mathcal{R}, \mathcal{B}, :, c)$ be a Peirce algebra. Let $\lambda \in \{$classical, non-additive, non-multiplicative$\}$. The following are equivalent.*

1. *$\mathcal{R}$ has a λ-representation,*
2. *$\mathcal{P}$ has a λ-representation,*
3. *$\mathcal{P}$ has a straight λ-representation.*

PROOF:

The implication (2) $\Rightarrow$ (1) is clear and (2) $\Leftrightarrow$ (3) is by lemma 16. We prove (1) $\Rightarrow$ (2).

Let $\mathcal{W}$ be a λ-representation of $\mathcal{R}$. Extend $\mathcal{W}$ to $\mathcal{W}'$ by interpreting the boolean elements as subsets of the domain of $\mathcal{W}$ like this:

$$b^{\mathcal{W}'} = \{x \in \mathcal{W} \mid \forall y\ ((x, y) \in 1_{\mathcal{R}}^{\mathcal{W}} \rightarrow (x, y) \in c(b)^{\mathcal{W}})\}$$

We must show that $\mathcal{W}'$ is a λ-representation of $\mathcal{P}$.

Claim: (a) $x \in b^{\mathcal{W}'}$ iff (b) $(x, x) \in c(b)^{\mathcal{W}}$ iff (c) $\exists y\ ((x, y) \in c(b)^{\mathcal{W}})$

Proof of claim: Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c) (note that $(x, x) \in 1'^{\mathcal{W}}_{\mathcal{R}} \subseteq 1_{\mathcal{R}}^{\mathcal{W}}$, for all $x \in \mathcal{W}$), so we must show that (c) $\Rightarrow$ (a). Assume $\exists y\ (x, y) \in c(b)^{\mathcal{W}}$. Take any z such that $(x, z) \in 1_{\mathcal{R}}^{\mathcal{W}}$. We have

$(y,x) \in (c(b)^{\smile})^{\mathcal{W}}$ and $(x,z) \in 1_{\mathcal{R}}^{\mathcal{W}}$ so $(x,z) \in (c(b); c(b)^{\smile}; 1_{\mathcal{R}})^{\mathcal{W}} \subseteq (c(b); 1_{\mathcal{R}})^{\mathcal{W}} = c(b)^{\mathcal{W}}$, by definition 6(2). This shows (c) $\Rightarrow$ (a) and proves the claim.

The restriction of $\mathcal{W}'$ to the boolean part is a boolean representation of $\mathcal{B}$, by definition 6(1).

To check that c is correctly represented, let $b \in \mathcal{B}$. If $(x,y) \in c(b)^{\mathcal{W}'}$ then by the claim $x \in b^{\mathcal{W}'}$. And conversely if $x \in b^{\mathcal{W}'}$ and $(x,y) \in 1_{\mathcal{R}}^{\mathcal{W}}$ then $(x,y) \in c(b)^{\mathcal{W}'}$, also by the claim. Thus $c(b)^{\mathcal{W}'} = \{(x,y) \in 1_{\mathcal{R}}^{\mathcal{W}} \mid x \in b^{\mathcal{W}'}\}$, as required.

Finally we check the Peircean operator. We have $x \in (r:b)^{\mathcal{W}'}$ iff $(x,x) \in c(r:b)^{\mathcal{W}} = (r; c(b))^{\mathcal{W}}$ (by the claim and by definition 6(4)) iff there is z such that $(x,z) \in r^{\mathcal{W}}$ and $(z,x) \in c(b)^{\mathcal{W}}$ iff there is z such that $(x,z) \in r^{\mathcal{W}}$ and $z \in b^{\mathcal{W}'}$. So the Peircean operator is correctly represented.

$\Box$

Next, we will provide an *interpretation* of a Peirce algebra in a two sorted algebra based solely on its relation algebra part. No such interpretation can be found for boolean modules (see [8, section 5.3] for a study of interpretations).

DEFINITION 18 *Let $\mathcal{R} = (R, 0, 1, -, +, 1', ^{\smile}, ;)$ be a relation algebra. Define a boolean algebra $\mathcal{B}(\mathcal{R}) = (\{r;1 \mid r \in \mathcal{R}\}, 0, 1, -, +)$. Define a boolean module $\mathcal{M}(\mathcal{R}) = (\mathcal{R}, \mathcal{B}(\mathcal{R}), ;)$, i.e. the Peircean operator is defined to be the relation algebra composition operator. Define a Peirce algebra $\mathcal{P}(\mathcal{R}) = (\mathcal{R}, \mathcal{B}(\mathcal{R}), ;, Id)$, i.e. right cylindrification is the identity map.*

It is easy to verify that $\mathcal{B}(\mathcal{R}), \mathcal{M}(\mathcal{R}), \mathcal{P}(\mathcal{R})$ are indeed, respectively, a boolean algebra, a boolean module, a Peirce algebra.

LEMMA 19 *Let $\lambda \in$ {classical, non-additive, non-multiplicative} and let $\mathcal{R}$ be a relation algebra. The following are equivalent.*

1. *$\mathcal{R}$ has a λ-representation.*
2. *$\mathcal{M}(\mathcal{R})$ has a λ-representation.*
3. *$\mathcal{M}(\mathcal{R})$ has a straight λ-representation.*
4. *$\mathcal{P}(\mathcal{R})$ has a λ-representation.*

PROOF:

We prove the implications cyclicly, from bottom to top. If $\mathcal{P}(\mathcal{R})$ has a λ-representation then by lemma 16 it has a straight λ-representation. The reduct to the boolean module signature is then a straight λ-representation of $\mathcal{M}(\mathcal{R})$. Obviously if $\mathcal{M}(\mathcal{R})$ has a straight λ-representation then it has a λ-representation. The reduct of a λ-representation of $\mathcal{M}(\mathcal{R})$ to the relation algebra signature is a λ-representation of $\mathcal{R}$. Finally, if $\mathcal{R}$ has a λ-representation then so does $\mathcal{P}(\mathcal{R})$, by lemma 17. $\Box$

THEOREM 20 *For any Peirce algebra $\mathcal{P} = (\mathcal{R}, \mathcal{B}, :, c)$ we have $\mathcal{P} \cong \mathcal{P}(\mathcal{R})$, i.e. any Peirce algebra whose relation algebra part is $\mathcal{R}$ is isomorphic to $\mathcal{P}(\mathcal{R})$.*

PROOF:

To define an isomorphism θ from $\mathcal{P}$ to $\mathcal{P}(\mathcal{R})$ we must define two bijections: one from $\mathcal{R}$ to $\mathcal{R}$ and the other from $\mathcal{B}$ to $\mathcal{B}(\mathcal{R})$. We let θ be the identity map from $\mathcal{R}$ to $\mathcal{R}$ and let θ be defined by right cylindrification c mapping $\mathcal{B}$ to $\mathcal{B}(\mathcal{R})$. Trivially the identity map is an isomorphism from $\mathcal{R}$ to $\mathcal{R}$. For any $b \in \mathcal{B}$ we have $\theta(b) = c(b) = c(b);1_{\mathcal{R}}$, so θ does map $\mathcal{B}$ into $\mathcal{B}(\mathcal{R})$. By definition 6(1), c is a boolean homomorphism. Definition 6(3) states that $c(b) : 1_{\mathcal{B}} = b$ and this implies that c is injective. For surjectivity, take any $r;1_{\mathcal{R}} = r;c(1_{\mathcal{B}}) \in \mathcal{B}(\mathcal{R})$. By definition 6(4), $r;1_{\mathcal{R}} = r;c(1_{\mathcal{B}}) = c(r : 1_{\mathcal{B}}) = \theta(r : 1_{\mathcal{B}})$, so θ maps onto $\mathcal{B}(\mathcal{R})$. Hence θ is a boolean isomorphism from $\mathcal{B}$ to $\mathcal{B}(\mathcal{R})$.

Now we check that θ preserves the Peircean operator and the right cylindrifier. For any $r \in \mathcal{R}$ and $b \in \mathcal{B}$, we have $\theta(r : b) = c(r : b) = r;c(b)$ by definition 6(4), and this is equal to $\theta(r);\theta(b)$. Since the Peircean operator is defined to be relation algebra composition in $\mathcal{P}(\mathcal{R})$ we see that the Peircean operator is preserved by θ. Finally, for any $b \in \mathcal{B}$ we have $\theta(c(b)) = c(b) = \theta(b) = c(\theta(b))$, since right cylindrification is the identity map over the boolean part of $\mathcal{P}(\mathcal{R})$. $\Box$

COROLLARY 21 *Neither the class of classically representable Peirce algebras nor the class of Peirce algebras with non-additive representations can be defined by finitely many axioms.*

PROOF:

Suppose, for contradiction, that F is a finite set of axioms defining the class of classically representable Peirce algebras (or the class of Peirce algebras with non-additive representations). For each formula $\phi \in F$ define ϕ' by deleting all occurrences of the right cylindrifier, c, and replacing each occurrence of the Peircean operator $:$ by the composition operator $;$. Observe, from the definition of $\mathcal{P}(\mathcal{R})$, that for any relation algebra $\mathcal{R}$ we have $\mathcal{P}(\mathcal{R}) \models \phi \iff \mathcal{P}(\mathcal{R}) \models \phi' \iff \mathcal{R} \models \phi'$. Let $F' = \{\phi' \mid \phi \in F\}$, a finite set of relation algebra formulas. By lemma 17 we have $\mathcal{R}$ is classically representable iff $\mathcal{P}(\mathcal{R})$ is classically representable and $\mathcal{R}$ has a non-additive representation iff $\mathcal{P}(\mathcal{R})$ has a non-additive representation. Hence, for all relation algebras $\mathcal{R}$, we have $\mathcal{R} \models F'$ if and only if $\mathcal{R}$ has a classical representation (respectively, non-additive representation). But it is known that no finite set of axioms defines the class of relation algebras with classical representations [17] and no finite set of axioms defines the class of relation algebras with non-additive representations [9]. $\Box$

COROLLARY 22 *Every Peirce algebra has a non-multiplicative representation.*

PROOF:

Let $\mathcal{P} = (\mathcal{R}, \mathcal{B}, :, c)$ be a Peirce algebra. By theorem 20, $\mathcal{P} \cong \mathcal{P}(\mathcal{R})$. By [11, theorem 4.22], $\mathcal{R}$ has a non-multiplicative representation. By lemma 17, $\mathcal{P}(\mathcal{R})$ also has a non-multiplicative representation and, therefore, so does $\mathcal{P}$. $\square$

5 Universal algebra and representations of boolean modules

DEFINITION 23 *An algebra is called* simple *if any homomorphism from that algebra is either an isomorphism or its range is a degenerate algebra. An algebra is called* subdirectly irreducible *if whenever it is isomorphic to a subalgebra of a direct product of component algebras and the isomorphism followed by any of the projection maps is a homomorphism onto a component algebra, it is necessarily isomorphic to one of the component algebras.*

Clearly, a simple algebra is subdirectly irreducible. If there is a discriminator for a class of algebras, as for example there is with relation algebras, then the other inclusion also holds: the class of subdirectly irreducible algebras is identical to the class of simple algebras.

This is not the case with boolean modules. Any subdirectly irreducible boolean module $(\mathcal{R}, \mathcal{B}, :)$ where $\mathcal{R}$ and $\mathcal{B}$ are not degenerate will *not* be simple, because the map from $(\mathcal{R}, \mathcal{B}, :)$ to $(\mathcal{R}, \mathbf{0}, :)$ defined in definition 5 is a many-one homomorphism whose range is not degenerate. [In contrast, a Peirce algebra is simple iff it is subdirectly irreducible iff its relation algebra part is simple.]

So there are important differences between the general algebraic theory of boolean modules and that of relation algebras or Peirce algebras.

LEMMA 24 *Let $\lambda \in \{$classical, non-additive, non-multiplicative, pseudo$\}$ and let $\mathcal{M}_i$ be a boolean module for each $i \in I$. Then each $\mathcal{M}_i$ has a λ-representation iff the direct product $\Pi_{i \in I}\mathcal{M}_i$ has a λ-representation. Each $\mathcal{M}_i$ has a straight λ-representation iff the direct product $\Pi_{i \in I}\mathcal{M}_i$ has a straight λ-representation. The same results hold for Peirce algebras.*

PROOF:

Let $\mathcal{M}_i = (\mathcal{R}_i, \mathcal{B}_i, :)$ (each i). If $\mathcal{W}$ is a (straight) λ-representation of $\Pi_i \mathcal{M}_i$ then $\mathcal{W}$ induces a (straight) λ-representation $\mathcal{W}_i$ of $\mathcal{M}_i$ (for each i) defined by $r_i^{\mathcal{W}_i} = (0, 0, \ldots, 0, r_i, 0, \ldots)^{\mathcal{W}}$ and $b_i^{\mathcal{W}_i} = (0, \ldots, b_i, 0, \ldots)^{\mathcal{W}}$ for $r_i \in \mathcal{R}_i$ and $b_i \in \mathcal{B}_i$, where the sequences are all zeros except at the i'th coordinates.

Conversely, let $\mathcal{W}_i$ be a (straight) λ-representation of $\mathcal{M}_i$, for each i. Rename the domains of the $\mathcal{W}_i$ so that they are pairwise disjoint. Define a (straight) λ-representation $\sum_i \mathcal{W}_i$ of $\Pi_i \mathcal{M}_i$ by $(r_i \mid i \in I)^{\sum_i \mathcal{W}_i} = \bigcup_{i \in I} r_i^{\mathcal{W}_i}$ and $(b_i \mid i \in I)^{\sum_i \mathcal{W}_i} = \bigcup_{i \in I} b_i^{\mathcal{W}_i}$, where $r_i \in \mathcal{R}_i$ and $b_i \in \mathcal{B}_i$ for each $i \in I$. $\square$

LEMMA 25 *Let $\lambda \in \{$classical, non-additive, non-multiplicative$\}$ and let $\mathcal{R}$ be a relation algebra. $\mathcal{R}$ has a λ-representation if and only if the boolean module $(\mathcal{R}, \mathbf{0}, :)$ of definition 5 has a λ-representation.*

PROOF:

Trivially, any λ-representation of $(\mathcal{R}, \mathbf{0}, :)$ induces an λ-representation of $\mathcal{R}$. Conversely, if $\mathcal{W}$ is an λ-representation of $\mathcal{R}$ then define a λ-representation $\mathcal{W}'$ of $(\mathcal{R}, \mathbf{0}, :)$ by letting $r^{\mathcal{W}'} = r^{\mathcal{W}}$ and $0_{\mathcal{B}}^{\mathcal{W}'} = \emptyset$. $\Box$

LEMMA 26 *Let $\lambda \in \{$classical, non-additive, non-multiplicative, pseudo$\}$. The class of boolean modules with straight λ-representations is not closed under homomorphic images.*

PROOF:

Take any non-trivial boolean module $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ with a straight λ-representation, e.g. let $\mathcal{M} = \mathcal{M}(U)$, the full boolean module on a set U (definition 14). The two sorted map from $(\mathcal{R}, \mathcal{B}, :)$ to $(\mathcal{R}, \mathbf{0}, :)$ of definition 5 is a homomorphism. But any pseudo-representation $\mathcal{W}$ of $(\mathcal{R}, \mathbf{0}, :)$ cannot be straight since $1_{\mathcal{R}}^{\mathcal{W}} \neq \emptyset$ (since $\mathcal{R}$ is not trivial) yet $1_{\mathcal{B}}^{\mathcal{W}} = 0_{\mathcal{B}}^{\mathcal{W}} = \emptyset$ since the boolean algebra $\mathbf{0}$ is trivial. $\Box$

So, what is the variety generated by the boolean modules with straight λ-representations? To approach this question we need some techniques and results of universal algebra, but a slight complication arises because boolean modules are *two-sorted* algebras and the standard theorems of universal algebra (e.g. Birkhoff's theorem) apply to one-sorted algebras. This difficulty is only superficial though as the relevant theorems of universal algebra do generalise to many-sorted algebras (see, for example, [12]).

THEOREM 27 *Let $\lambda \in \{$pseudo, non-additive, non-multiplicative, classical$\}$.*

1. *The class of boolean modules with square λ-representations and the class of Peirce algebras with square λ-representations can be axiomatised by universal formulas.*

2. *The class of boolean modules with λ-representations and the class of Peirce algebras with λ-representations are quasi-varieties. The class of boolean modules/Peirce algebras with straight λ-representations is a quasi-variety.*

3. *The class of boolean modules with classical representations and the class of Peirce algebras with classical representations are equationally definable varieties.*

PROOF:

Trivially, all these classes are closed under subalgebras (and isomorphic copies). And each class can be shown to be *pseudo-elementary* — the class can be defined in a three-sorted language, where the first two sorts are for the relation algebra part and the boolean algebra part of a boolean module (or Peirce algebra) and the third sort is for points in a representation of the boolean module or Peirce algebra ([7, Section 9.3] shows how to write out such a definition, in more detail). It follows, by Łoś' theorem [8, theorem 9.5.1], that each class is closed under ultraproducts. Since each class is closed under subalgebras and ultraproducts, each class is universally axiomatisable [3, Theorem 2.20].

If we have a sequence of boolean modules (or Peirce algebras) with λ-representations then a disjoint union of these component representations forms a λ-representation of the direct product of the sequence, by lemma 24. If the component representations are straight then so is the disjoint union. Hence the class of boolean modules (Peirce algebras) with λ-representations and the class of boolean modules (Peirce algebras) with straight λ-representations are closed under direct products. By a theorem of Mal'cev [16, theorem 3], all these classes are quasi-varieties.

To show that the class of classically representable boolean modules (Peirce algebras) forms a variety, it remains to show that the class is closed under homomorphic images. The proof is based on the proof of [7, theorem 3.37], but we take account of the boolean sort too.

Let $\mathcal{M} = (\mathcal{R}, \mathcal{A}, :)$, $\mathcal{N} = (\mathcal{S}, \mathcal{B}, :)$ be boolean modules, let $\mathcal{W}$ be a classical representation of $\mathcal{M}$ and let $h : \mathcal{M} \to \mathcal{N}$ be any homomorphism. We can assume, by lemma 10, that $\mathcal{W}$ is an $||\mathcal{M}||$-saturated representation of $\mathcal{M}$ so by lemma 11, for any ultrafilter γ of $\mathcal{R}$, we have $\bigcap_{r\in\gamma} r^{\mathcal{W}} \neq \emptyset$. Define

$$\begin{aligned}
ker_B(h) &= \{a \in \mathcal{A} \mid h(a) = 0_{\mathcal{B}}\} \\
ker_R(h) &= \{r \in \mathcal{R} \mid h(r) = 0_{\mathcal{R}}\} \\
K_B &= \bigcup_{a \in ker_B(h)} a^{\mathcal{W}} \\
K_R &= \{w \in \mathcal{W} \mid (w, w) \in r^{\mathcal{W}} \text{ for some } r \in ker_R(h)\} \\
E_B &= 1_{\mathcal{B}}^{\mathcal{W}} \setminus K_B \\
E_R &= \mathcal{W} \setminus K_R
\end{aligned}$$

Claim 1: $E_B \subseteq E_R$. In order to prove this claim, we must show that $1_{\mathcal{B}}^{\mathcal{W}} \cap K_R \subseteq K_B$. For any $w \in 1_{\mathcal{B}}^{\mathcal{W}} \cap K_R$ there is $r \in \mathcal{R}$ with $h(r) = 0_{\mathcal{R}}$ and $(w, w) \in r^{\mathcal{W}}$. So $w \in (r : 1_{\mathcal{B}})^{\mathcal{W}}$ and $h(r : 1_{\mathcal{B}}) = h(r) : h(1_{\mathcal{B}}) = 0_{\mathcal{B}}$, hence $w \in K_B$. This proves the claim.

Claim 2: if $v \in E_B$ and $(v, w) \in 1_{\mathcal{R}}^{\mathcal{W}}$ then $w \in E_B$. For claim 2, $(v, w) \in 1_{\mathcal{R}}^{\mathcal{W}}$ implies $(w, v) \in 1_{\mathcal{R}}^{\mathcal{W}}$ and given $v \in E_B \subseteq 1_{\mathcal{B}}^{\mathcal{W}}$ we see that $w \in (1_{\mathcal{R}} : 1_{\mathcal{B}})^{\mathcal{W}} = 1_{\mathcal{B}}^{\mathcal{W}}$. If $w \in K_B$ there would be $a \in \mathcal{A}$ with $h(a) = 0$ and $w \in a^{\mathcal{W}}$, but then $v \in (1_{\mathcal{R}} : a)^{\mathcal{W}}$ and $h(1_{\mathcal{R}} : a) = 0$ so $v \in K_B$, contradicting our assumption $v \in E_B$. Hence $w \in 1_{\mathcal{B}}^{\mathcal{W}}$ and $w \notin K_B$ so $w \in E_B$. This proves claim 2.

Define an $\mathcal{N}$-structure $\mathcal{V}$ with domain E_R by $h(r)^{\mathcal{V}} = r^{\mathcal{W}} \cap (E_R \times E_R)$ and $h(a)^{\mathcal{V}} = a^{\mathcal{W}} \cap E_B$, for $r \in \mathcal{R}$ and $a \in \mathcal{A}$. By [7, Theorem 3.37], the relation algebra part of this definition is well-defined and is a representation of $\mathcal{R}$. The definition we have here of the relation algebra part of the structure $\mathcal{V}$ is exactly the same as the structure defined in [7, theorem 3.37]. Saturation of $\mathcal{W}$ is used in the proof. The proof that $\mathcal{V}$ is well-defined on the boolean algebra part and a representation of it is similar, but rather easier, and we omit it as well.

It remains to check that the Peircean operator is preserved by $\mathcal{V}$. Let $r \in \mathcal{R}$ and $a \in \mathcal{A}$. We have $v \in (h(r) : h(a))^{\mathcal{V}} = h(r : a)^{\mathcal{V}}$

$$\begin{array}{lll}
\iff & v \in E_B \wedge v \in (r : a)^{\mathcal{W}} & \text{def. of } \mathcal{V} \\
\iff & v \in E_B \wedge \exists w\ ((v, w) \in r^{\mathcal{W}} \wedge w \in a^{\mathcal{W}}) & \mathcal{W} \text{ is a rep.} \\
\iff & v \in E_B \wedge \exists w\ ((v, w) \in r^{\mathcal{W}} \wedge w \in a^{\mathcal{W}} \cap E_B) & \text{claim 2} \\
\iff & v \in E_B \wedge \exists w\ ((v, w) \in (r^{\mathcal{W}} \cap (E_R \times E_R)) \wedge w \in h(a)^{\mathcal{V}}) & \text{claim 1} \\
\iff & \exists w\ ((v, w) \in h(r)^{\mathcal{V}} \wedge w \in h(a)^{\mathcal{V}}) & \text{def. of } \mathcal{V}
\end{array}$$

as required. Thus $\mathcal{V}$ is a classical representation of $\mathcal{N}$.

So the class of classically representable boolean modules is closed under taking subalgebras, direct products and homomorphic images. By [12], this class is an equationally definable variety of two sorted algebras.

A similar proof shows that the classically representable Peirce algebra also forms a variety, but this is not necessary. By lemma 17, an equational axiomatisation of the representable relation algebras together with the equations defining Peirce algebras (definitions 3 and 6) defines the class of classically representable Peirce algebras. □

PROBLEM 28 *Which of the following classes are equational varieties: pseudo-representable boolean modules, non-additively representable boolean modules, non-multiplicatively representable boolean modules? Repeat the question but restrict to straight representations.*

LEMMA 29

1. *The closure of the class of full boolean modules under the taking of subalgebras is the class of boolean modules with square classical representations.*
2. *The variety generated by the full boolean modules is the class of classically representable boolean modules.*

PROOF:

For (i) it is clear that a subalgebra of a boolean module with a square classical representation also has a square classical representation. And if U is the domain of a square classical representation of $\mathcal{M}$ then $\mathcal{M}$ is a subalgebra of the full boolean module $\mathcal{M}(U)$ (definition 14).

For (ii), by theorem 27, the variety generated by the full boolean modules is contained in the variety of classically representable boolean modules. Conversely, let $\mathcal{W}$ be a classical representation of the boolean module $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$. ${1_{\mathcal{R}}}^{\mathcal{W}}$ is an equivalence relation over the domain of $\mathcal{W}$. For each equivalence class U either $U \subseteq {1_{\mathcal{B}}}^{\mathcal{W}}$ or $U \cap {1_{\mathcal{B}}}^{\mathcal{W}} = \emptyset$. In the former case let $\mathcal{M}_U = \mathcal{M}(U)$, the full boolean module over U, and in the latter case let $\mathcal{M}_U = (\Re(U), \mathbf{0}, :)$, where $\Re(U)$ is the full relation algebra over U. We claim that $\mathcal{M}$ is a subalgebra of $\Pi_U \mathcal{M}_U$, where the product is taken over all equivalence classes U. For this claim, let h be a two sorted map from $\mathcal{M}$ into $\Pi_U \mathcal{M}_U$ defined by

$$\begin{array}{rcl} h(r) & = & (r^{\mathcal{W}} \cap (U \times U) \mid \text{equiv. classes } U) \\ h(b) & = & (b^{\mathcal{W}} \cap U \mid \text{equiv. classes } U) \end{array}$$

for $r \in \mathcal{R}$ and $b \in \mathcal{B}$. Now check that h is an injective homomorphism, so $\mathcal{M}$ is a subalgebra of $\Pi_U \mathcal{M}_U$. Since each $\mathcal{M}_U$ is a homomorphic image of a full boolean module we see that $\mathcal{M}$ is in the variety generated by the full boolean modules. $\Box$

6 Every Boolean Module has a Non-Multiplicative Representation

The remainder of this paper concerns axiomatisations of the representation classes for boolean modules (see definition 12). Brink gives a sufficient condition (which he calls 'bijectivity') for non-multiplicative representability.

THEOREM 30 ([1, Theorem 5.6]) *If a boolean module $(\mathcal{R}, \mathcal{B}, :)$ satisfies*

$$[\forall a \in \mathcal{B}\ (r : a = s : a)] \to (r = s)$$

for all $r, s \in \mathcal{R}$, then it has a non-multiplicative representation.

After proving this theorem Brink asks whether all boolean module possess non-multiplicative representations. It turns out they do. The trick here is to do with the fact, already observed, that the domain of the boolean unit might be strictly contained in the domain of the relation algebra unit. This allows us to divide a non-multiplicative representation of a boolean module into two parts — one part will be a non-multiplicative representation of the relation algebra part and will be outside the domain of the boolean unit; the other part will be a Stone representation of the boolean part with a homomorphic image of the relation algebra part grafted on.

LEMMA 31 *Let $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ be a boolean module. Let $\mathcal{S}$ be an $||\mathcal{M}||$-saturated boolean representation of $\mathcal{B}$ such that $1^{\mathcal{S}}_{\mathcal{B}} = \mathcal{S}$ and if $\forall b \in \mathcal{B}\ (x \in b^{\mathcal{S}} \leftrightarrow y \in b^{\mathcal{S}})$ then $x = y$, for any $x, y \in \mathcal{S}$ ($\mathcal{S}$ exists, by lemma 10).*

Define an $\mathcal{M}$-structure $\mathcal{T}$, with the same domain as $\mathcal{S}$, by letting $b^{\mathcal{T}} = b^{\mathcal{S}}$ for $b \in \mathcal{B}$, and

$$r^{\mathcal{T}} = \{(x, y) \in \mathcal{S} \times \mathcal{S} \mid \forall b \in \mathcal{B}\ (y \in b^{\mathcal{S}} \to x \in (r : b)^{\mathcal{S}})\}$$

for $r \in \mathcal{R}$. $\mathcal{T}$ respects all the operators of $\mathcal{M}$ except perhaps the relation algebra $\cdot$ and $-$. $\mathcal{T}$ is faithful over the boolean part $\mathcal{B}$.

PROOF:

It is clear that all the operators for $\mathcal{B}$ are respected by $\mathcal{T}$ and that $\mathcal{T}$ is faithful over $\mathcal{B}$, since $\mathcal{S}$ is a boolean representation of $\mathcal{B}$. We check the relation algebra operators one by one.

Well, $(x, y) \in 0^{\mathcal{T}}_{\mathcal{R}}$ iff $\forall b \in \mathcal{B}\ (y \in b^{\mathcal{S}} \to x \in (0_{\mathcal{R}} : b)^{\mathcal{S}} = 0^{\mathcal{S}}_{\mathcal{B}})$. Now $y \in 1^{\mathcal{S}}_{\mathcal{B}}$ but $x \notin 0^{\mathcal{S}}_{\mathcal{B}}$, so $(x, y) \notin 0^{\mathcal{S}}_{\mathcal{R}}$. Hence $0^{\mathcal{T}}_{\mathcal{R}} = \emptyset$.

If $(x,y)\in(r+_{\mathcal{R}}s)^{\mathcal{T}}$ then $\forall b\in\mathcal{B}\ (y\in b^{\mathcal{S}}\to x\in((r+_{\mathcal{R}}s):b)^{\mathcal{S}}=(r:b)^{\mathcal{S}}\cup(s:b)^{\mathcal{S}})$. We claim that either $\forall b\in\mathcal{B}\ (y\in b^{\mathcal{S}}\to x\in(r:b)^{\mathcal{S}})$ or $\forall b\in\mathcal{B}\ (y\in b^{\mathcal{S}}\to x\in(s:b)^{\mathcal{S}})$. If not, there would be $b_1,b_2\in\mathcal{B}$ such that $y\in b_1^{\mathcal{S}},\ y\in b_2^{\mathcal{S}}, x\notin(r:b_1)^{\mathcal{S}}$ and $x\notin(s:b_2)^{\mathcal{S}}$. But then, let $b=b_1\cdot b_2\in\mathcal{B}$. We have $y\in b^{\mathcal{S}}$ and $x\notin(r:b)^{\mathcal{S}}\cup(s:b)^{\mathcal{S}}$, contradicting the first sentence of this paragraph. Hence, by the claim, either $(x,y)\in r^{\mathcal{T}}$ or $(x,y)\in s^{\mathcal{T}}$. Conversely, if $(x,y)\in r^{\mathcal{T}}$ or $(x,y)\in s^{\mathcal{T}}$ it is clear that $(x,y)\in(r+_{\mathcal{R}}s)^{\mathcal{T}}$, by monotonicity of :.

If $(x,y)\in 1'^{\mathcal{T}}$ then $\forall b\in\mathcal{B}\ (y\in b^{\mathcal{S}}\to x\in(1':b)^{\mathcal{S}}=b^{\mathcal{S}})$. By the assumed properties of the boolean representation $\mathcal{S}$, this implies that $x=y$. Conversely $(x,x)\in 1'^{\mathcal{T}}$, so the identity is respected by $\mathcal{T}$.

Suppose $(x,y)\in r^{\mathcal{T}}$. Let $a\in\mathcal{B}$ be arbitrary subject to $x\in a^{\mathcal{S}}$. For any b with $y\in b^{\mathcal{S}}$ we have $x\in(r:b)^{\mathcal{T}}$ so $(r:b)\cdot a\neq 0$. By lemma 4(2) this implies $(r^{\smile}:a)\cdot b\neq 0$. Since this holds for all b with $y\in b^{\mathcal{S}}$ it follows that $y\in(r^{\smile}:a)^{\mathcal{T}}$ (use the fact that for any $c\in\mathcal{C}$ we have either $y\in c^{\mathcal{S}}$ or $y\in(-c)^{\mathcal{S}}$). Hence $(y,x)\in(r^{\smile})^{\mathcal{T}}$. So converse is respected.

If $(x,y)\in r^{\mathcal{T}}$ and $(y,z)\in s^{\mathcal{T}}$ then $\forall b\in\mathcal{B}\ (y\in b^{\mathcal{S}}\to x\in(r:b)^{\mathcal{S}}$ and $\forall c\in\mathcal{B}\ (z\in c^{\mathcal{S}}\to y\in(s:c)^{\mathcal{S}})$. So $\forall c\in\mathcal{B}\ (z\in c^{\mathcal{S}}\to y\in(s:c)^{\mathcal{S}}\to x\in(r:(s:c))^{\mathcal{S}}=((r;s):c)^{\mathcal{S}})$, so $(x,z)\in(r;s)^{\mathcal{T}}$.

Conversely, let $(x,z)\in(r;s)^{\mathcal{T}}$. We need saturation of $\mathcal{S}$ here. For any $a,c\in\mathcal{B}$ if $x\in a^{\mathcal{S}}\wedge z\in c^{\mathcal{S}}$ then $x\in a^{\mathcal{S}}\cap((r;s):c)^{\mathcal{S}}$ so $a\cdot(r:(s:c))\neq 0$. By lemma 4(2), $(r^{\smile}:a)\cdot(s:c)\neq 0$. The set of elements $\{r^{\smile}:a\mid x\in a^{\mathcal{X}}\}\cup\{s:c\mid z\in c^{\mathcal{S}}\}$ has the finite intersection property so by saturation (lemma 11), there is $y\in\mathcal{S}$ such that for all $a,c\in\mathcal{B}$ with $x\in a^{\mathcal{S}}$ and $z\in c^{\mathcal{S}}$ we have $y\in((r^{\smile}:a)\cdot(s:c))^{\mathcal{S}}$. Then $(y,z)\in s^{\mathcal{T}}$ and $(y,x)\in r^{\smile\mathcal{T}}$ and since converse is represented correctly, $(x,y)\in r^{\mathcal{T}}$. So ; is respected.

Finally we check the Peircean operator. Suppose $x,y\in\mathcal{S},\ y\in b^{\mathcal{S}}$ and $(x,y)\in r^{\mathcal{T}}$. We must show that $x\in(r:b)^{\mathcal{S}}$, but this is true by the given definition of $r^{\mathcal{T}}$. Conversely, suppose $x\in(r:b)^{\mathcal{S}}$. We must show that there is $y\in b^{\mathcal{S}}$ with $(x,y)\in r^{\mathcal{T}}$. By saturation, there is $y\in\mathcal{S}$ with $y\in\bigcap\{(r^{\smile}:a)^{\mathcal{S}}\mid a^{\mathcal{S}}\ni x\}$. For any $a,b'\in\mathcal{B}$ with $y\in b'^{\mathcal{S}}$ and $x\in a^{\mathcal{S}}$ we know that $y\in(r^{\smile}:a)^{\mathcal{S}}\cap b'^{\mathcal{S}}$ and since the boolean representation $\mathcal{S}$ represents all the boolean operators correctly, $y\in(r^{\smile}:a\cdot b')^{\mathcal{S}}$. Now, $(r^{\smile}:a)\cdot b'\neq 0\Rightarrow a\cdot(r:b')\neq 0$. Since this holds for all a with $x\in a^{\mathcal{S}}$ it follows that $x\in(r:b')^{\mathcal{S}}$, so $(x,y)\in r^{\mathcal{T}}$. Thus the Peircean operator : is respected by $\mathcal{T}$.

$\Box$

Note that $\mathcal{T}$ can fail to preserve the relation algebra operator $\cdot_{\mathcal{R}}$ correctly, since $(r\cdot_{\mathcal{R}}s):b$ can be strictly less than $(r:b)\cdot_{\mathcal{B}}(s:b)$, for $b\in\mathcal{B},\ r,s\in\mathcal{R}$.

THEOREM 32 *Every boolean module has a non-multiplicative representation.*

PROOF:

Let $\mathcal{M}=(\mathcal{R},\mathcal{B},:)$ be a boolean module. By [11, theorem 4.22] $\mathcal{R}$ has a non-multiplicative relation algebra representation. By lemma 25, $(\mathcal{R},\mathbf{0},:)$ has a non-multiplicative representation, $\mathcal{H}$ say. There is a homomorphism h from $\mathcal{M}$ to $(\mathcal{R}, \mathbf{0}, :)$ which acts as the identity on $\mathcal{R}$ (see definition 5), so the $\mathcal{M}$-structure $\mathcal{F}$ defined by $\mathcal{H} \circ h$, or more explicitly $b^{\mathcal{F}} = \emptyset$, $r^{\mathcal{F}} = (h(r))^{\mathcal{H}}$ for $b \in \mathcal{B}$, $r \in \mathcal{R}$, preserves all the boolean module operators. Although $\mathcal{F}$ is not faithful over the boolean part, it is faithful over the relation algebra part $\mathcal{R}$, since h acts as the identity on $\mathcal{R}$.

By lemma 31, there is an $\mathcal{M}$-structure $\mathcal{T}$ preserving all the operators except perhaps the relation algebra $+$ and $-$ and $\mathcal{T}$ interprets $\mathcal{B}$ faithfully.

We can assume that the domain of $\mathcal{F}$ is disjoint from the domain of $\mathcal{T}$. As in lemma 24, we define an $\mathcal{M}$-structure $\mathcal{F}+\mathcal{T}$ by

$$\begin{aligned} b^{\mathcal{F}+\mathcal{T}} &= b^{\mathcal{T}} \\ r^{\mathcal{F}+\mathcal{T}} &= r^{\mathcal{F}} \cup r^{\mathcal{T}} \end{aligned}$$

for $b \in \mathcal{B}$, $r \in \mathcal{R}$. Since both $\mathcal{F}$ and $\mathcal{T}$ preserve all the operators except perhaps the relation algebra $+$ and $-$, so does $\mathcal{F}+\mathcal{T}$. Since $\mathcal{F}$ is faithful over the relation algebra part and $\mathcal{T}$ is faithful over the boolean algebra part it follows that $\mathcal{F}+\mathcal{T}$ is faithful over all of $\mathcal{M}$. Thus $\mathcal{F}+\mathcal{T}$ is a non-multiplicative representation of $\mathcal{M}$. $\Box$

You might feel that we have cheated a bit here, by separating two disjoint representations, one essentially for the boolean part and one for the relation algebra part. We consider straight representations, where the domains of the two parts of a boolean module must coincide, later.

7 Unrepresentable boolean modules

DEFINITION 33 *Let $3 \leq n < \omega$. The* Lyndon Algebra *$\mathcal{L}_n$ [13] has $n+2$ atoms (minimal, non-zero elements) $A = \{1', a_0, \ldots, a_n\}$ and therefore 2^{n+2} elements altogether. Since all operators are completely additive in a relation algebra it suffices to define the operators only in the case where they are applied to atoms. We specify the operators as they act on the atoms of $\mathcal{L}_n$ as follows. The identity $1'$ is itself an atom. All atoms are self-converse (hence all elements of $\mathcal{L}_n$ are self-converse). For composition,*

1. $1'; x = x; 1' = x$ *(any atom x),*
2. $a_i; a_i = 1' + a_i$,
3. $a_i; a_j = \sum_{k \leq n,\, k \neq i,j} a_k = -(1' + a_i + a_j)$

where $i \neq j \leq n$.

Lyndon showed that if there is a projective geometry $\mathcal{P}$ of finite order n then we can obtain a classical representation of $\mathcal{L}_n$ by arbitrarily enumerating the $n+1$ points of some line l (the line at infinity) as $(l_0, \ldots, l_n)$, deleting l from $\mathcal{P}$ to give an affine geometry $\mathcal{A}$, and letting $a_i^{\mathcal{A}} = \{(x, y) \in \mathcal{A} \times \mathcal{A} \mid (x, y, l_i)$ are distinct and collinear$\}$ and $1'^{\mathcal{A}} = \{(x, x) \mid x \in \mathcal{A}\}$.

THEOREM 34 (Lyndon, [13, theorem 1]) *Let $n \geq 3$ be finite. The Lyndon algebra $\mathcal{L}_n$ is classically representable iff there is a projective plane of order n. Every square, classical representation of $\mathcal{L}_n$ is isomorphic to the representation based on the affine geometry of order n defined above. If $\mathcal{W}$ is any classical representation of $\mathcal{L}_n$ then $||\{y \mid (x,y) \in a_i^{\mathcal{W}}\}|| = n-1$, for any $x \in \mathcal{W}$ and any $i \leq n$.*

The next lemma is not needed here but will be useful later.

LEMMA 35 *For $n \geq 3$ the Lyndon algebra $\mathcal{L}_n$ has a non-additive representation.*

PROOF:

For each i with $2 \leq i \leq n$, consider the partial map g_i from $\mathcal{L}_n$ to $\mathcal{L}_3$ defined by $g_i(1') = 1'$, $g_i(a_0) = a_0$, $g_i(a_1) = a_1$, $g_i(a_i) = a_2$, $g_i(-(1' + a_0 + a_1 + a_i)) = a_3$. Extend g_i to a total map $h_i : \mathcal{L}_n \to \mathcal{L}_3$ by letting

$$h_i(\alpha) = \sum\{g_i(x) \mid x \in \{1', a_0, a_1, a_i, -(1' + a_0 + a_1 + a_i)\},\ x \leq \alpha\}$$

for all $\alpha \in \mathcal{L}_n$. h_i preserves all operators except perhaps negation and sum, and $h_i(1'), h_i(a_0), h_i(a_1), h_i(a_i) \neq 0$. Let $\mathcal{W}$ be a classical representation of $\mathcal{L}_3$ ($\mathcal{W}$ exists since there is a projective plane of order 3). For each i with $2 \leq i \leq n$ there is an $\mathcal{L}_n$ structure $\mathcal{V}_i$ with domain $\mathcal{W} \times \{i\}$ defined by $\alpha^{\mathcal{V}_i} = (h_i(\alpha)^{\mathcal{W}}, i)$. Note that the domains of the $\mathcal{V}_i$s are disjoint. Now define the structure $\mathcal{V}$ whose domain is the union of the domains of the $\mathcal{V}_i$s, for $i \leq n$, by $\alpha^{\mathcal{V}} = \bigcup_{2 \leq i \leq n} \alpha^{\mathcal{V}_i}$. $\mathcal{V}$ is a faithful $\mathcal{L}_n$ structure, since $a_i^{\mathcal{V}} \neq \emptyset$ for $2 \leq i \leq n$ and $1'^{\mathcal{V}_i}, a_0^{\mathcal{V}_i}, a_1^{\mathcal{V}_i} \neq \emptyset$ (all i), and it is a non-additive representation since $\mathcal{V}$ respects all the operators except perhaps negation and sum.

□

There are infinitely many values of n for which there is a projective plane of that order (e.g. any prime) and infinitely many for which there is not. By taking a non-principal ultraproduct of the $\mathcal{L}_n$ where n ranged over values for which there is no projective plane of that order, Monk showed that although the component algebras were not representable, the ultraproduct was a representable relation algebra. He deduced that **RRA** could not be finitely axiomatised [17]. This construction was the model for various non-finite axiomatisability results that followed, including the current one.

In our case it is the representable $\mathcal{L}_n$'s that are useful. Any infinite set of these will do, so we now restrict to the case where n is an odd prime. We define boolean modules $\mathcal{M}_p$, for all odd primes p. The relation algebra part is the Lyndon algebra $\mathcal{L}_p$ and the boolean algebra part $\mathcal{B}_{p+1}$ has $p+1$ atoms $\{b_0, \ldots, b_p\}$. As before, the Peircean operator is completely additive (lemma 4(1)), so we need only specify it when applied to a relation algebra atom and a boolean algebra atom. For this,

$$\begin{aligned} 1' : b_k &= b_k \\ a_i : b_k &= 1_{\mathcal{B}} \end{aligned}$$

where $i, k \leq p$ and $1_{\mathcal{B}}$ is the unit of the boolean part of $\mathcal{M}_p$. It is not hard to check, using the definition of composition for $\mathcal{L}_p$, that $\mathcal{M}_p$ satisfies all the axioms for boolean modules.

LEMMA 36 *$\mathcal{M}_p$ has no classical representations and no straight non-multiplicative representations.*

PROOF:

Suppose, for contradiction, that $\mathcal{W}$ is a classical representation of $\mathcal{M}_p$. Pick $x_0 \in b_0^{\mathcal{W}}$. For each $k \leq p$, since $a_0 : b_k \geq b_0$, there is $x_k \in b_k^{\mathcal{W}}$ with $(x_0, x_k) \in a_0^{\mathcal{W}}$. Since $\mathcal{W}$ is a representation of the boolean part of $\mathcal{M}_p$, all the x_k's must be distinct. So $||\{y \mid (x_0, y) \in a_0^{\mathcal{W}}\}|| \geq p$, contradicting theorem 34.

Suppose, for contradiction, that $\mathcal{W}$ is a straight non-multiplicative representation of $\mathcal{M}_p$. Since the algebra is finite and since $+_{\mathcal{R}}$ is correctly represented, for any points x, y with $(x, y) \in 1_{\mathcal{R}}^{\mathcal{W}}$ we have $(x, y) \in \alpha^{\mathcal{W}}$ for some atom α of the relation algebra part. But it is conceivable and possible in a non-multiplicative representation $\mathcal{W}$ that (x, y) belongs to the interpretation of two distinct atoms, say $(x, y) \in \alpha^{\mathcal{W}} \cap \beta^{\mathcal{W}}$ for some atoms $\alpha \neq \beta$. If this happens, it follows that $(x, x) \in (\alpha; \beta^{\smile})^{\mathcal{W}} \subseteq 0'^{\mathcal{W}}$, where $0' =_{def} 1_{\mathcal{R}} - 1'$, hence $(x, x) \in 1'^{\mathcal{W}} \cap 0'^{\mathcal{W}}$. If the representation respected the meet operator $\cdot_{\mathcal{R}}$ this would lead to a contradiction because we could conclude that $(x, x) \in (1' \cdot 0')^{\mathcal{W}} = 0^{\mathcal{W}} = \emptyset$, but the meet operator need not be respected in a non-multiplicative representation. So $(x, y) \in \alpha^{\mathcal{W}} \cap \beta^{\mathcal{W}}$ is possible, in general.

However there is $y \in \mathcal{W}$ with $(y, y) \in 1'^{\mathcal{W}} \setminus 0'^{\mathcal{W}}$, since $0' \not\geq 1'$. Now for distinct atoms α, β we have $\alpha; \beta \leq 0'$ (using the fact that all atoms are self-converse), so $(y, y) \notin (\alpha; \beta)^{\mathcal{W}}$, hence

$$\forall x \in \mathcal{W}\ \exists \text{ unique atom } \alpha \text{ such that } (x, y) \in \alpha^{\mathcal{W}} \tag{2}$$

As $a_1; a_1 \geq 1'$ there is z_1 with $(y, z_1) \in a_1^{\mathcal{W}}$. See figure 1. For each i with $1 < i \leq p$, since $a_i; a_0 \geq a_1$, there is z_i with $(y, z_i) \in a_i^{\mathcal{W}}$ and $(z_i, z_1) \in a_0^{\mathcal{W}}$.

For each $x \in \mathcal{W}$ we know that $(x, x) \in 1'^{\mathcal{W}} \subseteq 1^{\mathcal{W}}$. By lemma 15, since $\mathcal{W}$ is straight, we deduce that $x \in 1_{\mathcal{B}}^{\mathcal{W}}$. Since $\mathcal{W}$ does respect $+_{\mathcal{B}}$, it follows that there is a boolean atom b_k with $x \in b_k^{\mathcal{W}}$. For $1 \leq i \leq p$ let $k_i \leq p$ such that $z_i \in b_{k_i}^{\mathcal{W}}$. Pick $k \leq p$ with $k \notin \{k_i \mid 1 \leq i \leq p\}$ (there are at most p values of k_i but $p+1$ possible choices for k). Since $b_{k_1} \leq a_0 : b_k$ there is $z \in b_k^{\mathcal{W}}$ with $(z_1, z) \in a_0^{\mathcal{W}}$. For $1 < i \leq p$, by considering the triangle (z, z_1, z_i) we have $(z, z_i) \in \beta^{\mathcal{W}}$ for some atom $\beta \leq a_0; a_0 = 1' + a_0$ and since $k \neq b_{k_i}$, $\beta = 1'$ is impossible (although $\mathcal{W}$ need not respect $\cdot_{\mathcal{R}}$ it does respect $\cdot_{\mathcal{B}}$), hence $\beta = a_0$. By (2), there is a unique atom γ such that $(z, y) \in \gamma^{\mathcal{W}}$. From the triangle (z, z_i, y) we see that $\gamma = 1'$, $\gamma = a_0$ or $\gamma = a_i$ are impossible, for $1 \leq i \leq p$. There aren't any other possible atoms for γ so this is a contradiction. $\square$

Let $\mathcal{M}$ be a non-principal ultrafilter of the $\mathcal{M}_p$s, for prime $p \geq 3$.

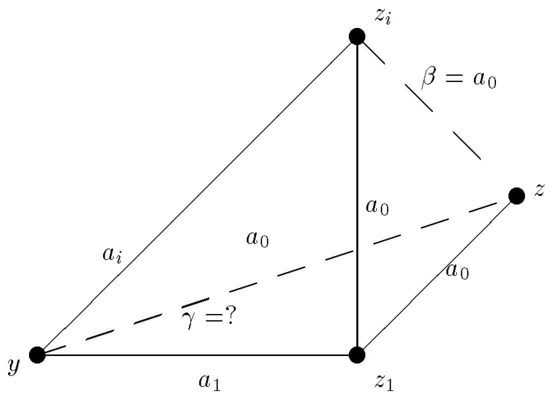

Figure 1:

LEMMA 37 *$\mathcal{M}$ has a classical, square representation.*

PROOF:

Let $\mathcal{N} = (\mathcal{R}, \mathcal{B}, :)$ be a countable elementary subalgebra of $\mathcal{M} = \Pi_D \mathcal{M}_p$ (where D is a non-principal ultrafilter over ω). By theorem 27 the class of boolean modules with classical, square representations is universally axiomatisable, certainly elementary, so if we can show that $\mathcal{N}$ has a classical, square representation then so does $\mathcal{M}$. By Łoś' theorem, every first-order property true in all the $\mathcal{M}_n$s is true in $\mathcal{M}$ and hence also in $\mathcal{N}$. So $\mathcal{N}$ is atomic, all its relation algebra atoms are self-converse, its identity $1'$ is an atom, for distinct relation algebra atoms $r, s \neq 1'$ we have $r;s = -(1' + r + s)$ and $r;r = 1' + r$, and for any boolean atom b we have $1' : b = b$, $r : b = 1_{\mathcal{B}}$. Thus the atoms of $\mathcal{R}$ behave like the atoms of the $\mathcal{L}_n$s, but $\mathcal{R}$ has countably many atoms. Enumerate the atoms of $\mathcal{R}$ as $1', a_0, a_1, \ldots$.

To find a classical, square representation of $\mathcal{N}$, let $\mathcal{P}$ be a countable projective plane of order 0, let l be any line of $\mathcal{P}$ (the line at infinity) and let $\mathcal{A}$ be the affine plane obtained from $\mathcal{P}$ by deleting l. Arbitrarily enumerate the points of l as $l_0, l_1, \ldots$ (l is countably infinite). Make $\mathcal{P}$ into a representation of $\mathcal{R}$ by letting

$$\begin{aligned} 1'^{\mathcal{P}} &= \{(x,x) \mid x \in \mathcal{A}\} \\ a_i^{\mathcal{P}} &= \{(x,y) \in \mathcal{A} \times \mathcal{A} \mid (x,y,l_i) \text{ are collinear and distinct}\} \end{aligned}$$

(cf. [13, theorem 1]). Extend this definition to arbitrary elements of $\mathcal{R}$ by letting

$$\alpha^{\mathcal{P}} = \bigcup_{\text{atoms } a \leq \alpha} a^{\mathcal{P}}$$

It remains to represent the boolean part in $\mathcal{P}$. For each $x \in \mathcal{A}$ we will find $b \in \mathsf{At}(\mathcal{B})$ and let $x \in b^{\mathcal{P}}$. We do this step by step. Initially no point of $\mathcal{A}$ is labelled by a boolean atom, but at each step one extra point is given a label. The number of steps is ω. At each stage, only finitely many points will have been labelled. Alternately, at each step, we carry out one or other of the following two tasks.

- Pick the first point (in some fixed enumeration of the points of $\mathcal{A}$) $x \in \mathcal{A}$ not previously chosen, pick an arbitrary atom $b \in \mathsf{At}(\mathcal{B})$ and extend the labelling so that $x \in b^{\mathcal{P}}$. This ensures that every point in $\mathcal{A}$ is eventually assigned some atom of $\mathcal{B}$.
- Pick the first line s and atom $b \in \mathsf{At}(\mathcal{B})$ (in some enumeration of the set of line-atom pairs) such that b does not occur as the label of any point on s. This is possible since $||\mathsf{At}(\mathcal{B})||$ is infinite but only finitely many atoms have been used so far. This ensures that every atom occurs eventually as the label of a point on every line.

Eventually, (i) every point is labelled by a unique atom and (ii) for every line and every atom there is a point on the line labelled by that atom. Now let $b^{\mathcal{P}} =_{def} \{p \in \mathcal{P} \mid \exists \beta \in \mathsf{At}(\mathcal{B})\ (\beta \leq b \wedge p \text{ is labelled by } \beta)\}$. This defines $\mathcal{P}$. Condition (i) is enough to make $\mathcal{P}$ into a boolean representation of $\mathcal{B}$ and condition (ii) ensures that the Peircean operator is correctly represented. Thus $\mathcal{P}$ is a classical, square representation of $\mathcal{N}$.
$\square$

THEOREM 38 *Let Γ be any set of sentences in the signature of boolean modules and let $Mod(\Gamma)$ be the class of all boolean module like structures $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$. If $Mod(\Gamma)$ is either*

1. *the class of all classically representable boolean modules,*
2. *the class of all boolean modules with square classical representations,*
3. *the class of all boolean modules with straight classical representations,*
4. *the class of all boolean modules with square non-multiplicative representations, or*
5. *the class of all boolean modules with straight non-multiplicative representations,*

then Γ contains infinitely many sentences not derivable from an axiomatisation of the representable relation algebras.

PROOF:

Suppose for contradiction that Γ contains only finitely many sentences not derivable from an axiomatisation of the representable relation algebras. Without loss $\Gamma \subseteq Th(\mathbf{RRA}) \cup \{\gamma\}$ for a single sentence γ. Then, since the ultraproduct $\mathcal{M}$ in the previous lemma has a square classical representation, $\mathcal{M} \models \Gamma$ for each of the alternatives. By Łoś' theorem we have $\mathcal{M}_p \models \Gamma$ for many values of p. But, since the relation algebra part of $\mathcal{M}_p$ is representable, this means $\mathcal{M}_p \models \Gamma$ so $\mathcal{M}_p$ has either a classical representation or a straight non-multiplicative representation, depending on which alternative is chosen. This contradicts lemma 36. $\square$

8 Non-additive representations

For a boolean module to possess a non-additive representation it is clearly necessary for its relation algebra part to have a non-additive representation. As with classical representations, the class of relation algebras with non-additive representations is already non-finitely axiomatisable [9, theorem 2.3]. We will show next that even if the relation algebra part of a boolean module has a non-additive representation it does not follow that the boolean module has such a representation. We will define boolean modules $\mathcal{M}_n$ with no non-additive representations, even though their relation algebra parts all possess square classical representations. We'll show that a non-principal ultraproduct of the $\mathcal{M}_n$'s has a square non-additive representation.

The relation algebra part of the $\mathcal{M}_n$'s is a simplified version of a rainbow algebra [7, chapter 16].

DEFINITION 39 *Let $m \geq 1$. The atomic relation algebra $\mathcal{R}_m$ has $3+m$ atoms, all self-converse. They are $\{1', \mathsf{y}, \rho, \mathsf{g}_i \mid i < m\}$. We call* y *the* yellow atom, ρ *the* red atom *and* g_i *a* green atom, *for each $i < m$. We define composition on the atoms by*

1. $1'; x = x; 1' = x$
2. $\mathsf{y}; \mathsf{y} = -\mathsf{y}$
3. $\mathsf{g}_i; \mathsf{g}_j = \begin{cases} \mathsf{y} + \rho & \text{if } i \neq j \\ 1' + \mathsf{y} + \rho & \text{if } i = j \end{cases}$
4. $\rho; \rho = 1$
5. $u; v = -1'$

where x is an arbitrary atom, $i, j < m$ are arbitrary and u, v are an arbitrary pair of atoms not dealt with in the previous lines, i.e. $u \neq 1'$, $v \neq 1'$ and u and v have different colours. [For those familiar with atom structures, the forbidden triples of atoms are permutations of $(1', x, y)$ (any atoms $x \neq y$) $(\mathsf{y}, \mathsf{y}, \mathsf{y})$ and $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k)$ (any $i, j, k < m$).] Composition of arbitrary elements of $\mathcal{R}_m$ can now be calculated, using complete additivity.

Let $n \geq 3$ and let $\mathcal{M}_n = (\mathcal{R}_{n+1}, \mathcal{B}_n, :)$ be the boolean module whose relation algebra part is $\mathcal{R}_{n+1}$ (defined above), whose boolean part is the boolean algebra with n atoms $b_0, \ldots, b_{n-1}$, and where the Peircean operator is defined on atoms by

6. $1' : b_i = b_i$
7. $\mathsf{y} : b_i = \mathsf{g}_j : b_i = 1_{\mathcal{B}_n}$
8. $\rho : b_i = -b_i$

for any $i < n$, $j < n+1$. This suffices to define the Peircean operator on non-atomic elements, by additivity.

LEMMA 40 *The relation algebra $\mathcal{R}_m$ has a square classical representation, for any m.*

PROOF:

The atom $\rho \in \mathcal{R}$ is a *flexible* atom (if $x \neq 1'$ and $y \neq 1'$ are atoms then $x; y \geq \rho$, consequently $\rho; y^{\smile} \geq x$ and $x^{\smile}; \rho \geq y$). By [5] or [14, theorem 5.19] $\mathcal{R}$ is representable. For any atom x of $\mathcal{R}$ we have $\rho; x; \rho = 1$ (since ρ is flexible) so $1; x; 1 = 1$ and $\mathcal{R}$ is *simple*. It is easy to check that any simple, representable relation algebra has a square representation (e.g. see [7, exercise 1, page 129]). $\Box$

LEMMA 41 *For $n \geq 3$, $\mathcal{M}_n$ is a boolean module.*

PROOF:

Since $\mathcal{R}_{n+1}$ is representable, it is a relation algebra. $\mathcal{B}_n$ is clearly a boolean algebra. It remains to check axioms (1)–(6) in definition 3. Since we are defining the Peircean operator on non-atomic elements by additivity, axioms 3(1), 3(2) and 3(5) hold. Axiom 3(4) is true, by definition of the Peircean operator in $\mathcal{M}_n$. For axiom 3(3) (namely, $(r; s) : b = r : s : b$) if $b = 0$ then since $r : 0 = 0$ is a trivial instance of lemma 4(1), the axiom reduces to $0 = 0$, which is true. Assume $b \neq 0$ and let $b_i \leq b$ be any atom of the boolean algebra. The cases where $r = 1'$ or $s = 1'$ are easily checked. In all other cases either $\mathsf{g}_0 \leq r; s$ or $\mathsf{y} \leq r; s$ (see definition 39(2)–(5)) so $(r; s) : b = 1$ (by 39(7)). On the right hand side, since $s \neq 1'$, we have $s : b \geq -b_i$, by 39(7) and 39(8), and so $r : s : b \geq -b_j$ for all $j \neq i$, $j < n$. Using the fact that $n \geq 3$ this implies $r : s : b = 1$ and the axiom holds. Finally we check axiom 3(6) ($r^{\smile} : (-(r : a)) \leq -a$). The cases where $r = 1'$ or $a = 0$ are easily checked, so suppose not. If $r = \mathsf{y}, \mathsf{g}_i$ (any i) or if a is not a boolean atom then the left hand side is 0 so the inequality holds. Suppose $r = \rho$ and $a = b_i$ for some $i < n$. The left hand side of the inequality is $\rho : (-(\rho : b_i)) = \rho : (-(-b_i)) = \rho : b_i = -b_i$ (use 39(8)) and this equals the right hand side of the inequality. Thus all the axioms in definition 3 hold. $\Box$

LEMMA 42 *For each n with $n < \omega$, the boolean module $\mathcal{M}_n$ has no non-additive representation.*

PROOF:

Suppose, for contradiction, that $\mathcal{W}$ is a non-additive representation of $\mathcal{M}_n$. In outline, we derive a contradiction by using the relation algebra part $\mathcal{R}_{n+1}$ to show that $\mathcal{W}$ includes a *red clique* with $n+1$ points (a set of points contained in $1^{\mathcal{W}}_{\mathcal{B}_n}$ such that every pair of distinct points from the set belongs to $\rho^{\mathcal{W}}$) and by using the boolean part and the Peircean operator to show that $\mathcal{W}$ cannot contain a red clique with more than n points.

For the first part of this argument, since $\mathcal{W}$ is a representation of $\mathcal{M}_n$, we know $b_0^{\mathcal{W}} \neq \emptyset$, so pick $x_0 \in b_0^{\mathcal{W}}$. We have $(x_0, x_0) \in 1'^{\mathcal{W}}$. Since $\mathsf{g}_0; \mathsf{g}_0 \geq 1'$, there is $x_1 \in \mathcal{W}$ with $(x_0, x_1) \in \mathsf{g}_0^{\mathcal{W}}$. For each $j < n+1$ since $\mathsf{g}_j; \mathsf{y} \geq \mathsf{g}_0$, there is $z_j \in \mathcal{W}$ with $(x_0, z_j) \in \mathsf{g}_j^{\mathcal{W}}$ and $(z_j, x_1) \in \mathsf{y}^{\mathcal{W}}$. For all distinct $j, k < n+1$ we have $(z_j, z_k) \in (\mathsf{g}_j; \mathsf{g}_k)^{\mathcal{W}} \cap (\mathsf{y}; \mathsf{y})^{\mathcal{W}} = (\mathsf{y} + \rho)^{\mathcal{W}} \cap (-\mathsf{y})^{\mathcal{W}} = \rho^{\mathcal{W}}$ (by 39(2) and 39(3)).

And $z_j \in (\mathsf{g}_j : b_0)^{\mathcal{W}} = 1_{\mathcal{B}}^{\mathcal{W}}$ (by 39(7)). Thus $\{z_j \mid j < n+1\}$ is a red clique.

Now, for each $j < n+1$, since $\mathcal{W}$ is a representation of the boolean part, there is an $i < n$ such that $z_j \in b_i^{\mathcal{W}}$. Hence there must be distinct $j, k < n+1$ and some $i < n$ such that $z_j, z_k \in b_i^{\mathcal{W}}$. But then $z_k \in (\rho : b_i)^{\mathcal{W}} \cap b_i^{\mathcal{W}} = (-b_i)^{\mathcal{W}} \cap b_i^{\mathcal{W}} = \emptyset$ (by definition 12(1) and 39(8)), a contradiction. Thus, $\mathcal{M}_n$ has no non-additive representation. $\Box$

LEMMA 43 *Any non-principal ultraproduct of the $\mathcal{M}_n$'s has a square non-additive representation.*

PROOF:

Let $\mathcal{M}$ be a non-principal ultraproduct of the $\mathcal{M}_n$'s and let $\mathcal{N} = (\mathcal{R}, \mathcal{B}, :)$ be a countable, elementary subalgebra of $\mathcal{M}$. By Łoś' theorem, any first-order property true in all the $\mathcal{M}_n$'s is preserved in the ultraproduct and hence also in $\mathcal{N}$. So $\mathcal{R}$ is an atomic symmetric relation algebra and $\mathcal{B}$ is an infinite atomic boolean algebra. We can rename the atoms of $\mathcal{R}$ so that each of them is either the identity $1'$, the yellow atom y, the red flexible atom ρ or some green atom g_α. As in each $\mathcal{R}_m$ we have $\mathsf{y};\mathsf{y} = -\mathsf{y}$, $\mathsf{g}_\alpha;\mathsf{g}_\alpha = 1'+\mathsf{y}+\rho$, $\mathsf{g}_\alpha;\mathsf{g}_\beta = \mathsf{y}+\rho$ (any distinct green atoms $\mathsf{g}_\alpha, \mathsf{g}_\beta$), $\rho;\rho = 1$ and for non-identity atoms u, v of different colours, $u;v = -1'$. Composition of arbitrary elements of $\mathcal{R}$ is calculated by complete additivity. For any boolean atom b we have $\rho : b = -b$ and for any other non-identity atom a we have $a : b = 1$. That is, the atom structure of $\mathcal{N}$ is similar to the atom structures of the $\mathcal{M}_n$s, the only difference being the number of atoms. Note that $||\mathsf{At}(\mathcal{B})|| = ||\mathsf{At}(\mathcal{R})|| = \omega$.

Now that we know how the atoms of $\mathcal{N}$ behave we wish to prove that $\mathcal{N}$ has a square non-additive representation. A standard way of proving representability is to define certain finite fragments of a claimed representation possessing all the universal properties of the required representation, and to extend these fragments step by step so that all the existential 'defects' get repaired and thus the limit of the sequence is indeed a representation. In [7] and elsewhere, these fragments are called *networks* and we use that term here, with modifications to make them appropriate for non-additive representations.

Define a *boolean module network over $\mathcal{N}$* (or just a network) N to be a complete graph where for all nodes $x, y \in N$ the edge (x, y) is labelled by an element $N(x, y) \in \mathcal{R} \setminus \{0\}$ and each node x is labelled by an element $N(x) \in \mathsf{At}(\mathcal{B})$, such that

- $N(x, y) = 1' \leftrightarrow (x = y)$,
- $N(x, y) = N(y, x)$ (note that the algebras under consideration are symmetric),
- $N(x, y); N(y, z) \geq N(x, z)$ and
- $N(x, y) : N(y) \geq N(x)$

for all nodes $x, y, z \in N$. We write $M \subseteq N$ if each node of M is a node of N and for all $x, y \in M$ we have $M(x, y) = N(x, y)$ and $M(x) = N(x)$.

We construct a square non-additive representation $\mathcal{W}$ of $\mathcal{N}$ step by step, by defining a countable, nested sequence of finite networks

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_i \subseteq \ldots$$

$(i < \omega)$. N_0 has just one node x and it is labelled $N_0(x,x) = 1'$, $N_0(x) = b$, where b is an arbitrary atom of $\mathcal{B}$.

For any finite network N, if there are nodes $x, y \in N$ and $s, t \in \mathcal{R}$ such that $N(x,y) \leq s;t$, $s \leq N(x,y);t^{\smile}$ and $t \leq s^{\smile};N(x,y)$, we define an extension $N^+ = N^+(N,x,y,s,t) \supseteq N$ as follows. If $s = 1'$ or $t = 1'$ we let $N^+ = N$. Otherwise N^+ includes exactly one new node, z say, not in N such that

$$\begin{aligned} N^+(x,z) = N^+(z,x) &= s \\ N^+(z,y) = N^+(y,z) &= t \\ N^+(w,z) = N^+(z,w) &= \rho \\ N^+(z) &= b' \end{aligned}$$

where $w \in N \setminus \{x,y\}$ is an arbitrary node and b' is an atom of $\mathcal{B}$ not in $\{N(w) \mid w \in N\}$. Since $\mathcal{B}$ has infinitely many atoms and N is finite, such an atom b' exists. It is simple to check that N^+ is a network, because ρ is a flexible atom and because b' is a new boolean atom.

More generally, if N has nodes x, y and there are s, t such that $N(x,y) \leq s;t$ then by [7, exercise 3.3.7(10)] there are $s_0 \leq s$, $t_0 \leq t$ with $N(x,y) \leq s_0;t_0$, $s_0 \leq N(x,y);t_0^{\smile}$ and $t_0 \leq s_0^{\smile};N(x,y)$ — for this, let $s_0 = s \cdot N(x,y);t^{\smile}$ and $t_0 = t \cdot s^{\smile};N(x,y)$. In this case we let $N^+ \supseteq N$ be defined as $N^+(N,x,y,s_0,t_0)$ of the previous paragraph. In all cases N^+ contains a node z such that $N^+(x,z) \leq s$ and $N^+(z,y) \leq t$ (for the case $s = 1'$ let $z = x$ and if $t = 1'$ let $z = y$).

For any finite network N, if there is a node $x \in N$, $s \in \mathcal{R}$ and an atom $c \in \mathsf{At}(\mathcal{B})$ such that $N(x) \leq s : c$ then we define another extension $N^* = N^*(N,x,s,c) \supseteq N$, as follows. If $s = 1'$ we let $N^* = N$. Otherwise define N^* with one extra node, z say, not in N by

$$\begin{aligned} N^*(z) &= c \\ N^*(w,z) = N^*(z,w) &= N(w,x);s \end{aligned}$$

where w is an arbitrary node of N. N^* is a network because if $w \in N \setminus \{x\}$ then $N^*(w,z)$ is above the product of two non-identity atoms and is therefore above atoms of at least two colours (see 39(2)–(5)). Hence, by 39(5), if $w \neq v \in N\setminus$ then $N^*(w,z);N^*(z,v) \geq -1'_{\mathcal{R}}$ and by 39(7), $N^*(w,z) : c = 1_{\mathcal{B}}$. Therefore $N^*(w,z);N^*(z,v) \geq N^*(w,v)$ and $N^*(w,z) : N^*(z) \geq N^*(w)$. Other permutation of w, v, z are similar and the case where $w = v$ is easy.

In the countable sequence of extensions $N_0 \subseteq N_1 \subseteq \ldots$ we schedule the extensions so that for any $i < \omega$, any $x, y \in N_i$ and any s, t with $s;t \geq N_i(x,y)$, there is a $j \geq i$ such that the extension

$N_{j+1} = N^+(N_j, x, y, s, t) \supseteq N_j$ includes a node $z \in N_{j+1}$ with $N_{j+1}(x, z) \leq s$ and $N_{j+1}(z, y) \leq t$, and for any $i < \omega$, any $x \in N_i$, any $c \in \mathsf{At}(\mathcal{B})$ and any $s \in \mathcal{R}$ such that $N_i(x) \leq s : c$, there is $j \geq i$ such that the extension $N_{j+1} = N^*(N_j, x, s, c) \supseteq N_j$ includes a node z witnessing $N_{j+1}(z) = c$ and $N_{j+1}(x, z) = s$. Since the boolean module $\mathcal{N}$ is countable, all networks are finite and the length of our sequence is infinite, it is possible to schedule all these extensions in.

Now let $N = \bigcup_{i<\omega} N_i$. This is the network whose nodes are all the nodes occurring in any N_i and with labelling $N(x) = N_i(x)$, $N(x, y) = N_i(x, y)$, where i is arbitrary such that $x, y \in N_i$. N is a network satisfying

I. for all $x, y \in N$, $s, t \in \mathcal{R}$ and $c \in \mathsf{At}(\mathcal{B})$, if $N(x, y) \leq s;t$ then there is $z \in N$ with $N(x, z) \leq s$, $N(z, y) \leq t$, and

II. if $N(x) \leq s : c$ then there is $z \in N$ with $N(z) = c$ and $N(x, z) \leq s$.

Now define a $\mathcal{N}$-structure W whose domain is the set of nodes of N, by

$$\begin{aligned} s^{\mathcal{W}} &= \{(x, y) \mid N(x, y) \leq s\} \\ b^{\mathcal{W}} &= \{x \in N \mid N(x) \leq b\} \end{aligned}$$

for $s \in \mathcal{R}$, $b \in \mathcal{B}$. Since each node of N is labelled by a boolean atom and each boolean atom labels some node of N, $\mathcal{W}$ is a boolean representation. By property I of N and since N is a network, $\mathcal{W}$ is a non-additive representation of $\mathcal{R}$. Property II and the fact that N is a network show that $\mathcal{W}$ respects the Peircean operator. Since N is completely labelled, $\mathcal{W}$ is a square non-additive representation of $\mathcal{N}$.

By theorem 27 we know that the class of boolean modules with non-additive square representations is universally axiomatisable, certainly elementary. Since $\mathcal{M} \equiv \mathcal{N}$ we see that $\mathcal{M}$ also has a non-additive, square representation. $\square$

Hence,

THEOREM 44 *Let K be any class of boolean modules with non-additive representations such that K contains all boolean modules with square non-additive representations. E.g. K could be the class of all boolean modules with non-additive representations, the class of all boolean modules with straight non-additive representations or the class of all boolean modules with square non-additive representations.*

Any axiomatisation of K contains infinitely many axioms not derivable from an axiomatisation of the class of relation algebras with non-additive representations.

PROOF:

Let Σ axiomatise the relation algebras with non-additive representations and suppose for contradiction that $\Sigma \cup \{\phi\}$ axiomatises K for some single formula ϕ, where K is as in the theorem. By lemma 43 the ultraproduct $\mathcal{M}$ has a square non-additive representation so it belongs to K, hence $\mathcal{M} \models \Sigma \cup \{\phi\}$. By Łoś' theorem $\mathcal{M}_n \models \phi$ for many values of $n < \omega$. By lemma 40, the relation algebra part of $\mathcal{M}_n$ is representable, so $\mathcal{M}_n \models \Sigma$. By assumption, $\mathcal{M}_n$ belongs to K, for many values of n, so it has a non-additive representation, contradicting lemma 42. Hence $\Sigma \cup \phi$ cannot axiomatise K. □

9 Straight and arbitrary representations of boolean modules

THEOREM 45 *Let $\lambda \in$ {classical, non-additive, non-multiplicative}. Any axiomatisation of the class of boolean modules with straight λ-representations includes infinitely many axioms not valid over the class of boolean modules with arbitrary λ-representations.*

PROOF:

Depending on λ, let $\langle(\mathcal{R}_i, \mathcal{B}_i, :) \mid i < \omega\rangle$ be a sequence of boolean modules with no straight λ-representations but where each relation algebra $\mathcal{R}_i$ has a λ-representation and a non-principal ultraproduct of these boolean modules has a straight λ-representation. For λ = 'classical' and λ = 'non-multiplicative' use the boolean modules based on the Lyndon algebras (section 7, see lemmas 36 and 37) and for λ = 'non-additive' use the boolean modules of definition 39 (see lemmas 40, 42 and 43). We have not proved that these boolean modules have even λ-representations, but we can get round that.

For each $i < \omega$, let $(\mathcal{S}_i, \mathcal{B}_i, :)$ be any boolean module, whose boolean part is $\mathcal{B}_i$, possessing a straight λ-representation. E.g. let X be a boolean representation of $\mathcal{B}_i$ and let $\mathcal{S}_i$ be the full relation algebra over the domain of X. Now consider the sequence of boolean modules $\mathcal{M}_i = (\mathcal{R}_i \times \mathcal{S}_i, \mathcal{B}_i, :)$ where the Peircean operator is defined by $(r, s) : b = s : b$, for $r \in \mathcal{R}_i,\ s \in \mathcal{S}_i,\ b \in \mathcal{B}_i$. We will show that each $\mathcal{M}_i$ has a λ-representation but no straight λ-representation, yet an ultraproduct of the $\mathcal{M}_i$s has a straight λ-representation.

The disjoint union of a relation algebra λ-representation of $\mathcal{R}_i$ and a boolean module λ-representation of $(\mathcal{S}_i, \mathcal{B}_i, :)$ provides a λ-representation of $\mathcal{M}_i$. Suppose, for contradiction, that $\mathcal{M}_i$ has a straight λ-representation, $\mathcal{W}$, say. Define a representation $\mathcal{V}$ of $(\mathcal{R}_i, \mathcal{B}_i, :)$ by $r^{\mathcal{V}} = (r, 0)^{\mathcal{W}}$ and $b^{\mathcal{V}} = b^{\mathcal{W}}$, for $r \in \mathcal{R}_i,\ b \in \mathcal{B}_i$. Now $(w, w) \in 1^{\mathcal{V}}_{\mathcal{R}_i} \Rightarrow (w, w) \in 1^{\mathcal{W}}_{\mathcal{R}_i}$ and since $\mathcal{W}$ is straight this implies $w \in 1^{\mathcal{W}}_{\mathcal{B}_i} = 1^{\mathcal{V}}_{\mathcal{B}_i}$. Thus, by lemma 15, $\mathcal{V}$ is a straight λ-representation of $(\mathcal{R}_i, \mathcal{B}_i, :)$, contrary to assumption. Therefore, $\mathcal{M}_i$ has no straight λ-representation.

Let $\mathcal{M}$ be a non-principal ultraproduct $\Pi_D \mathcal{M}_i$ of the $\mathcal{M}_i$s. $\mathcal{M} = \Pi_D(\mathcal{R}_i \times \mathcal{S}_i, \mathcal{B}_i, :) \cong (\Pi_D \mathcal{R}_i \times \Pi_D \mathcal{S}_i, \Pi_D \mathcal{B}_i, :)$. Let $\mathcal{V}$ be a straight λ-representation of $\Pi_D(\mathcal{R}_i, \mathcal{B}_i, :)$ and let $\mathcal{W}$ be a straight λ-representation of $\Pi_D(\mathcal{S}_i, \mathcal{B}_i, :)$, assume the domains of $\mathcal{V}, \mathcal{W}$ are disjoint. Now define a straight λ-representation $\mathcal{X}$ of $\mathcal{M}$ by $(r, s)^{\mathcal{X}} = r^{\mathcal{V}} \cup s^{\mathcal{W}}$ for $r \in \Pi_D \mathcal{R}_i$ and $s \in \Pi_D \mathcal{S}_i$, and $b^{\mathcal{X}} = b^{\mathcal{V}} \cup b^{\mathcal{W}}$.

Hence each $\mathcal{M}_i$ has a λ-representation but no straight λ-representation, yet an ultraproduct of the $\mathcal{M}_i$'s has a straight λ-representation. The theorem now follows, by Łoś' theorem. $\Box$

10 Summary of main results

DEFINITION 46 *Let $\lambda \in$ {classical, non-additive, non-multiplicative}, but use the abbreviations {class, non-+, non-×}. Define* $\mathbf{RRA}_\lambda$ *to be the class of Peirce type algebras whose relation algebra part has a λ-representation, define* $\mathbf{RBM}_\lambda$ *to be the class of Peirce type algebras whose boolean module reduct has an λ-representation, and define* $\mathbf{RPA}_\lambda$ *to be the class of λ-representable Peirce algebras. Let* $\mathbf{SRBM}_\lambda$ *be the class of Peirce type algebras whose boolean module reduct has a straight λ-representation.*

THEOREM 47 *Let $\lambda \in$ {classical, non-additive, non-multiplicative}.*

1. *A Peirce algebra has a λ-representation if and only if it has a straight λ-representation if and only if its relation algebra part has a λ-representation.*

2. *The variety generated by the class of all full boolean modules is the same as the variety generated by the boolean modules with straight classical representations which is the same as the class of boolean modules with classical representations.*

3.
$$\begin{array}{ccccccc}
\mathbf{RRA}_{non\text{-}\times} & \supset & \mathbf{RBM}_{non\text{-}\times} & \Supset & \mathbf{SRBM}_{non\text{-}\times} & \supset & \mathbf{RPA}_{non\text{-}\times} \\
\Cup & & \Cup & & \Cup & & \Cup \\
\mathbf{RRA}_{non\text{-}+} & \Supset & \mathbf{RBM}_{non\text{-}+} & \Supset & \mathbf{SRBM}_{non\text{-}+} & \supset & \mathbf{RPA}_{non\text{-}+} \\
\Cup & & \Cup & & \Cup & & \Cup \\
\mathbf{RRA}_{class} & \Supset & \mathbf{RBM}_{class} & \Supset & \mathbf{SRBM}_{class} & \supset & \mathbf{RPA}_{class}
\end{array}$$

   All inclusions are strict. When we write $K \Supset L$ we mean that $K \supset L$ and L cannot be defined within K using only finitely many axioms. For each λ, $\mathbf{RPA}_\lambda$ *can be defined within* $\mathbf{RRA}_\lambda$ *by the finite set of equations defining Peirce algebras (definitions 3 and 6).* $\mathbf{RBM}_{non\text{-}\times}$ *can be defined, within* $\mathbf{RRA}_{non\text{-}\times}$*, by the finite set of equations defining boolean modules (definition 3).*

PROOF:

1. Lemma 17.
2. By lemma 29, the variety generated by the full boolean modules is the class of classically representable boolean modules. The boolean modules with straight classical representations are obviously classically representable and contain all the full boolean modules.
3. Note that the algebras in these classes are not assumed to satisfy the axioms for boolean modules or Peirce algebras. Indeed

if we restrict each class to Peirce algebras then by lemma 17 all four classes in any one of the rows are equal.

Each class is a quasi-variety, by theorem 27. For strictness of inclusions, any Peirce type algebra where the relation algebra part is λ-representable but the Peircean operator fails any of the axioms of definition 3 will belong to $\mathbf{RRA}_\lambda \setminus \mathbf{RBM}_\lambda$. Similarly, for strictness of the inclusion $\mathbf{SRBM}_\lambda \supset \mathbf{RPA}_\lambda$, take any boolean module with a straight λ-representation and define a right cylindrifier in any way so as to make some of the axioms in definition 6 fail.

$\mathbf{RPA}_\lambda$ is defined within $\mathbf{RRA}_\lambda$ by the axioms for boolean modules and Peirce algebras, by lemma 17. $\mathbf{RBM}_{\text{non-}\times}$ is defined within $\mathbf{RRA}_{\text{non-}\times}$ by the axioms for boolean modules, by corollary 22.

Now we prove the non-finitely axiomatisable inclusions. For $\lambda \in \{\text{non-}+, \text{class}\}$ we have $\mathbf{RBM}_\lambda$ is not finitely axiomatisable within $\mathbf{RRA}_\lambda$, by theorems 38 and 44. And for any λ, $\mathbf{SRBM}_\lambda$ cannot be defined within $\mathbf{RBM}_\lambda$ using finitely many axioms, by theorem 45. This proves all the horizontal inclusions.

Finally, we prove the vertical inclusions. By [9], there is a sequence $\mathcal{R}_i$ of relation algebras with no non-additive representations whose ultraproduct $\Pi_D \mathcal{R}_i$ does have a non-additive representation. By corollary 22 and lemma 19, for each i, the Peirce algebra $\mathcal{P}(\mathcal{R}_i)$ has a non-multiplicative representation but no non-additive representation yet the ultraproduct $\Pi_D(\mathcal{P}(\mathcal{R}_i)) \cong \mathcal{P}(\Pi_D(\mathcal{R}_i))$ has a straight, non-additive representations. Hence, for $K \in \{\mathbf{RRA}, \mathbf{RBM}, \mathbf{SRBM}, \mathbf{RPA}\}$, $\mathcal{P}(\mathcal{R}_i) \in K_{\text{non-}\times} \setminus K_{\text{non-}+}$ but $\Pi_D(\mathcal{P}(\mathcal{R}_i)) \in K_{\text{non-}+}$. By Loś' theorem, $K_{\text{non-}\times} \Supset K_{\text{non-}+}$.

By lemma 35 the algebras $\mathcal{L}_p$ for $p \geq 3$ have non-additive representations but by lemma 36 they do not have classical representations, yet by lemma 37, an ultraproduct of them has a classical representation. By lemma 19, each of $\mathcal{P}(\mathcal{L}_p)$ has a straight non-additive representation but no classical representation yet an ultraproduct of them has a classical representation. It follows that $K_{\text{non-}+} \Supset K_{\text{class}}$, for $K \in \{\mathbf{RRA}, \mathbf{RBM}, \mathbf{SRBM}, \mathbf{RPA}\}$.

□

PROBLEM 48 *We do not know if the classes of boolean modules/Peirce algebras with non-multiplicative representations forms a variety (see problem 28). If it is not a variety, we would like to know if it generates a finitely based variety or not. Repeat this problem for non-additive representations.*

PROBLEM 49 *Let $\mathcal{M} = (\mathcal{R}, \mathcal{B}, :)$ be a finite boolean module (i.e. both $\mathcal{R}$ and $\mathcal{B}$ are finite). If $\mathcal{R}$ has a representation on a finite base must $\mathcal{M}$ have a representation on a finite base?*

PROBLEM 50 *A* relativised representation *$\mathcal{W}$ of a boolean module $(\mathcal{R}, \mathcal{B}, :)$ is a structure and a reflexive, symmetric unit element $V \subseteq \mathcal{W} \times \mathcal{W}$, such that*

$\mathcal{W}$ is exactly like a classical representation of the boolean module, except that composition is relativized to the unit V, i.e. $(v,w) \in (r;s)^{\mathcal{W}} \iff (v,w) \in V \wedge \exists u\, ((v,u) \in r^{\mathcal{W}} \wedge (u,w) \in s^{\mathcal{W}})$. It is known that every relation algebra has a relativized representation, indeed every weakly associative algebra has one and every algebra of the type of relation algebras with a relativized representation is a weakly associative algebra [14].

Does every boolean module have a relativized representation? Generalise the definition of boolean modules so that the associativity of composition is replaced by a weaker axiom but also, perhaps, the axiom $(r;s) : b = r : (s : b))$ is also weakened. Try to prove that these axioms define the class of boolean modules with relativized representations.

PROBLEM 51 *There are various extensions to the basic description logic. Three well-known extensions are*

- Nominals — *propositions which must be interpreted as singleton points,*
- Counting quantifiers $\exists_{\geq n} R$,
- Transitive closure R^*.

In each case devise a suitable extension to the signature of boolean modules and investigate axiomatisations for the representation class.

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