Mathematics — the Pure Science?

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Abstract

The philosophy of mathematics provides a severe test for a materialist explanation of science. This is because mathematics is mostly abstract and mathematical theory is rarely tested directly in practice. All the main schools of the philosophy of mathematics — platonism, logicism, intuitionism, formalism — are varieties of idealism. Nevertheless all human ideas, including mathematical ideas, originate from our experience of the world and are rooted in reality.

In the history of mathematics it can be seen that problems facing society have given a great impetus to the development of the subject. Thus the rise of trade and changes in technology have each led to great advances, though purely internal contradictions within mathematics have also been of considerable importance.

Formal mathematical reasoning, like classical logic, has been highly successful in the evolution of science but is inadequate for reasoning about indeterministic processes in a state of change. Thus formal logic and dialectics should be thought of as complementary and both are required for a fully scientific understanding of the world.

Introduction

Marxism provides a coherent understanding of the philosophy of science. According to the materialist view all our ideas come not from heaven or some special realm of thoughts but from the world we live in. This is as true of science as it is of religion or anything else. Marx's view was more subtle than that of deterministic materialism in that he saw how our ideas did not simply reflect the real world passively, but that they could consciously shape our environment in order to satisfy our needs.

Science, however, is different to other forms of thought because, in contrast to religion, it can discover and make sense of the way the world behaves. Newton's theory of gravitation, for example, was not just an idea in Newton's mind — it contained at least a partial truth that allowed humans to explain and control a whole range of phenomena, from the falling of apples to the motion of planets. Science is not a fixed methodology but a developing process, nevertheless two prominent features can be identified:

- scientific ideas are not to be taken on trust, but are open to rational disagreement and debate
- scientific theories must be tested in practice, modified and corrected where necessary.

Of course this scientific ideal is rarely matched by reality, but there is a strong contrast between the scientific method and the way religious ideas are developed. With religion there is an emphasis on *faith* — i.e. the unquestioning acceptance of dogma. Many of the ideas cannot be tested in practice (how can you tell whether you go to heaven when you die?) and those religious ideas which can be tested usually fail that test (the world was not made in seven days!) The two features of scientific thought (above) help to explain the enormous advance that science made over its predecessors. It has been a vital ingredient in a process that has extended the understanding and control of humans over every aspect of our environment.

The philosophy of mathematics poses a number of problems for Marxists. Firstly, mathematics is the most abstract of subjects — the symbols in mathematics rarely represent concrete objects, they may sometimes stand for numbers, but frequently they refer to completely formal structures. This means that a demonstration of the materialist root of mathematical ideas will be harder to produce than for other sciences. Secondly, mathematics is not usually considered an experimental subject, and so the second feature of the scientific method appears not to hold. Thus, if mathematics is not tested in practice, what claim does it have to be different to religion or any other mystical belief?

Yet, mathematics is an integral part of virtually every science. Contrary to one of the myths, science is not a mechanical process where masses of data are collected, patterns observed and eventually theories emerge. In many cases the scientific theory is developed on the basis of mathematics and logic far in advance of any experimental justification. A good example of this is Einstein's general theory of relativity, developed mathematically way ahead of the experimental science, though subsequent experimental work has confirmed the predictions of the theory to an extraordinary degree of accuracy. A second example is in *Chaos Theory* where pure mathematicians such as Cantor, Julia and Fatou had produced a theoretical framework for the subject long before the computer technology had been developed to make it applicable. So any weakness in the methodology in mathematics is also a weakness in the way science is done.

This article has two main points. The principal theme is that even the most abstract ideas, those of pure mathematics, arise from experience of the real world. Yet the dominant views of mathematics will be shown to be varieties of *idealism*: they all regard concepts like numbers and equations as having an existence or meaning quite independent of the real world. Each of these views will be shown to lead to considerable problems. Only materialism can give a coherent picture of mathematics, as a body of ideas that has arisen from the world and which has developed to solve real problems facing society. The second theme is to look at formal reasoning, like mathematics and formal logic, to assess the strengths and limitations of this method and to compare it with the dialectic framework. It will be argued that formal reasoning and dialectics are not rival alternatives but complementary. Both are necessary components of a fully scientific method of understanding the world.

First let us briefly sketch the main views of mathematics.

Platonism

Perhaps the most basic entities in mathematics are numbers. So exactly what are they, in particular are they *synthetic*, i.e. man-made, or *objective* — existing in reality? If they are real, then where or what are they? In the Platonist view, named in honour of the Greek philosopher, all mathematical objects (like numbers, equilateral triangles, theorems, etc.) have an objective existence independent of human consciousness. But they exist not in our physical universe (after all, have you ever seen a number - not the symbol but the *actual* number?) but in a separate *Platonist realm* of mathematical entities, thus raising mathematics to the level of metaphysics. It follows that mathematics is a process of *discovery* not *invention*. Each question, provided it is phrased unambiguously, has a definite answer even if that answer is not known to us. So unsolved questions like the Goldbach conjecture[1] must be either true of false, it is just our human ignorance which stops us from knowing which.

It was Platonism that dominated right up to the twentieth century. This philosophy was a central pillar in a hierarchical view of science. It was argued that science dealt with absolute truth because each subject could be reduced to a more fundamental one (so biology could be reduced to chemistry which could be reduced to physics, etc.) and that this process of reduction rested on the rock solid foundations of pure mathematics and logic.

In the twentieth century Platonists have had to modify their ideas to some extent but still Carl Hempel argues that the basic formulas of mathematics and logic are 'true *a priori*, which is to indicate that their truth is logically independent of, or logically prior to, any experiential evidence' [9]. Or read [17] which includes, among many fascinating insights, an orthodox exposition of the Platonist case.

Marxists might classify Platonism as a form of *objective idealism* : concepts and ideas (like numbers) are given an objective reality separate from human thought. But this objective truth in mathematics is not to be found in the nature of the physical world we inhabit, but in some other world of numbers and ideas. The question arises: if numbers and mathematics are located neither in our heads nor in the world then how do we come to have mathematical knowledge if its truth resides in a world that we can neither visit nor see? Some Platonists argue that humans have an 'intuition' from which we can learn mathematics directly. But as Hilary Putnam put it

> This appeal to mysterious faculties seems both unhelpful as epistemology and

[1]The Goldbach conjecture says that every even number greater than 3 is the sum of two prime numbers: $4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 5 + 5, 12 = 5 + 7$, etc.

unpersuasive as science. What neural process, after all, could be described as the perception of a mathematical object? [18]

Platonism received its first major set-back from one of its arch proponents: Plato's predecessor Pythagoras [6'th century BC]. Obviously Pythagoras did not use the term 'Platonism' - a term that was not used until the twentieth century — but his approach to mathematics conformed to the philosophy just outlined. In Pythagoras' time numbers and arithmetic were considered the field where absolute truth was most certain to be found. Geometry also, to a lesser extent, provided one of the foundations of objective truth. At around the time of Pythagoras, there was the discovery of *irrational numbers*. (There is some doubt as to whether it was Pythagoras who actually made this discovery. See [20].) An irrational number, like $\sqrt{2}$ or π, cannot be written as a fraction. Though $\frac{22}{7}$ is a good approximation, it can be shown that no fraction can be exactly equal to π. Here the problem is explained using modern terminology but the Greeks did not know about irrational numbers nor even fractions, the numbers they used were the ordinary counting numbers 1, 2, 3, ... They were interested in *similar triangles* and could prove that two triangles were similar by showing that the ratio of the sides (a:b:c) was the same for both triangles. However for a right angled, isosceles triangle it is not possible to find the ratio of the sides as the ratio of three integers. The sides were called *incommensurable* which means that they had no common measure.

Numbers like this were considered as irrational, and probably not numbers at all. Thereafter it was geometry that became the more solid base for certain knowledge. Also, the unexpected and shocking discovery of irrational numbers led to the development of formal proofs in mathematics, epitomised by Euclid's *Elements*, because it was felt that every statement required a rigorous justification, nothing could be taken as obvious.

Much later, Platonism suffered an increasing series of attacks. One of these was the invention (discovery?) of complex numbers. They were originally considered as purely imaginary fictions or ideal elements, which were very useful for the solution of cubic equations. Nevertheless complex numbers were increasingly used in a wide variety of contexts, perhaps the most concrete demonstration of their value is in the description of wave-forms in quantum physics by complex functions. Gradually the view emerged that complex numbers were not fictions but had a real existence of their own. The extension of this view was that any mathematical construction, provided it was consistent, existed in the Platonist realm.

Later, mathematicians like Lobachevsky, Bolyai, Gauss, Riemann and others showed that Euclidean geometry was not the one and true geometry, but one among many alternative geometries. In an interesting example of a genuine experiment in mathematics, Gauss questioned the assumption that our universe had a Euclidean geometry by taking three triangulation points and summing the angles of the triangle. In fact, given the experimental accuracy that was available, he found the sum to be $180°$, consistent with a Euclidean geometry. It was much later, with Einstein's general relativity, that it was shown that the geometry of the universe is not Euclidean but curved.

But the greatest crisis and the eventual overthrow of Platonism occurred in the mathematical revolution[2] which happened between 1870 and 1935.

The Crisis of Foundations

The crisis took place in many parts of mathematics but was particularly acute in set theory and logic. In 1874 Cantor developed a theory called *transfinite arithmetic* which dealt with the arithmetic of infinite sets. It was based on the naive set theory now taught in schools, for example there is an axiom, the *axiom of comprehension*, which says: given a well-defined property P there is a set S which consists of all objects which satisfy the property P. This is written

$$S = \{x : P(x)\}.$$

Thus the set of all even numbers and the set of all living creatures exist, provided we can always tell whether an object has these properties or not. In this way, sets were also admitted to the Platonist realm.

Cantor's theory was an immensely powerful theory and is still the subject of much research today. However, it led to a number of paradoxes the most famous of which is called Russell's paradox. Russell defines a property P which a set may or may not have. $P(S)$ holds if S *is not an element of itself.* The set of even numbers, for example, satisfies this property, because the set of even numbers is not, itself, an even number. Most sets in fact have this property, and it is always possible to tell whether a given set satisfies the property or not. So there must be a set R where

$$R = \{x : P(x)\}.$$

Now the paradox. We ask whether R is a member of R. The only objects which belong to R are those which do not belong to themselves, so if R belongs to R then R does not belong to R. Conversely, if R does not belong to R then it follows that R belongs to R.

The conclusion must be drawn that despite having a perfectly clear definition, the set R does not exist. The axiom of comprehension leads to an inconsistency. This paradox marked a major set-back particularly for Frege's Logicist project (below), but led to fruitful developments in axiomatic set theory. It might be added, in defence of Cantor, that he was aware of contradictions like this and was usually very careful in his formulations of the axioms, but others, including Frege, certainly adopted the axiom of comprehension.

The crisis in Logic was equally profound, initiated by the work of Frege, who developed the modern system of formal logic. (Frege's logic, now referred to as the predicate logic was a great advance of the traditional Aristotelian *syllogism.* Indeed it could be shown that the syllogism was not universally valid, despite the fact that it had been accepted virtually without question, for over two thousand years [7]. Frege appeared to be unaware of this consequence of his work, or afraid to make it explicit.) Frege started a project, the

[2] For a good debate as to whether revolutions are possible in mathematics read [8].

so-called *Logicist* project which attempted to eliminate all the unjustifiable assumptions in mathematics and to reduce it to pure logic. Here any theorem in mathematics was to be proved using the simple rules of logic from axioms that were self evident truths. The study of set theory was particularly important again, because numbers were defined as sets[3] and the theorems of arithmetic were to be derived from some axioms of set theory. Frege wrote

> The firmest method of proof is obviously the purely logical one, which, disregarding the particular characteristics of things, is based solely upon the laws on which all knowledge rests. (Preface to [5] page 103)

He then continues, in the same article, to attempt to demonstrate that arithmetic and probably geometry, differential and integral calculus can be handled by this very rigorous method of deduction. To quote Frege again, 'arithmetic is a branch of logic and need not borrow any ground of proof whatever from experience or intuition.' This project was taken up by the English philosophers Russell and Whitehead, who produced a massive tome *Principia Mathematica* which tried to prove all the important theorems of mathematics in formal logic. This was an attempt to secure the foundations of Science by showing that mathematics dealt only with an unchanging and absolute truth.

This project failed, but it is interesting to see that it was not because of a philosophical attack, but a contradiction within the subject, in the language of logic and pure mathematics. The failure came in the form of the Incompleteness theorem of Kurt Gödel which can be stated roughly as follows.

THEOREM *In any consistent, recursively enumerable, formal logic sufficient for arithmetic there will be true statements for which there exists no proof.*

'Sufficient for arithmetic' means that the language should have symbols for addition and multiplication together with appropriate axioms for these. The theorem demonstrates the inadequacy of formal logic because no formal system is capable of proving all the true formulas.

Work related to this also showed that basic notions, like 'set' or 'number' cannot be identified by a formal definition. This is because for any axiomatisation of set theory (for example the Zermelo-Fraenkel axioms) apart from the *intuitive* model of set theory there will also be *non-standard* models satisfying the axioms but either including as sets objects which you had not intended to count as sets, or missing out some sets which you wanted included. The philosophical implications of this turn out to be far-reaching [18]. (Incidentally, this philosophical problem about non-standard interpretations of the axioms has been put to good use in the subject known as *Non-Standard Analysis* which uses models containing, as well as all the ordinary real numbers, infinitely large and infinitesimally small numbers. So, for example in calculus the differential $\frac{dy}{dx}$ is usually defined to be *limit* of a sequence of fractions $\frac{\delta y}{\delta x}$ as δx tends to zero. The concept of a limit can be handled

[3]The number 0 can be defined as the empty set $\emptyset$. 1 is a set with a single element $\{\emptyset\}$ and in general $n+1=\{0,1,2,...,n\}$. In this system all the sets are built up from the empty set using more and more brackets, but this is enough to do arithmetic.

rigorously, but it is not entirely straight-forward. In non-standard analysis dx is taken to be an infinitesimal, i.e. a non-standard real number, smaller than all the 'standard' positive reals but greater than zero. $\frac{dy}{dx}$ is then just a non-standard fraction and the 'standard part' of $\frac{dy}{dx}$ is defined to be the differential. In this way we obtain a mathematical respectability to Leibnitz's notion of the infinitesimal,or *fluxion*, in calculus.)

A vast collection of new problems arose: the undecidability of the halting problem in algorithms, the independence of the axioms of set theory, even the undecidability of the consistency of arithmetic.

Suffice it to say that the foundations were no longer considered to be quite so solid. It is perhaps worth adding that as far as formal logic is concerned it is possible to reconstruct a Platonist framework for mathematics while avoiding the contradictions [2]. It is rather like the struggle between the Ptolomeic and the Galilean view of the solar system. Given enough cycles and epi-cycles the Ptolomeic system could explain everything that the Galilean system could. But the latter system benefited from its relative simplicity and turned out to be a better framework for later developments like Kepler's elliptical motion and eventually Newton's theory of gravitation. Similarly, in the philosophy of mathematics the Platonist model lost its authority partly because it became increasingly complicated to defend but, equally important, because a revolutionary momentum built up in which attempts to defend the old system were largely ignored. With the foundations of mathematics and logic cracked open, Logicism was defeated and the Platonist philosophy was also weakened. But what was to be put in its place?

Logicism, Formalism, Intuitionism

The three main branches in the philosophy of mathematics in the twentieth century are sketched in outline. The first of these, Logicism, has already been mentioned and was terminated abruptly by the work of Gödel (above). The second tendency appeared more radical: the formalists denied all meaning to mathematics and claimed that mathematics was only about symbols, rules for manipulating the symbols and the (symbolic) results which could be obtained by this manipulation. For them, questions about the truth or meaning of mathematical formulas were nonsensical. One could only say that a formula followed from certain other formulas using an agreed set of rules of derivation. Hilbert, the chief architect of Formalism, said

> Still it is consistent with our finitary viewpoint to deny any meaning to logical symbols, just as we denied meaning to mathematical symbols, and to declare that the formulas of the logical calculus are ideal statements which mean nothing in themselves. [11]

Formalists have no explanation of the tremendous power of mathematics applied to physical science. If mathematics has no meaning then why are the predictions of, say, quantum electrodynamics accurate to over twenty decimal places? Why is the correspondence between science and the world so good? Furthermore, however much the formalists deny any meaning to the symbols and formulas they use, in practice they appear to be highly influenced by the implications and meanings of their choice of language, axioms and inference rules. For the most part the language they adopt is not of random hieroglyphs but the language of arithmetic, set-theory and logic. And after the discovery of the paradoxes in set-theory the formalists too removed the axiom of comprehension from their axiom system.

The intuitionists have a different notion of truth. To them, a proposition can be said to be true only if someone has rigorously demonstrated its truth. Undecided questions, like the Goldbach conjecture, are neither true nor false. Intuitionists do not accept the law of the excluded middle in logic which says that every proposition is either true or false. Thus a standard technique in mathematics, proof by contradiction, is not allowed by them. One version of proof by contradiction works like this: in order to prove a formula ϕ first assume that ϕ is false. From this assumption deduce a contradiction (like $1 = 2$). If this can be done then infer that the assumption, of ϕ being false, does not hold and therefore ϕ is true. The intuitionists, however, will not accept this type of argument.

Intuitionism differs from standard mathematics in that an object may be said to exist only if an actual construction of it can be demonstrated. Thus irrational numbers like $\sqrt{2}$ and π are fine because there are algorithms which calculate their decimal expansions to any required accuracy. But consider the following argument, due to Cantor, about the cardinality of the set of irrational numbers. Cantor showed that it is possible to compare the cardinalities even of infinite sets. The counting numbers, for example, have the same cardinality as the set of fractions because there is a way of matching one counting number with one fraction so that none are counted twice and there are none left over. However Cantor showed, using proof by contradiction, that the set of real numbers has a strictly larger cardinality because there is no way of matching the counting numbers with the reals. Sets like the reals or the irrationals are called *uncountable*. However, this argument does not satisfy the intuitionist. They do not accept that such sets exist, they want to see them actually constructed.

Intuitionism is a form of *subjectivism* because mathematics is seen as a construction of individual humans. This is certainly true of Brouwer's exposition, though other intuitionists do present more sophisticated versions. Here mathematics is not a process of *discovery* (as in Platonism) but of *invention*. The complex number $\sqrt{-1}$ was not discovered but constructed, as indeed were all other numbers. Pythagoras' theorem only became true after Pythagoras gave the proof.

Frege (himself a Platonist) criticised this subjectivism saying:

> If the number two were a [subjective — RH] idea, then it would have straight away to be private to me only. Another man's idea is, *ex vi termini* [from the power of the boundary line], another idea. We should have to speak of my two and your two, of one two and all twos. ...and in the course of millennia these might evolve, for all we could tell, to such a pitch that two of them would make five. [6]

We can now counter-pose two views. At one extreme we have the Platonists who believe in the objectivity of mathematics. They regard mathematics as being independent of human consciousness, but go further and consider it to be independent even of our physical universe. To them the laws of mathematics and logic are valid not just in our world, but are necessarily true in any possible universe. At the other extreme we can place the intuitionists who argue that mathematics is a process of human invention. Maths is to do with the way humans think, it has nothing to do with objective truth or real properties of the world. An intuitionist might make a contrast between the *discovery* of the law of gravitation and the *invention* of the theory of calculus. All of these views deny a link between the meaning of mathematics and the objective properties of the universe we inhabit. So the problem facing all these approaches is to explain why mathematics is such a useful subject, why highly abstract theories often lead to powerful applications in the rest of science.

Marxist understanding of mathematics

To illuminate this problem of the nature of mathematical truth we must now turn to the Marxist approach. Here mathematics is not viewed as a purely objective reflection of the world nor as a purely subjective human construction but it is produced by a dialectic interaction between the material world and human conciousness: mathematics is based on human *experience*.

Marx [16] and Trotsky were both interested in mathematics but were unable to pursue it to any degree because of other, more pressing issues they faced. It was Engels who developed the materialist view of science furthest. Here we can find the beginnings of a materialist understanding of mathematics [4, 3].

To start with it must be said that many of the mathematical problems addressed by Engels in Anti-Dühring have been superseded by later developments in the subject. For example Engels' discussion of the infinite is only marginally less confused than that of Dühring and it was not until the work of Cantor that a rigorous treatment of infinite arithmetic was given. For example, Engels argues "... beginning and end necessarily belong together, like the North Pole and the South Pole, and that if the end is left out, the beginning just becomes the end — the one end which the series has; and vice versa. ... all mathematical series positive or negative, must start with 1, or they cannot be used for calculation. " ([3] page 63) However the negative integers form an infinite series with an end but no beginning, the positive integers have a beginning but no end, the integers have neither and in fact there is a vast collection of different order types of infinite sequences.

Infinitesimals, similarly, are of great interest to Engels, but cause much confusion. Consider the following quote ([3] p 175) "Therefore, $\frac{dy}{dx}$, the ratio between the differentials of x and y, is equal to $\frac{0}{0}$, but $\frac{0}{0}$ taken as the expression of $\frac{y}{x}$". This doesn't make much sense mathematically and the problem of infinitesimals had been fully and rigorously reduced to the treatment of finite numbers, some years *earlier*, by Weierstrass. Indeed Engels argues that until the development of the theory of infinitesimals in calculus, mathematics and logic were the only areas which could claim to deal with absolute and final truth. (Though we have already seen that even this part of science has an evolution, is conditional and needs to be considered dialectically.) But once this step beyond the finite was taken mathematics, too, fell from grace. However, this is not a helpful way of analysing the subject and is not in accordance with the state of mathematics today. These criticisms do not detract from the central points that Engels makes, which are sketched, in outline, as follows.

First on the objectivity of mathematics. Engels attacks Dühring's notion that mathematics is a free creation of the imagination independent of the world and our particular experience.

> The concepts of number and form have been derived from no source other than the world of reality. The ten fingers on which men learnt to count, that is, to carry out the first arithmetic operation, are anything but a free creation of the mind. Counting requires not only objects that can be counted, but also the ability to abstract from all properties of the objects being considered except their number — and this ability is the product of a long historical development based on experience. Like the concept of number, so the concept of form is derived exclusively from the external world and does not arise in the mind as a product of pure thought. There must have been things which had shape and whose shapes were compared before anyone could arrive at the concept of form ([3] page 47).

It should be added that mathematics does not always derive directly from nature, but can arise out of science and even from mathematics itself. Thus elementary arithmetic is abstracted from the very concrete task of counting and comparing physical objects (like fingers), group theory is abstracted from algebraic equations and geometric symmetries. Tensor theory owed its origin to the attempt by Riemann, in the mid-19th century, at solving the problem of unifying gravity and electromagnetism, a premature attempt which, nevertheless, led ultimately to Einstein's more successful work in relativity. In this way mathematics can become very abstract indeed. Nevertheless mathematics, like all other ideas, ultimately arises from experience.

Engels goes on to consider the strengths and limitations of mathematics in science. When we first reflect on nature and history

> we see the picture of an endless maze of connections and interactions, in which nothing remains what, where and as it was, but everything moves, changes, comes into being and passes away. ([3] page 24)

In order to try to make sense of this, the first task is that of *analysis* — the separation of the different components, their isolation in the laboratory, and the detailed study of one phenomenon while ignoring all others. That is the great achievement of natural science. Mathematics and formal logic are the highest abstractions of this process.

So, in the foundations of mathematics the basic elements reflect this analytic approach. In classical logic a proposition p possesses a definite truth value — true or false, there is never any doubt about it. And once p is made true (or false) it is fixed for all time. Similarly in set theory, if we have a set S, we can always tell whether an object x belongs to S or not and the membership of S is fixed.

But consider (as Engels does) the set of all living things. For everyday purposes (leaving aside the house of Lords) it is perfectly clear what this means, but there is a problem in accurately defining the boundary. At exactly what stage can we say that a complex protein is not just a chemical but alive? When do we say that a person is dead — when the heart stops, when there is brain-death, when rigor mortis sets in or what? Of course we can make a definition of 'being alive' but the definition will be arbitrary to some extent. Secondly, every living thing embodies a struggle for existence. A living organism may be in transition to death. The true proposition p is *becoming* false. So the set of living things is not fixed, its membership is constantly changing. Furthermore, as medical science progresses even the definitions and boundaries will evolve and get extended.

Set theory and classical logic thus fail to capture this process of transformation. So, it must be borne in mind that the properties and laws discovered by this analytic method are not absolute and final, but *contingent* on the very narrow conditions that were imposed on the experiment. The second task, therefore, is that of *synthesis*. Here it is the relations between things which concern us, the way in which a particular event fits into the totality. A property is no longer considered as definitely true or false, but dependent on other factors and therefore in a state of change. The abstraction of this wider view is *dialectics*.

It would be a mistake to oversimplify this point. It is not true that mathematics can only deal with static or deterministic events. The differential calculus, for example, is devoted to the study of change, while probability theory allows the handling of indeterminacy. Mathematics starts off as a rigid, static discipline but finds that this is inadequate to deal with a range of phenomena. The subject is forced to expand in order to deal with this, but it is not achieved without difficulty — it took 200 years before the methods used in the calculus were given a satisfactory justification.

But even at its most advanced level, the understanding of the world through mathematics involves the reduction of some process to a formal string of symbols (a formula). This might be a differential equation (for processes which change in time) or a stochastic matrix (for indeterminate processes) but the history, or the probability distribution, is fixed for all time by a single formula. This inevitably leads to problems when a qualitatively new process evolves. So in economics, for example, quite sophisticated models using differential equations and probabilities can take into account as many factors as you like and are run through a computer. The economists are always surprised when their predictions fail to materialise as the system moves into a new phase, governed by new laws. No doubt mathematics will evolve to take more of this type of development into account, but it always involves the reduction from the living world to a dead formula.

Dialectics provides the philosophical framework for reasoning about change and interaction and the process of synthesis.

Mathematics — the theory and the practice

From this viewpoint it is possible to answer an earlier question — is mathematics an experimental subject? Indeed it is, but many of the experiments are rudimentary and are repeated every generation by children using fingers or counters to discover and verify the elementary rules of arithmetic (e.g. $3 + 4 = 7$, $x \times y = y \times x$, etc.). As the mathematician G. H. Hardy put it

> The theory of numbers, more than any other branch of mathematics, began by being an experimental science. Its most famous theorems have all been conjectured, sometimes a hundred years or more before they were proved; and they have been suggested by the evidence of a mass of computations.

It is plausible that the basic laws of logic, geometry and probability are also developed, very early in a child's development, in a similar, experimental way.

Nevertheless, there is a contrast between mathematics and natural science. In the former the most elementary laws may be obtained on the basis of experiment, but an enormous edifice is built up from here and it is very rare for experiments to be performed at the higher level. With natural science, typically, a whole theory develops, often using mathematics in the exposition of the theory, but then predictions are made and experiments are done to test that theory. It is the testing in practice that ensures that science unveils an objective understanding even if, as we have seen already, that is only a partial understanding. *Indeed in the Marxist view the whole notion of objective truth can be defined as the correspondence between theory and practice.*

So in order for higher mathematics to claim objectivity there must be some correspondence between theory and practice. Now the *practice* of mathematics can be in its applications to other sciences, its applications to some other part of mathematics itself or to a range of human activities. The theory of the differential calculus, for example, has innumerable applications to almost every science and, indeed, was developed at least partly in response to problems in other fields. Group theory is applied more often to subjects within mathematics: number theory, geometry, topology, etc., but also to sub-atomic physics. Cryptography developed directly from a problem facing society: the military need to send and intercept secret messages during the second world war [12].

At its most abstract level mathematics deals with objects and structures which cannot correspond directly with any real practice. Infinite sets, for example, are unlikely to refer to any real objects or events. (If we assume a bounded universe and a quantisation of all the dimensions then infinite sets cannot correspond to any set of physical objects or events.) However the study of infinite sets is *useful* to mathematics, firstly because the language of infinite set theory allows us to handle finite objects quite neatly (e.g. when we talk of the set of all even numbers, or when we deal with irrational numbers like π by means of an infinite series of approximations 3.141...) and secondly because it clarifies the deductive method in an area where intuition is of very limited value. It is at this highly abstract level that the most profound contradictions have been found and where the understanding of truth and meaning are most problematic

As a final example of this consider Cantor's so-called *continuum hypothesis* which says that there is no set whose cardinality lies strictly between that of the integers and the real numbers. It has been shown that the continuum hypothesis cannot be proved from the other axioms, nor can it be proved false — it is *independent* of the axioms of set theory. Some mathematicians (e.g. Gödel) hoped that a new, intuitively obvious axiom would come to light and solve this type of problem, but this has not happened and most mathematicians now think that it never will. The Platonist has no problem with the continuum hypothesis: (s)he believes that it is definitely true (or definitely false). The human brain may be incapable of ever discovering the truth, but there *is* a truth. Intuitionists, too, are not troubled because they do not believe in the existence of the set of real numbers as a completed totality, as this cannot be constructed. Therefore, the continuum hypothesis is without meaning for them, or more precisely, the continuum hypothesis is trivially true for intuitionists since they do not accept the existence of the set of real numbers. The formalists could argue that the continuum hypothesis could be made true or false *at will.* To them, mathematics is a process of making formal definitions and conventions — there is no meaning attached to it.

The materialist should pause before saying anything rash, but then perhaps put the following case.

Elementary mathematics, and indeed the vast bulk of mathematics, derives more or less directly from experience. It is tested by its logical consistency moreover by its applications to a number of other subjects and activities. Therefore, when we say, for example, that $5 + 5 = 10$ or that there are infinitely many prime numbers, we refer to an *objective truth.* It does not depend on the individual who says it, nor on the type of society that made the discovery. At the most abstract level, though, mathematics does not correspond to any real process and so propositions like the continuum hypothesis cannot be construed as objective truths, but are indeed a matter of definition or convention. As Putnam put it

> Urging this relativism is not advocating *unbridled* relativism; I do not doubt that there are some objective (if evolving) canons of rationality; I simply doubt that we would regard them as settling this sort of question let alone as singling out *one* unique 'rationally acceptable set theory'. ([18] page 430.)

If it were the case that the majority of mathematics were of this very abstract type, and if mathematics were principally an internal subject with few connections to other sciences, then it would be very hard to argue that mathematics were scientific. However, neither of these conditions are met: a large part of mathematics deals directly with applications and even at the most abstract level the mathematician is always delighted if their work links in with some distantly related area.

Mathematics is a tool devised to help us solve certain problems. The suggestion that mathematics has to be useful in order to be scientific would be met with as much enthusiasm from pure mathematicians as the publication of the transcript of the Camilla tape in the national newspapers received from Prince Charles. Many pure mathematicians consider their work to exist in a rarefied level of the stratosphere, unsullied by the menial tasks facing other scientists — but there's no helping that!

It was the 19'th century French mathematician Poincaré who argued, correctly, that *Mathematics is a human construction but not an arbitrary construction.* A mathematical theory arises from experience — experience of the world, of science or experience of mathematics itself. Such a theory can be tested in two ways: (i) it is tested for *rational consistency* — this is why proof is so important in mathematics — but consistency is something which also arises from experience; (ii) it is tested by its *use* — a theory which does not solve any significant problem is quickly forgotten.

Mathematics, though a construction, refers to objective truth to the extent that it corresponds to the real world and helps to solve real problems.

How mathematics developed

In the materialist view, then, to understand what mathematics means it is necessary to look at its history and see how it fits in with the general development of society and, in particular, what it was used for. There have been a number of worthwhile attempts to map out such a history [13, 14, 19] not just as a history of ideas (as it is usually presented) but as one aspect of the history of class society. Here restrictions on space allow only some brief remarks about three of the more significant factors in that history: the rise of trade, problems of technology and contradictions internal to mathematics. As should be clear already the history of mathematics is a highly complex process and it is in no way suggested that these are the sole determining factors.

The rise of trade

Hunter-gatherer societies may have only counted as high as three (though the evidence is not conclusive), and we can assume that this was sufficient to allow them to perform the very simple tasks they faced. The first class societies, based on farming, needed to count their livestock and to measure the lengths of the seasons and so they could count much higher. But it may be that the rise of trade, right through history, has given the most significant impetus to the development of arithmetic, geometry and algebra. This is for three main reasons:

- The rise of trade and the inadequacy of simple barter has led to the creation of money and arithmetic [13, 6]. The latter was needed in order to quantify money and goods and generally to keep accounts. A more developed system of trade requires a more sophisticated system of mathematics.
- Trading societies are able to gain knowledge from a very wide variety of sources. The ancient Greeks, for example, borrowed their number system from the Egyptians and had learned from Babylonian mathematics.
- The difficult problem of navigating by the stars has led to the development of geometry. Related to this are problems of cartography and surveying. The

Alexandrians, like Archimedes (who was actually Sicilian, but studied in Alexandria), Euclid and Ptolomy, were able to make some very significant advances, like trigonometry, and made tolerably accurate estimates of the diameters of the Earth and Moon as well as the distance between them.

So it is not a coincidence that the great trading societies all made contributions to the advance of mathematics: the Sumerians and Phoenicians who started counting in groups; the Hebrews, Romans and Greeks who used abstract symbols for numbers; the Chinese who probably invented the abacus and may have discovered Pythagoras' theorem before him; the Mayans who invented the numeral 0; the Babylonians who had a place value system; the Hindus who invented fractions and the modern number system; the great Arabic trading empire of the medieval period that developed algebra (algebra first occurs in the works of al-Khwarizmi (*c.* 780–*c.* 850)) and algorithms for multiplication, division, etc.; Europe since the 17'th Century where, amongst other things, differential calculus was invented. In the twentieth century the enormous expansion of trade has meant that large companies require huge banks of the most up to date computers, expert statisticians, mathematicians and even artificial intelligence.

Technology

The drawing of scaled diagrams for the construction of buildings was clearly important for the early development of geometry. The Egyptian are known to have used Pythagorean triangles and a relatively advanced number system for the building of the pyramids, but the ideological function of the priesthood meant that this was kept secret and very little written evidence remains. By contrast the later Greek philosophers scorned all concern with practical problems, and so Platonism was quite a natural consequence of this. Hogben argues, perhaps taking the point too far, that it was because of this separation of theory and practice that Greek mathematics stagnated [13]. Interestingly, Kline [14] argues the opposite: it was the *abstract* nature of Greek mathematics that gave it such generality and power. Certainly it is accepted that the Greek contribution to mathematics was considerable, but when faced with paradoxes, like *Zeno's paradox* or the problem of irrational numbers mentioned earlier, instead of pushing mathematics forward, they retreated into philosophy and failed to solve the problems. It was the Alexandrians, with their practical interest in technology (like Archimedes pump, various war machines involving cog-wheels, catapults, etc.) who were able to make real advances in geometry (the circular function *sine* and *cosine*, an estimation of π correct to two decimal places) and arithmetic (where they solved Zeno's paradox and could sum infinite series). It seems the interplay between theoretic and practical considerations typical in mathematics, is more subtle that either of the two cited books acknowledges.

Hessen [10] convincingly demonstrates how Newtonian mechanics and calculus were required to solve very concrete technical problems. Mining, water transport, arms manufacture and the metal industry were crucial.

In the twentieth century the effect of technology on mathematics can be seen clearly in subjects as diverse as number theory, logic, complexity and chaos theory all of which have been closely connected to the development of computer technology. More generally science under capitalism (which was dependent on the knowledge of other societies, notably Hindu mathematics and the influx of Arabic science into Spain, Italy and Southern France from about the 12'th Century) has made an accelerating progress reflecting the development of technology and industry.

Internal Contradictions

As has already been said, mathematics does not simply deal with the practical problems facing society, it also learns from other sciences and other parts of mathematics. Examples of very fruitful developments caused by internal contradictions include the following: the formalisation of logic in ancient Greece caused by the discovery of irrational numbers; Descartes' synthesis of algebraic geometry (co-ordinate geometry) out of the previously unrelated subjects of algebra and geometry; axiomatic set theory following the discovery of paradoxes in set theory; Gödel's synthesis of arithmetic and logic etc. etc.

One final point to add: it is not just the *history* of mathematics that proceeds dialectically, even the *content* can be conceived in this way. In a brilliant essay [15] Lakatos shows that even the process of mathematical proof is an evolving and dialectic one. The mathematical proof of a particular result is shown to have an evolution caused by the tension between the rational deduction (proof) and the counter-example *both of which can exist simultaneously.*

Conclusion

Formal logic and dialectics complement each other. Formal logic is the abstraction from those sciences which deal with mechanical, deterministic processes in isolation from other processes. In a field where there is an absolute, unchanging truth, mathematics and formal logic are well suited. However, it should be remembered that this is always at best an approximation as all phenomena involve inter-relations with other processes and are in a state of change. And when we identify an objective truth in the world, this is only a partial truth and will generally depend on other contingent factors. Mathematics is good at explaining phenomena which have some invariant property or behave deterministically. For example, were it not for the fact that planetary motion is relatively stable over fairly long periods of time, then all the theories about circular or elliptic motion would not have made much sense of things. More fundamentally, mathematics has been so hugely successful precisely because the first stage of understanding must involve the separation of the components, the study of the detail — in short *analysis* — and mathematics has evolved to fit this process.

Dialectical materialism, on the other hand, is the abstraction from those sciences which consider all the connections between different events and understands how new processes are constantly arising. These new processes may involve qualitative leaps and require new theories and laws to explain them. Dialectics is not magic[4] though, and does not tell us what those theories are. To discover these requires thorough investigation and scientific research.

The philosophy of mathematics is not the most vital issue facing Marxists today, but its clarification can help us argue that the materialist framework is the correct one for making sense of every aspect of the world. We can counter the arguments of those who say that objective truth is to be found only in the 'hard' sciences and that trying to make sense of history and society today leads only to subjective opinion and waffle. No indeed, mathematics is an enormously powerful tool and a beautifully elegant subject in its own right, but it is not the foundation of all knowledge and it is not the only tool.

It is too wasteful keeping mathematics in its ivory tower where only a small elite are granted the privilege of access. It deprives those outside of a powerful technique for understanding and, hence, changing the world. And it deprives mathematics of a vast pool of experience and intelligence which could raise the subject to hitherto unexplored heights.

[4] One who appeared to believe in such magic powers was Hegel: in 1800 he argued, on the basis of philosophy alone, that the number of planets could not be other than seven. It was simply bad luck that in the following year the eighth planet, Ceres, was discovered.

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