Relation Algebras of Intervals

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Abstract

Given a representation of a relation algebra we construct relation algebras of *pairs* and of *intervals*. If the representation happens to be *complete, homogeneous* and *fully universal* then the pair and interval algebras can be constructed direct from the relation algebra. If, further, the original relation algebra is ω-categorical we show that the interval algebra is too. The complexity of relation algebras is studied and it is shown that every pair algebra with infinite representations is intractable. Applications include constructing an interval algebra that combines metric and interval expressivity.

1 Introduction

There has been considerable interest in reasoning systems that can handle intervals, particularly for temporal reasoning. For many applications it turns out that formalisms based on points lack the expressive power required to describe the situation adequately. Using intervals instead of points as the basic entities significantly increases the expressive power but, in general, involves a loss of tractability. Interval reasoning is important in all those applications that involve interfering processors, multi-agents or interactions with the environment. The application might require us to say that "one process takes place *while* another property holds" or "two actions have *disjoint* duration". One of the most powerful algebraic tools for temporal reasoning is relation algebra. This has given some very general results about the decidability and completeness of systems of binary relations (for a good survey see [Ń91], see also [Mon64, Lyn50, Lyn56, And89]) and might also be useful for considering questions of complexity. A background knowledge in relation algebra is certainly an advantage when reading this paper, though terms are defined as they are introduced. Background reading in relation algebra includes, amongst many others, the previously cited works and [JT48, Tar55, LM87, Mad89]. A good history of the study of relation algebras may be found in [Mad91].

The idea in this paper is to see how relation algebras can be used to handle interval reasoning.

Section 2 gives the basic definitions for relation algebras and their representations together with some properties of representations. In section 4 we show how to take a relation algebra - intended to consist of binary relations on *points* - and build *pair* and *interval algebras* from it. In section 5 it is shown that a pair or interval algebra is ω categorical if the original point algebra is. In section 6 we show that virtually all pair algebras are intractable. A number of concepts from model theory are used in this construction, like homogeneity and universality, but they are defined in the text.

Although it has a somewhat theoretical flavour, this work is very applicable. A number of attempts have been made to combine qualitative interval reasoning with quantitative metric expressivity. Section 7 starts from a point-based metric system and gives a construction of an interval algebra which achieves that combination and has some advantage over its competitors.

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2 Basics

Definitions

- A *proper relation algebra* is a set of binary relations over some domain, closed under the boolean operations, converse, composition and containing the identity relation.
- A *relation algebra* $\mathcal{A}$ is a tuple $(A, \vee, -, 1, 0, Id, \smile, *)$ which obeys the Tarski axioms [JT48].
- A relation algebra is called *simple* if for all $0 \neq a \in \mathcal{A}$ we have $1 * a * 1 = 1$. Any relation algebra can be decomposed into a sum of simple ones.
- An *integral* relation algebra satisfies $1 * a = a * 1 = 1$ for all non-zero $a \in \mathcal{A}$.
- A *representation* X of $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to a proper relation algebra. The element Id must be mapped to the identity $\{(d, d) : d \in D\}$ (D is the domain of the reresentation), $\smile, *$ are interpretted as converse and composition respectively.

  Where there is no confusion we use the same letter X to stand for the isomorphism and the domain of the representation, thus $x \in X$ means that x is a point in the domain of the representation. For a simple relation algebra the unit 1 always gets represented as a sum of disjoint, complete graphs each of which is a representation on its own. In this paper we only consider simple relation algebras and assume that a representation consists of a single component i.e. for any pair of points in the representation the unit relation holds between them.
- An *atomic representation* has the further property that for any two points x_1, x_2 in the domain of the representation there is a unique atomic relation from $\mathcal{A}$ that holds between them. It can be proved that for atomic relation algebras a representation is atomic if and only if it is *complete* - i.e. arbitrary unions are preserved. For finite relation algebras every representation is a complete representation.
- An *atomic $\mathcal{A}$-network* N is a finite directed graph with each edge (m, n) labelled by an atom of $\mathcal{A}$, say $N(m, n)$ and *transitively closed*: for any three nodes l, m, n of N we have

  $$N(l, m) * N(m, n) \geq N(l, n)$$

  A general *$\mathcal{A}$-network* is defined similarly, but it is no longer assumed that each edge is labelled with a single atomic relation. The label on an edge should, however, be restricted to recursively definable disjuncts i.e. $N(n, m)$ is always a recursive union of atoms[2].

Fact Not every relation algebra has a representation [Lyn50] and indeed there is no finite set of axioms which characterises the representable relation algebras [Mon64].

- An *automorphism* θ of the representation is a permuation of the representation preserving all the relations i.e. $\forall x, y \in X, \forall a \in \mathcal{A}\ (x, y) \models a \Leftrightarrow (x\theta, y\theta) \models a$.
- A *local isomorphism* h of a representation is a finite map $h : \bar{x} \to \bar{y}$ for some tuples $\bar{x}, \bar{y}$ in the representation , preserving all the relations that hold between each pair of points.
- A representation is said to be *homogeneous* if every local isomorphism extends to a full automorphism of the representation.
- A representation X is called *universal* if it has the following property: for all atomic networks N if N embeds in any representation of $\mathcal{A}$ then it embeds in N. If *all* atomic $\mathcal{A}$-networks embed in the representation it is called *fully universal.*

[2] It is possible to construct a first-order language $L(\mathcal{A})$ with one binary relation symbol for each element of $\mathcal{A}$ and a first-order theory $Th(\mathcal{A})$ whose models are exactly the representations of $\mathcal{A}$. If the networks have only finite disjunct of atoms on the edges then each network is equivalent to a certain first-order existential sentence and the network embeds in some representation if and only if the sentence is consistent with $Th(\mathcal{A})$.

Definition A representation is *normal* if it is complete, fully universal and homogeneous. In [Hir94] it is shown that a relation algebra has a normal representation if and only if its atomic networks form an *amalgamation class* but the concept of amalgamation is not needed here.

Examples

- Let $\mathcal{P}$ be the 'point algebra' consisting of three atoms $Id, <, >$ with $<^{\smile} = >$ and composition defined by $< * < = <$, $< * > = 1$. It follows from this composition table that any representation of $\mathcal{P}$ must be a dense linear order without endpoints. So any countable representation is isomorphic to the rationals with their usual ordering. This representation turns out to be normal. To show homogeneity, let ρ be any local isomorphism i.e. a finite order preserving partial map from $\mathbb{Q}$ to $\mathbb{Q}$. Use a back and forth construction to extend ρ to a full automorphism. That $\mathbb{Q}$ is fully universal follows from the fact that any atomic $\mathcal{P}$-network is effectively a finite linear order and therefore embeds in $\mathbb{Q}$.
- The Allen interval algebra $\mathcal{I}$ has thirteen atoms $Id, <$, meets, overlaps, starts, during, ends plus the converses of the last six. The composition table can be found in [All83]. We will see later that it has only one countable representation namely ordered pairs of rationals (p, q) with p less than q. This representation also turns out to be normal.
- The metric point system $\mathcal{M}$ of [DMP91] consists of finite unions of basic relations $R(i)$ where i is a real interval with rational endpoints (open, closed or semi-open). The identity is $R([0, 0])$, complement is given by[3] $-R([p, q]) = R((-\infty, p)) \vee R((q, \infty))$, converse is $R([p, q])^{\smile} = R([-q, -p])$ and composition $R([p, q]) * R([r, s]) = R([p + r, q + s])$. A representation of $\mathcal{M}$ can be constructed on either the real numbers or the rationals by letting $(a, b) \in R(i)$ if and only if the distance $b - a$ belongs to the interval i. The representation based on the reals is not atomic (equivalently not complete) as the pair $(0, \pi)$ are not related by any atom, but the representation based on the rationals is a complete representation.

3 Intervals

3.1 The Idea

There seem to be two views [vB83] of intervals: they may be considered as *convex sets* of points or simply as *ordered pairs* of points (the endpoints of the interval). Here we take the latter approach. First we take a representation X of a relation algebra $\mathcal{A}$ and build a *representation of pairs* X^2 whose domain includes all pairs i.e.[4] $X \times X$. If the relation algebra happens to have a normal representation X then we give a simple construction for an *algebra of pairs* $\mathcal{A}^2$ directly from $\mathcal{A}$ - independent of its representations.

Next, we fix one atomic relation r and define an r-interval (or simply interval) (x, y) to be a pair related by r. The exemplary case is $<$ in the point algebra $\mathcal{P}$ where an interval is a pair (x, y) with $x < y$. The representation of pairs X^2 contains a *representation of intervals* X^2_r whose domain is the set of all r-intervals. Also $\mathcal{A}^2$ contains a subalgebra $\mathcal{A}^2_r$ called an *interval algebra.* Again, $\mathcal{A}^2_r$ is calculated direct from $\mathcal{A}$ and is built out of certain two by two matrices of atoms of $\mathcal{A}$. In this way we can build the Allen Interval Algebra from the point algebra $\mathcal{P}$.

3.2 Representations

Notation Let X be a complete representation. For any pair of points $x, y \in X$ let $\overleftarrow{X}(x, y)$ denote the unique atom that holds between x and y in X. If $\bar{x}$ is a tuple of points in X, let $\overleftarrow{X}(\bar{x})$ denote the atomic network with $\bar{x}$ as its set of nodes and each edge (x_i, x_j) labelled by the atom $\overleftarrow{X}(x_i, x_j)$.

[3] With similar definitions for open and semi-open intervals.

[4] Recall that we use the same letter X to denote the representation and the domain of the representation.

We want to define some binary relations on pairs from X. Which binary relations are naturally definable from X? Well, two pairs $\bar{\alpha} = (\alpha_1, \alpha_2)$, $\bar{\beta} = (\beta_1, \beta_2)$ define a quadrangle of atomic relations $\overleftarrow{X}(\bar{\alpha}, \bar{\beta})$. So, for each transitively closed atomic four-network N (N has four nodes) we can define a binary relation R_N on pairs by

$$R_N(\bar{\alpha}, \bar{\beta}) \Leftrightarrow \overleftarrow{X}(\bar{\alpha}, \bar{\beta}) \cong N$$

Taking all such N it is possible to form a proper relation algebra of binary relations on pairs from X by closing under unions, complement relative to $X \times X$ and composition. But note that if X is not fully universal then some networks R_N may be the zero relation. Also, if X is not homogeneous then the non-zero relations R_N may not be atomic as illustrated by the following example.

Example Let $\mathcal{A}$ be the relation algebra with atoms Id, e, d, all self-converse, and composition defined by

$*$	Id	e	d
Id	Id	e	d
e	e	$Id \vee e$	d
d	d	d	1

In any representation X of $\mathcal{A}$ the relation $Id \vee e$ is an equivalence relation and partitions X into at least three clusters each of size at least three. Two distinct elements are related by e if they are in the same cluster and by d otherwise. A normal representation of $\mathcal{A}$ consists of infinitely many infinite clusters. But if all the clusters have size three then the following network N defines the zero relation.

Now consider a representation X where all the clusters have size four or six. This is not homogeneous as mapping the four elements of a four-cluster into a six-cluster is a local isomorphism that cannot extend to an automorphism. Consider the relation $R_N \cap (R_N * R_N)$, where N is the network in the diagram above. This can only hold on a distinct pair of pairs taken from a cluster of size six. Therefore R_N is not an atom.

Later, we will consider normal representations so these problems will not arise. But for an arbitrary representation X what would be a natural choice for the atoms of a proper relation algebra of binary relations on pairs from X?

Definitions Let X be a representation of $\mathcal{A}$.

1. The *representation of pairs* X^2 is a proper relation algebra with domain $X \times X$. The atoms of X^2 are the orbits of pairs of pairs from X under the automorphism group on X. So two pairs $(\bar{x}, \bar{y}) = ((x_1, x_2), (y_1, y_2))$ are related by the same atomic relation as two pairs $(\bar{w}, \bar{z})$ if and only if there is an automorphism of X sending $\bar{x}, \bar{y}$ to $\bar{w}, \bar{z}$ respectively. (In the previous example the element R_N split into two atoms: those where the two distinct pairs lie in a cluster of size four and those which lie in a cluster of size six.) It is simple to check that the converse of an atom is an atom; that the identity is a union of atoms and the composition of two atoms is a union of atoms. This follows from the equations

$$(\bar{x_1}, \bar{y_1}) \sim (\bar{x_2}, \bar{y_2}) \quad \Rightarrow \quad (\bar{y_1}, \bar{x_1}) \sim (\bar{y_2}, \bar{x_2})$$

$$\begin{aligned}(\bar{x}, \bar{x}) \sim (\bar{y}, \bar{z}) \quad &\Rightarrow \quad \bar{y} = \bar{z} \\ [x, z] \quad &\leq \quad [x, y] * [y, z]\end{aligned}$$

where $\bar{u} \sim \bar{v}$ denotes the existence of an automorphism sending $\bar{u}$ to $\bar{v}$ and $[\bar{u}]$ is the $\sim$-class of $\bar{u}$. Hence X_r^2 is a proper relation algebra. X^2 may then be defined as all unions of these atoms (alternately it could be defined as only the finite and co-finite unions). The unit of X^2 is $X \times X$. Thus the structure of X^2 derives naturally from X.

2. The *representation of intervals* X_r^2 is a proper relation algebra with domain $X(r)$ i.e. all pairs from X related by the atom r. The atoms of X_r^2 are the atoms of X^2 restricted to r-intervals, so a pair of intervals (i, j) are related by the same atom as two intervals (k, l) if and only if there is an automorphism of X sending i, j to k, l respectively. The unit of X_r^2 is $X(r) \times X(r)$.

4 Pair and Interval Algebras

If the representation X in the previous construction is not homogeneous then X^2, X_r^2 can both be considerably bigger than X. More importantly, they depend heavily on the representation X. In this section we assume that $\mathcal{A}$ has a normal representation. This will yield a simpler construction of a pair algebra $\mathcal{A}^2$ and an interval algebra $\mathcal{A}_r^2$ which can be constructed direct from the relation algebra, independent of its representations.

4.1 Pairs

Definitions

- Let $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$ be any two pairs. Since X is homogeneous the atomic relation in X^2 that holds on a pair of pairs is determined by the isomorphism type of the atomic network defined by the four points. So for *any* pair of pairs $(\bar{u}, \bar{v})$ such that $\overleftarrow{X}(\bar{u}, \bar{v}) \cong \overleftarrow{X}(\bar{x}, \bar{y})$ we have $(\bar{u}, \bar{v}) \sim (\bar{x}, \bar{y})$.

- The *atomic pair relations* of $\mathcal{A}^2$ are the set of all isomorphism classes of atomic networks of size four. Because X is fully universal all the atomic pair relations embed in X^2 and homogeneity guarantees that the occurrences of an atomic pair relation in X^2 form an atom of X^2.

- It is convenient to denote the isomorphism type of the atomic network

  as $A_{r,s} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{r,s}$. This atomic pair relation relates an r-interval $\bar{\alpha} = (\alpha_1, \alpha_2)$ to an s-interval $\bar{\beta}$ and the endpoints are related by a_{11}, a_{12}, a_{21} and a_{22} as in the diagram[5]. Thus $\overleftarrow{X}(\alpha_i, \beta_j) = a_{ij}$ $(i, j = 1, 2)$.

[5] Although it is true that converse can be calculated by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{R,S}^{\smile} = \begin{bmatrix} A^{\smile} & C^{\smile} \\ B^{\smile} & D^{\smile} \end{bmatrix}_{S,R}$ you have to be careful about composition. The composition of two relations of this type cannot be calucated by matrix multiplication, but is defined as the union of all the products of atomic pair relations contained within the respective relations.

- We can use this notation with non-atomic entries. Thus
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{R,S} =_{\text{def}} \bigcup\{\text{atomic pair relations } \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{r,s} : a \le A, b \le B, \ldots, s \le S\}$$
- We can now define the structure $\mathcal{A}^2$. It consists of unions of the atomic pair relations and the identity, converse and composition are defined by
$$\begin{aligned} Id_{\mathcal{A}^2} &= \begin{bmatrix} Id_{\mathcal{A}} & 1_{\mathcal{A}} \\ 1_{\mathcal{A}} & Id_{\mathcal{A}} \end{bmatrix}_{1_{\mathcal{A}},1_{\mathcal{A}}} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{r,s}^{\smile} &= \begin{bmatrix} a^{\smile} & c^{\smile} \\ b^{\smile} & d^{\smile} \end{bmatrix}_{s,r} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{r,s} * \begin{bmatrix} e & f \\ g & h \end{bmatrix}_{t,u} &= \begin{cases} \begin{bmatrix} a*e \wedge b*g & a*f \wedge b*h \\ c*e \wedge d*h & c*f \wedge d*h \end{bmatrix}_{r,u} & \text{if } s=t \\ 0 & \text{if } s \neq t \end{cases} \end{aligned}$$
The composition is ordinary matrix multiplication with $\wedge, *$ in place of $+, \times$ respectively plus a test for matching subscripts.

THEOREM 1 *Let X be a normal representation*[6] *of $\mathcal{A}$. Then $\mathcal{A}^2$ forms a relation algebra and X^2 is a normal representation of it.*

PROOF:

There is a potential problem because there is no distribution law for intersection ($a*(b\cap c) = a*b \,\cap\, a*c$ is not generally true in relation algebras) and consequently the matrix multiplication may not be associative. But the theorem can be proved using the fact that X^2 is a proper relation algebra, by showing that there is a natural isomorphism Ξ from $\mathcal{A}^2$ to X^2.

For each atomic pair relation M_{rs} let $\Xi(M_{rs})$ be the set of pairs of pairs $(\bar{\alpha}, \bar{\beta})$ such that the atomic network formed by the four points is isomorphic to M_{rs} that is $\overleftarrow{X}(\bar{\alpha}, \bar{\beta}) \cong M_{rs}$. Ξ can be extended to non-atomic relations so that it preserves unions. Since X is fully universal, every atomic network of size four embeds in X so $\Xi(M_{rs}) = 0$ cannot happen. Thus Ξ is injective and it is easy to see that it is surjective too. It is not hard to check that identity and converse are preserved. The crucial argument is to show that Ξ preserves composition.

LEMMA 2 *Let X be a normal representation of $\mathcal{A}$, let M_{rs} and N_{ut} be atomic pair relations and let $\bar{\alpha} = (\alpha_1, \alpha_2)$ and $\bar{\beta} = (\beta_1, \beta_2)$ be any pairs in X. Then $(\bar{\alpha}, \bar{\beta}) \models M_{rs} * N_{ut}$ (the product defined above) iff there is some pair $\bar{\gamma}$ such that $(\bar{\alpha}, \bar{\gamma}) \models M_{rs}$ and $(\bar{\gamma}, \bar{\beta}) \models N_{ut}$.*

Proof of Lemma

There are two cases to check: $s \neq t$ and $s = t$. First let $s \neq t$. For all pairs $\bar{\alpha}$ and $\bar{\beta}$
$$\begin{aligned} &\exists\bar{\gamma} : (\bar{\alpha}, \bar{\gamma}) \models M_{rs} \wedge (\bar{\gamma}, \bar{\beta}) \models N_{tu} \\ &\Leftrightarrow \ \bar{\gamma} \text{ is an } s\text{-interval and } \bar{\gamma} \text{ is a } t\text{-interval} \\ &\Leftrightarrow \ \perp \\ &\Leftrightarrow \ (\bar{\alpha}, \bar{\beta}) \models M_{rs} * N_{tu} = 0 \end{aligned}$$
Now let $s = t$. The proof from right to left comes first i.e.
$$(\bar{\alpha}, \bar{\beta}) \models M_{rs} * N_{su} \Leftarrow \exists\bar{\gamma} \text{ with } (\bar{\alpha}, \bar{\gamma}) \models M_{rs} \wedge (\bar{\gamma}, \bar{\beta}) \models N_{su}$$

[6] In the proof it will be sufficient that X is universal over the class of all atomic networks of size six. Similarly we need only require that local isomorphisms of size four extend to full automorphisms - full universality and homogeneity are not strictly necessary.

This implication always holds in any representation. $(\bar{\alpha}, \bar{\gamma}) \models M_{rs}$ means that $\bar{\alpha}$ is an r-interval and $\bar{\gamma}$ is an s-interval and that $(\alpha_i, \bar{\gamma}_j) \models M(ij)$ $(i, j = 1, 2)$ - the (i, j)'th entry in the two by two matrix M. Similarly $\bar{\beta}$ is a u-interval and $(\gamma_j, \beta_k) \models N(jk)$ $(j, k = 1, 2)$. Considering the triangle $(\alpha_i, \gamma_j, \beta_k)$ in the representation X (see diagram below), it must be the case that $(\alpha_i, \beta_k) \models M(ij) * N(jk)$ $(i, j, k = 1, 2)$.

Therefore $(\alpha_i, \beta_k) \models M(i1) * N(1k) \cap M(i2) * N(2k)$ which is the entry calculated by the matrix product. Hence $(\bar{\alpha}, \bar{\beta}) \models M_{rs} * N_{su}$.

Now to prove the implication from left to right: suppose; $(\bar{\alpha}, \bar{\beta}) \models M_{rs} * N_{su}$. Let the atomic pair relation between $\bar{\alpha}$ and $\bar{\beta}$ be T_{ru} $(T_{ru} \subseteq M_{rs} * N_{su})$. Since M_{rs} is a pair relation, the atomic network

is consistent. By considering each triangle, the following atomic network is transitively closed.

Universality guarantees that this network embeds in the representation and homogeneity ensures that the local isomorphism from this occurrence to extends to a full automorphism of the representation. Therefore there is an s-interval $\bar{\gamma}$ such that $(\bar{\alpha}, \bar{\gamma}) \models M_{rs}$ and $(\bar{\gamma}, \bar{\beta}) \models N_{su}$ as required.

Proof of Theorem (continued).
We have shown that $\mathcal{A}^2$ is isomorphic to X^2 if X is normal. Therefore $\mathcal{A}^2$ is a relation algebra and X^2 is a representation of it. It remains to show that X^2 is itself normal. Clearly X^2 is atomic as any pair of pairs in X^2 is related by an atomic pair relation - this follows from the completeness of X. It is easy to show that it is fully universal by

1. taking any atomic pair network (the nodes of the network represent pairs)
2. converting to an atomic network of points by splitting each node into two and putting the appropriate atoms on the edges
3. using the universality of X to find the network in X and
4. converting back to X^2 thus showing that the original network embeds in X^2.

Homogeneity is handled in much the same way.
$\square$

4.2 Intervals

Let r be any atom, X a representation of $\mathcal{A}$. We have defined the interval algebra X^2_r to be the proper relation algebra formed by restricting the proper relation algebra X^2 to a domain consisting only of r-intervals. So X^2_r consists of binary relations on r-intervals. If $\mathcal{A}$ has a normal representation then the pair algebra $\mathcal{A}^2$ can be constructed. Now we will show that $\mathcal{A}^2$ contains a subalgebra $\mathcal{A}^2_r$ and that X^2_r is a normal representation of it.

Definition An *equivalence element* $e \in \mathcal{A}$ satisfies $e = e * e = e^{\smile}$.

An equivalence element e defines a subalgebra $\mathcal{A}_e$ of $\mathcal{A}$ consisting of all elements $a \leq e$. Union, converse and composition are unchanged but the identity is $Id_{\mathcal{A}} \wedge e$, the unit is e and complement is relative to e.

Now $\mathcal{A}^2$ contains an equivalence element $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{r,r}$. $\mathcal{A}^2_r$ is defined to be the subalgebra contained in this equivalence element and consists of all elements of $\mathcal{A}^2$ indexed by r, r. Under the representation X^2 of $\mathcal{A}^2$ any element of $\mathcal{A}^2_r$ is represented as a binary relation on r-intervals. Since the indices are now fixed we do not need to write them down explicitly any more. So the *atomic interval relations* are all the two by two matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\begin{aligned} a * c^{\smile} &\geq r \\ b * d^{\smile} &\geq r \\ a^{\smile} * b &\geq r \\ c^{\smile} * d &\geq r. \end{aligned}$$

In other words the network below must be transitively closed.

As with pair algebras we can use the same matrix notation with non-atomic entries to denote the union of all the atomic interval relations contained in it.

THEOREM 3 *Let $\mathcal{A}$ be a relation algebra with a normal representation X.*

1. *$\mathcal{A}_r^2$ forms a relation algebra. The identity relation is* $\begin{bmatrix} 1_{\mathcal{A}} & r \\ r^{\smile} & 1_{\mathcal{A}} \end{bmatrix}$ *and the converse*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{\smile} = \begin{bmatrix} a_{11}{}^{\smile} & a_{21}{}^{\smile} \\ a_{12}{}^{\smile} & a_{22}{}^{\smile} \end{bmatrix}.$$

The composition of any two atomic interval relations is calculated thus:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11} * b_{11}) \cap (a_{12} * b_{21}) & (a_{11} * b_{12}) \cap (a_{12} * b_{22}) \\ (a_{21} * b_{11}) \cap (a_{22} * b_{21}) & (a_{21} * b_{12}) \cap (a_{22} * b_{22}) \end{bmatrix}.$$

in other words, ordinary matrix multiplication with $\cap, *$ in place of $+, \times$ respectively.

2. *X_r^2 forms a universal, homogeneous representation of $\mathcal{A}_r^2$.*

PROOF:

$\mathcal{A}_r^2$ is a subalgebra of $\mathcal{A}^2$ and the restriction of the representation X^2 to $\mathcal{A}_r^2$ is X_r^2.

□

Note If the representation X fails to be normal then X_r^2 may not be a representation of $\mathcal{A}_r^2$. If $\mathcal{A}$ does not possess any normal representation then there is no guarantee that the matrix product is associative. Thus $\mathcal{A}_r^2$ may or may not form a relation algebra. Even if it does form a relation algebra it does not follow that X_r^2 is a representation of it.

COROLLARY 4

1. *The Allen Interval Algebra can be constructed from the point algebra $\mathcal{P}$ and has a normal representation as ordered pairs of rationals. It is isomorphic to the relation algebra with atomic relations the set of two by two matrices* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with elements $<, >$ and $=$ such that $a * c^{\smile} \geq$ '$<$'; $b * d^{\smile} \geq$ '$<$' etc. The atomic interval relations and their corresponding matrices are*

$$\text{'equals'} \begin{bmatrix} = & < \\ > & = \end{bmatrix};$$

$$\text{'before'} \begin{bmatrix} < & < \\ < & < \end{bmatrix};$$

$$\text{'meets'} \begin{bmatrix} < & < \\ = & < \end{bmatrix};$$

$$\text{'overlaps'} \begin{bmatrix} < & < \\ > & < \end{bmatrix};$$

$$\text{`starts'} \begin{bmatrix} = & < \\ > & < \end{bmatrix};$$

$$\text{`during'} \begin{bmatrix} > & < \\ > & < \end{bmatrix};$$

$$\text{`ends'} \begin{bmatrix} > & < \\ > & = \end{bmatrix}$$

plus the converses of the last six.

2. *It is possible to take the interval algebra, fix any one atomic relation say 'overlaps' and then define a relation algebra of 'intervals of intervals'. Here an interval will be any pair of intervals i, j such that i overlaps j.*

The construction of an interval algebra from a suitable relation algebra can always be done this way, but there is one case that we consider to be degenerate. An element $r \in \mathcal{A}$ is called *non-singular* if $r * r^{\smile} = Id_{\mathcal{A}}$.

THEOREM 5 *Let $\mathcal{A}$ be any relation algebra and r a non-singular atom of $\mathcal{A}$. Then the interval algebra $\mathcal{A}^2_r$ is isomorphic to $\mathcal{A}$.*

PROOF:

An atomic interval relation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must satisfy $b \leq a * r,\ c \leq r^{\smile} * a,\ d \leq r^{\smile} * a * r$ for atoms a, b, c, d. But since r is non-singular $a * r$ is an atom and so are $r^{\smile} * a$ and $r^{\smile} * a * r$. Therefore, the only atomic interval relations are of the form $\begin{bmatrix} a & a * r \\ r^{\smile} * a & r^{\smile} * a * r \end{bmatrix}$. The mapping which takes this matrix and sends it to the atom a is the required isomorphism □

THEOREM 6

1. *If $\mathcal{A}$ is* integral *then so is the interval algebra $\mathcal{A}^2_r$*

2. *If $\mathcal{A}$ has representations of size bigger than two then the pair algebra $\mathcal{A}^2$ is not integral.*

PROOF:

1. It can be shown that a relation algebra is integral if and only if the identity is an atom. But if $Id_{\mathcal{A}}$ is an atom then so is the identity of $\mathcal{A}^2_r$ namely $\begin{bmatrix} Id & r \\ r^{\smile} & Id \end{bmatrix}$ since all the entries are atomic.
2. The identity of $\mathcal{A}^2$ is not atomic since for each atom $r \in \mathcal{A}$ it contains the atom $\begin{bmatrix} Id & r \\ r^{\smile} & Id \end{bmatrix}_{r,r}$.

□

5 Points from Intervals

So far we have shown how to build a pair algebra and interval algebras from a point algebra. It will be useful if we can work backwards too: given a representation of a pair or interval algebra we would like to retrieve the points from the representation. For a representation of a pair algebra this can be done by identifying pairs of the form (x, x) with the point x. However, this won't work with intervals because an interval is always related by the atom r whereas (x, x) is related by the identity. So instead we recover the points in a different way.

THEOREM 7

1. *Let Y be a representation of the pair algebra $\mathcal{A}^2$. There is a representation X of $\mathcal{A}$ with $Y \cong X^2$. Such an X is unique up to isomorphism.*
2. *Let Y be a representation of the interval algebra $\mathcal{A}^2_r$. There is a representation X of $\mathcal{A}$ with $Y \cong X^2_r$. Such an X is unique up to isomorphism.*

PROOF:

1. Let Y be a representation of the pair algebra $\mathcal{A}^2$. Define a domain D to consist of all the elements $x \in Y$ such that $(x, x) \models \begin{bmatrix} Id & Id \\ Id & Id \end{bmatrix}_{Id,Id}$ - x is intended to be a pair of equal points. Define a representation X of $\mathcal{A}$ with domain D by mapping the atom $r \in \mathcal{A}$ to $Y(\begin{bmatrix} r & r \\ r & r \end{bmatrix}_{Id,Id})$. This will be a binary relation on the domain D and is clearly a representation since it is a restriction of Y. For the isomorphism $Y \cong X^2$ let $y \in Y$ be arbitrary.

   There are unique points x_1, x_2 as in the diagram satisfying
$$(y, x_1) \models \begin{bmatrix} Id & Id \\ 1 & 1 \end{bmatrix}_{1,Id}$$
   and
$$(y, x_2) \models \begin{bmatrix} 1 & 1 \\ Id & Id \end{bmatrix}_{1,Id}.$$
   The required isomorphism maps y to the pair (x_1, x_2).
   For uniqueness, let Z be any representation of $\mathcal{A}$ such that there is an isomorphism Ξ from Y to Z^2. Let $x \in X \subset Y$. Applying Ξ to x can only give a pair of the form $(z, z) \in Z^2$ - this follows from the definition of X and the fact that Ξ is an isomorphism. The mapping which sends x to z is an isomorphism from X to Z.
2. The relation
$$same - start = \begin{bmatrix} Id_{\mathcal{A}} & r \\ r^{\smile} & r^{\smile} * r \end{bmatrix}$$
   defines an equivalence relation on Y. Let the set of equivalence classes be D, these will form the points for the representation (X, D) of $\mathcal{A}$. Next we represent the atoms of $\mathcal{A}$. If a is an atom and $p, q \in D$, let
$$(p, q) \in X(a) \Leftrightarrow \forall i \in p, j \in q \;\; [i \begin{bmatrix} a & a * r \\ r^{\smile} * a & r^{\smile} * a * r \end{bmatrix} j]$$

It is a routine exercise to check that this defines a representation, $(X, D)^7$, of $\mathcal{A}$. To get the isomorphism take any interval $i \in Y$. Let Ξ map i to the pair $(p, q) \in X^2_r$ where p is the equivalence class of intervals to which i belongs and q is the set $\{j \in Y : i \text{ meets } j\}$ and 'meets' is the interval relation $\begin{bmatrix} r & r*r \\ Id_{\mathcal{A}} & r \end{bmatrix}$. Check that q is a point in X (i.e. an equivalence class under 'same-start') and that the pair (p, q) belongs to $X(r)$, in other words that (p, q) is an interval in X^2_r. It is not hard to show that Ξ respects all the operations.
Ξ is injective, for if $\Xi(i) = \Xi(j) = (p, q)$ then $i, j \in p$ so i and j are related by $\begin{bmatrix} Id_{\mathcal{A}} & r \\ r^{\smile} & r^{\smile} * r \end{bmatrix}$. By considering the definition of q we see that for all intervals k, i meets k if and only if j meets k. Hence i and j are also related by

$$\begin{bmatrix} r & r*r \\ Id_{\mathcal{A}} & r \end{bmatrix} * \begin{bmatrix} r & r*r \\ Id_{\mathcal{A}} & r \end{bmatrix}^{\smile} \leq \begin{bmatrix} r*r^{\smile} & r \\ r^{\smile} & Id_{\mathcal{A}} \end{bmatrix}.$$

Intersecting the two relations between i and j we deduce that i relates to j via

$$\begin{bmatrix} Id_{\mathcal{A}} & r \\ r^{\smile} & Id_{\mathcal{A}} \end{bmatrix}$$

which is the identity relation on intervals. Hence $i = j$.
To show that Ξ is surjective, let (p, q) be any pair of points from X related by r. Let i and j be any representative elements of p and q (respectively). i relates to j by $\begin{bmatrix} r & r*r \\ r^{\smile} * r & r^{\smile} * r * r \end{bmatrix}$ but since

$$\begin{bmatrix} r & r*r \\ r^{\smile} * r & r^{\smile} * r * r \end{bmatrix} = \begin{bmatrix} Id_{\mathcal{A}} & r \\ r^{\smile} & r^{\smile} * r \end{bmatrix} * \begin{bmatrix} r & r*r \\ Id_{\mathcal{A}} & r \end{bmatrix}$$

there must be an interval k such that the i and k are related by $\begin{bmatrix} Id_{\mathcal{A}} & r \\ r^{\smile} & r^{\smile} * r \end{bmatrix}$ and k and j are related by $\begin{bmatrix} r & r*r \\ Id_{\mathcal{A}} & r \end{bmatrix}$. This interval k satisfies $\Xi(k) = (p, q)$.
We have shown that the interval representation Y is isomorphic to X^2_r where X is the representation of $\mathcal{A}$ constructed above. To show that X is unique up to isomorphism, suppose $\Upsilon : Z^2_r \cong Y$ where Z is a representation of $\mathcal{A}$. For all $z \in Z$ map z to

$$\{i \in Y : \exists w \in Z[(z, w) \in Z^2_r \wedge \Upsilon(z, w) = i]\}.$$

Check that z maps to an element of X and that the map is an isomorphism.

□

COROLLARY 8 *If $\mathcal{A}$ is categorical in some infinite cardinality κ then $\mathcal{A}^2$ and $\mathcal{A}^2_r$ are κ-categorical too.*

PROOF:

Follows from theorem 7. □

Note The converse does not always hold: the interval algebra $\mathcal{A}^2_r$ can be κ-categorical but the point algebra $\mathcal{A}$ may not be. It is true that $\mathcal{A}$ can have only one representation X of cardinality κ such that X^2_r is a representation of $\mathcal{A}^2_r$ but it may have other representations too (either not fully universal, inhomogeneous or not complete).

The next corollary was proved first in [LM87] but follows here from a more general result.

[7] Recall that we can drop the D and simply call the representation X.

COROLLARY 9 *The Allen interval algebra $\mathcal{A}$ is ω-categorical.*

PROOF:

We have already seen that $\mathcal{P}$ is ω-categorical. So corollary 8 gives the result.
$\square$

PROBLEM 1 *We have shown how to recover a representation of $\mathcal{A}$ from a representation of $\mathcal{A}^2$ (or $\mathcal{A}^2_r$). Is it possible to recover $\mathcal{A}$ from $\mathcal{A}^2$ directly, without considering its representations? Of course we have defined $\mathcal{A}^2$ using a special notation - two by two matrices with indices, so $\mathcal{A}$ is isomorphic to the elements indexed by Id, Id - but we really want to do this algebraicly. Formally, if $\mathcal{B} \cong \mathcal{A}^2$ then is there a relation algebra $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D}$ is definable from $\mathcal{B}$ and $\mathcal{D} \cong \mathcal{A}$?*

6 Complexity of Interval Algebras

The REALLY BIG COMPLEXITY PROBLEM (RBCP) for Relation Algebra is to clearly map out which relation algebras are tractable and which are intractable. Let us make this more precise. When we talk about the complexity of a set of L-formulas Σ over a class of L-structures K we are thinking of the following question: for each $\phi \in \Sigma$ is ϕ satisfied in some structure from K? The complexity is measured in terms of the length of ϕ. If Σ contains a countably infinite number of different symbols then we have to be careful about the length of the representation of each symbol, but for countable languages most complexity classes are indifferent[8] to these distinctions.

Considering now the complexity of a relation algebra $\mathcal{A}$ we want to know whether certain formulas are satisfiable in a representation of $\mathcal{A}$. The formulas we consider are networks - a network N is equivalent to a first-order existential sentence. So we want to know for which $\mathcal{A}$ is there an algorithm that decides, in time polynomial in the size of a network, whether the network embeds in some representation of $\mathcal{A}$.

In this direction there are few known results: the point algebras $\mathcal{P}$ and $\mathcal{M}$ (see page 3) have cubic time algorithms for satisfiability but the Allen interval algebra $\mathcal{I}$ is NP-complete [VK86, VKvB89]. The intractability of the Allen Interval Algebra has been problematic in temporal reasoning and in applications to databases and planning [AK83]. It might be hoped that there are other interval algebras that are tractable and yet more expressive than point-based relation algebras. In this section we give no succour to that hope and show that *all* pair algebras are intractable if they have infinite representations, but leave open the conjecture that all non-degenerate interval algebras are intractable too.

PROBLEM 2 (Decidability) *It is not clear, and seems rather unlikely, that for each $\mathcal{A}$ the problem of testing the satisfiability of even atomic $\mathcal{A}$-networks is decidable. So an open problem is to find one fixed relation algebra $\mathcal{A}$ such that the class of all atomic, satisfiable $\mathcal{A}$-networks is undecidable. Of course, the decidability of the atomic network problem implies the decidability of the general network satisfaction problem - for a general network simply try all possible*[9] *atomic refinements and if one of them is consistent then so is the original network.*

We now move on to the question of complexity with a basic lemma:

LEMMA 10 *Let $\mathcal{A} \subseteq \mathcal{B}$ be relation algebras such that every representation of $\mathcal{A}$ embeds in some representation of $\mathcal{B}$. Then the network satisfaction problem for $\mathcal{A}$* reduces *to the network satisfaction problem for $\mathcal{B}$.*

PROOF:

[8] A word of caution: the complexity can be *reduced* if the representation of symbols is very long. Testing whether a number, n, is prime can be done in polynomial time if n is represented as $III\ldots I$ (n Is) but the complexity is worse in the usual decimal notation, assuming $P \neq NP$.

[9] When considering infinite, atomic relation algebras we should assume that there is only a finite disjunction of atoms on each edge of a network. It is necessary that the relations on an edge are at least recursive for there to be a meaningful definition of complexity.

It is always the case that for any representation X of $\mathcal{B}$ the *reduct* of X to $\mathcal{A}$ is a representation of $\mathcal{A}$. Since we are also assuming that any representation of $\mathcal{A}$ embeds in some representation of $\mathcal{B}$ it follows that an $\mathcal{A}$-network N embeds in a representation of $\mathcal{A}$ if and only if it embeds in a representation of $\mathcal{B}$. So, given an $\mathcal{A}$-network N first consider N as a $\mathcal{B}$-network then decide whether N is satisfiable in a representation of $\mathcal{B}$ and this will tell you whether N is satisfiable in a representation of $\mathcal{A}$. $\Box$

We want to prove that virtually all the pair algebras are intractable and we do this by first constructing the simplest possible pair algebra $\mathcal{C}^2$, showing that this is NP-complete and then applying the lemma.

Let $\mathcal{C}$ be the relation algebra with just two atoms $Id, \sharp$ where $\sharp * \sharp = 1$. The atomic networks of this relation algebra form an amalgamation class so we can build a pair algebra $\mathcal{C}^2$. A normal representation of this has domain $S \times S$ where S is any infinite set - i.e. the domain consists of pairs from S.

The atomic interval relations are

$$e_1 = \begin{bmatrix} Id & \sharp \\ \sharp & Id \end{bmatrix}_{\sharp,\sharp} \qquad \text{swap} = \begin{bmatrix} \sharp & Id \\ Id & \sharp \end{bmatrix}_{\sharp,\sharp} \qquad \text{disjoint} = \begin{bmatrix} \sharp & \sharp \\ \sharp & \sharp \end{bmatrix}_{\sharp,\sharp}$$

$$\text{meets} = \begin{bmatrix} \sharp & \sharp \\ Id & \sharp \end{bmatrix}_{\sharp,\sharp} \qquad \text{met-by} = \begin{bmatrix} \sharp & Id \\ \sharp & \sharp \end{bmatrix}_{\sharp,\sharp} \qquad \text{same-start} = \begin{bmatrix} Id & \sharp \\ \sharp & \sharp \end{bmatrix}_{\sharp,\sharp}$$

$$\text{same-end} = \begin{bmatrix} \sharp & \sharp \\ \sharp & Id \end{bmatrix}_{\sharp,\sharp}$$

plus three relations with subscript $Id, \sharp$ three with $\sharp, Id$ and two with subscript Id, Id

$$\begin{bmatrix} Id & \sharp \\ Id & \sharp \end{bmatrix}_{Id,\sharp} \qquad \begin{bmatrix} \sharp & Id \\ \sharp & Id \end{bmatrix}_{Id,\sharp} \qquad \begin{bmatrix} \sharp & \sharp \\ \sharp & \sharp \end{bmatrix}_{Id,\sharp}$$

$$\begin{bmatrix} Id & Id \\ \sharp & \sharp \end{bmatrix}_{\sharp,Id} \qquad \begin{bmatrix} \sharp & \sharp \\ Id & Id \end{bmatrix}_{\sharp,Id} \qquad \begin{bmatrix} \sharp & \sharp \\ \sharp & \sharp \end{bmatrix}_{\sharp,Id}$$

$$e_2 = \begin{bmatrix} Id & Id \\ Id & Id \end{bmatrix}_{Id,Id} \qquad \begin{bmatrix} \sharp & \sharp \\ \sharp & \sharp \end{bmatrix}_{Id,Id}$$

- fifteen atomic pair relations in all. The identity $Id_{\mathcal{C}^2} = e_1 \vee e_2$. The first seven listed above form the atoms of the interval algebra $\mathcal{C}^2_\sharp$. The composition table for this pair algebra can be calculated 'by hand' e.g. 'same-start' $*$ 'same-start' = 'same-start' and 'meets' $*$ 'swap' = 'same-end'.

THEOREM 11 *The network consistency problem for the relation algebra $\mathcal{C}^2$ is NP complete.*

PROOF:

The proof also shows that the interval algebra $\mathcal{C}^2_\sharp$ is NP complete.

Any transitively closed atomic network in $\mathcal{C}^2$ is consistent so the network consistency problem must be in NP - non-deterministically choose an atom from each edge and see if the network is transitively closed. We show it is NP complete by reducing the Hamiltonian circuit problem to it.

Let G be any undirected finite graph, i.e. a finite set of nodes and edges. Let the number of nodes of G be n. We will build a $\mathcal{C}^2$-network N in such a way that N is consistent if and only if G contains a Hamiltonian circuit. The construction of N will be done in time polynomial in the size of G.

1. Turn G into a directed graph by arbitrarily choosing a direction for each edge. Call it G'
2. Make a $\mathcal{C}^2$-network M by making one node for each edge of G' and setting $M(e, f)$ to the relation that actually holds in G' between the two edges e and f. For example, if e and f are disjoint in G' let $M(e, f)$ = 'disjoint'. Note that (a) M is consistent (b) for distinct edges e and f the relation between them cannot include Id or 'swap' and (c) All the relations used are from $\mathcal{C}^2_\sharp$ so G' is a $\mathcal{C}^2_\sharp$-network.

3. Extend M to M^+ by adding n new nodes $x_1, \ldots, x_n$ in such a way that each x_i is constrained to be equal or the 'swap' of one of the original nodes of M and so that it is still consistent for x_i to be equal or the 'swap' of *any* of the nodes of M. This construction is given later.
4. Add to the network the assertions

$$x_i \text{ 'shares exactly one end-point with' } x_{i+1}$$

for $i = 1 \ldots n-1$,

$$x_n \text{ 'shares exactly one end-point with' } x_1$$

and for all the other pairs x_i, x_j

$$x_i \text{ 'disj' } x_j.$$

Call this network N.

If G does contain a Hamiltonian circuit then the x_i can be chosen to be the edges of a Hamiltonian circuit and N is therefore consistent. Conversely, if N is consistent then in any model the construction enforces that the intervals x_i form a Hamiltonian circuit on a graph isomorphic to G.

It remains to show how to perform the construction in part 3. Let N be any consistent $\mathcal{C}^2$-network such that for distinct nodes a and b the relation $N(a,b)$ does not include equality or 'swap'. We show how to add extra nodes to N including a node x so that x must coincide with one of the nodes of N and x can consistently coincide with any node of N. a coincides with b means that they have the same endpoints, though possibly in the opposite order. The size of the extension will be bound by a polynomial in the size of N.

First group the nodes of N in pairs (possibly with an odd one left). For each pair a, b we add the nodes a', b', w and x_{ab} and set

$$a' \; \{Id \vee \text{ 'swap' }\} \; a,$$

$$b' \; \{Id \vee \text{ 'swap' }\} \; b$$

and

$$a' \; \{ \text{ 'disj' or 'starts' } \} \; b'.$$

This constrains a', b' to lie on the same edges as a and b (respectively) though possibly in the opposite directions, and a', b' look like one of the two diagrams below.

This is where we need the assumption that they don't share two endpoints. Now let

$$w \; \{ \text{ meets or met-by } \} \; a', b'$$

so w must join the two 'top ends' of a' and b'. Finally let

$$x_{ab} \; \{ \text{ meets or ends } \} \; w$$

and

$$x_{ab}\ \{Id \vee \text{`disj' or 'starts'}\}\ a', b'.$$

x_{ab} finishes at one or the other endpoint of w (so it can't be disjoint from both a' and b') and the second constraint forces x_{ab} to be equal to either a' or b'. Either choice is consistent.

We now have a set of new nodes of the form x_{ab}, about half as many as we started with and distinct nodes x_{ab} and x_{cd} still share at most one endpoint. Therefore we can repeat the whole procedure and construct new nodes x_{abcd} that must coincide with one of x_{ab} or x_{cd} i.e. they coincide with one of a, b, c or d. This process is repeated about $\log(n)$ times until there is a node x constrained to be any one of the original nodes of the graph. This is done for each of the nodes x_i. □

COROLLARY 12 *The complexity of the network satisfaction problem for any pair algebra $\mathcal{A}^2$ with infinite representations is NP hard.*

PROOF:

The proof is based on lemma 10. Since $\mathcal{A}$ has representations of size bigger than two, $\mathcal{A}^2$ must have a subalgebra isomorphic to $\mathcal{C}^2$ (in theorem 11). For example, this subalgebra includes the element $\begin{bmatrix} -Id & Id \\ -Id & -Id \end{bmatrix}_{-Id,-Id}$ (where $-Id$ is the complement of the identity relation of $\mathcal{A}$) which corresponds to the atom 'meets' of $\mathcal{C}^2$. $\mathcal{A}^2$ has similar elements corresponding to the other atoms of $\mathcal{C}^2$. It remains to show that any representation of $\mathcal{C}^2$ embeds in a representation of $\mathcal{A}^2$. Since $\mathcal{A}^2$ has infinite representations, by the Löwenheim - Skolem theorem[10], it has representations arbitrarily big. So for any representation X of $\mathcal{C}^2$ take a representation Y of $\mathcal{A}^2$ at least as big as X. Use theorem 7 to find representations x of $\mathcal{C}$ and y of $\mathcal{A}$ such that $X \cong x^2$ and $Y \cong y^2$. y is still as big as x, so x can be embedded in y any way you like provided distinct points remain distinct. This embedding determines an embedding of X into Y. Now use lemma 10. □

PROBLEM 3 *Note, by compactness, that if $\mathcal{A}^2$ does not have infinite representations then its representations have sizes with a uniform finite bound n say. An $\mathcal{A}^2$-network N without equality on any edge is certainly inconsistent if it has more than n nodes. If it has n or less nodes then its consistency can be tested in constant time by picking one atom from each edge and seeing if it embeds in any of the representations. There are at most 2^n possible choices and in each case there are only a finite number of non-isomorphic representations (and each representation is finite) to check. It seems, then, that the network satisfaction problem is tractable. Howewever, we have not been able to prove the tractability of the satisfiability problem for pair networks where the equality relation is allowed.*

PROBLEM 4 *The situation with interval algebras is less clear. For non-singular atoms $r \in \mathcal{A}$ the interval algebra $\mathcal{A}^2_r \cong \mathcal{A}$ and this is considered to be a degenerate case. But the following conjecture remains unproved: the complexity of the network satisfaction problem for any non-degenerate interval algebra with infinite representations is NP hard.*

7 Intervals with metrics

The metric point system $\mathcal{M}$ has a normal representation, namely $\mathbb{Q}$ and so the construction of theorem 3 produces a relation algebra of intervals with metrics. But this is a rather uninteresting algebra of intervals as an interval here is defined by a single, fixed atomic relation. That means

[10]For any relation algebra $\mathcal{A}$ it is possible to make a first-order language $L = L(\mathcal{A})$ with one binary relation symbol for each element of $\mathcal{A}$ and then define an L-theory Σ whose models are exactly the representations of $\mathcal{A}$.

that all intervals have to be of the same size, an over-restricted definition. An interval is more usually considered as any pair of points with the first one less than the second. In order to deal with these it is necessary to consider non-atomic networks.

7.1 Definition of $\mathcal{M}^2$

1. An *interval* is a pair of rationals (p, q) such that $p < q$.

2. An *elementary metric interval relation* A is a two by two matrix $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11}, A_{12}, A_{21} and A_{22} are intervals with rationals endpoints such that the $\mathcal{M}$-network

   is transitively closed. $(p^-, p^+)\ A\ (q^-, q^+)$ asserts that $q^- - p^- \in A_{11}$, $q^+ - p^- \in A_{12}$ etc.

3. Such relations are composed according to the rule

   $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} (A_{11} + B_{11}) \cap (A_{12} + B_{21}) & (A_{11} + B_{12}) \cap (A_{12} + B_{22}) \\ (A_{21} + B_{11}) \cap (A_{22} + B_{21}) & (A_{21} + B_{12}) \cap (A_{22} + B_{22}) \end{bmatrix}$$

   in other words ordinary matrix multiplication with addition of intervals and intersection instead of multiplication and addition respectively.

4. The identity is $\begin{bmatrix} [0,0] & (0,\infty) \\ (-\infty,0) & [0,0] \end{bmatrix}$ (note that this is not atomic) and the converse of a relation $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is $\begin{bmatrix} -A_{11} & -A_{21} \\ -A_{12} & -A_{22} \end{bmatrix}$.

5. More general metric interval relations can be formed as disjuncts of elementary ones. The complement of an elementary relation will typically be a disjunct. Non-elementary matrices can be introduced e.g. $R \vee S$. The product

   $$(R \vee S).(T \vee U) = RT \vee RU \vee ST \vee SU.$$

   Thus the disjunct $i\{<,>\}j$ can be expressed as

   $$i \begin{bmatrix} (-\infty,0) & (-\infty,0) \\ (-\infty,0) & (-\infty,0) \end{bmatrix} \vee \begin{bmatrix} (0,\infty) & (0,\infty) \\ (0,\infty) & (0,\infty) \end{bmatrix} j.$$

   Note that this could not be expressed as a network in the point algebras $\mathcal{P}$ or $\mathcal{M}$.

6. The efficiency will be enhanced if a disjunction is reduced by the rule: $M \vee N \Rightarrow N$ if $M \subseteq N$ (where $M \subseteq N$ if each of the four entries of M is a subset of the corresponding entry of N). Also for matrices with three of the four entries equal

   $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \vee \begin{bmatrix} A & B \\ C & D' \end{bmatrix} \Rightarrow \begin{bmatrix} A & B \\ C & D \vee D' \end{bmatrix}$$

   provided $D \vee D'$ is an interval.

As before, it is necessary to check that composition is associative and this is done by showing that matrix product is isomorphic to composition of relations. The critical section of the proof takes two intervals $\bar{\alpha}$ and $\bar{\beta}$ related by the matrix product $R * S$. It is required to show that there exists a third interval $\bar{\gamma}$ such that $\bar{\alpha}$ and $\bar{\gamma}$ are related by R and $\bar{\gamma}$ and $\bar{\beta}$ are related by S. But this follows from the fact that a simple $\mathcal{M}$-network N (a transitively closed $\mathcal{M}$-network with only one interval on each edge) has the *extension property* - for any subnetwork L of N it can be shown that any embedding of L into $\mathbb{Q}$ can be extended to an embedding of N to $\mathbb{Q}$ (see [Hir94] for the details).

7.2 Expressive Power

This system is capable of expressing all of Allen's interval relations e.g. 'overlaps' is written as $\begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix}$. A constraint on the duration of i can be expressed using this format as $i \begin{bmatrix} [0,0] & [d,e] \\ [-e,-d] & [0,0] \end{bmatrix} i$ where d and e are respectively lower and upper bounds on the duration of i. Thus, the qualitative expressive power of Allen's system is combined with the quantitative power of the metric system $\mathcal{M}$ of Dechter, Meiri and Pearl.

7.3 Complexity of $\mathcal{M}^2$

It is possible to use any of the algorithms from the literature in conjunction with this language, but for the present suppose we use a fixpoint algorithm to calculate the transitive closure of a network

$$\begin{array}{l} \text{Repeat} \\ \qquad N := N^2 \wedge N \\ \text{Until } N \subseteq N^2 \end{array}$$

where

$$N^2(i,j) = \bigwedge_k N(i,k) * N(k,j).$$

This is equivalent to the Allen propagation algorithm.

$\mathcal{M}^2$ is a highly expressive language and the worst case complexity of checking the consistency of a network will be at least as bad as its two sublanguages $\mathcal{M}$ and $\mathcal{A}$, i.e. it is NP-hard. In fact, a non-deterministic Turing machine could solve the problem in polynomial time since the non-disjunctive case can be solved in cubic time (below) so consistency checking for $\mathcal{M}^2$ is NP-complete.

But if we restrict to certain fragments of the full language we obtain the following results.

- A network with only elementary metric interval relations on the arcs (no disjuncts) can be checked in cubic time. This follows from the proof in [DMP91] that computing the transitive closure of an $\mathcal{M}$-network with only one interval on each arc (in their terminology an STP), can be done in cubic time, and computes the minimal network. In turn this result follows from the extension property mentioned earlier.

- If all the relations on the arcs of the network are *pure Allen relations* i.e. equivalent to a union of some of the thirteen primitive interval relations, then the matrix product (which is calculated in constant time) will produce the same result as the Allen transitivity table. Therefore the same complexity results will hold i.e. consistency checking is NP-complete, but the Allen propagation algorithm provides a useful approximation in cubic time.

- For general metric constraints with disjunctions, the problem is NP-complete. Dechter, Meiri and Pearl left open the problem of whether the fixpoint algorithm must terminate at all. If the metric values are commensurate (the ratios are rational) then without loss it may be assumed that all the metric constraints have integer bounds. In this case the fixpoint algorithm will certainly terminate and if the number of integers lying in any constraint has a fixed bound then the algorithm will terminate in cubic time. The argument is exactly the same when analysing $\mathcal{M}^2$ except the bound must apply to the number of atomic matrices with integer entries, within the constraint. We answer the general termination question affirmatively in the following section

The remaining problem is that calculating the fixpoint (transitive closure) is not a *complete* deductive mechanism. It is easy to devise inconsistent but transitively closed networks in $\mathcal{M}$ and hence also in $\mathcal{M}^2$. However, computing the transitive closure may give a useful first approximation for a consistency checker which then proceeds by brute force to test each choice of disjunctions for consistency. (The constraint technology will clearly be useful here.)

7.4 Proof that the fixpoint algorithm terminates on Temporal Constraint Problems (TCPs)

A TCP is defined in [DMP91] as an $\mathcal{M}'$-network where the metric constraint on an edge is a finite union of intervals, possibly with irrational endpoints. e.g.

Let us gloss over the problem that algorithms do not handle real numbers as these are not all finitely representable and answer an open problem raised in [DMP91]

THEOREM 13 *The fixpoint algorithm (page 18) always terminates finitely.*

PROOF:

One iteration of the algorithm takes some triangle (a, b, c) from the network N and replaces $N(a, c)$ by $N(a, c) \cap (N(a, b) * N(b, c))$. This composition is calculated by addition of intervals. Let $S(N)$ be the (finite) set of all endpoints of intervals mentioned in N. If a number occurs more than once in $S(N)$ then label each instance separately in order to distinguish them.
Claim: at each stage the relation between nodes n and m is either $\emptyset$ (inconsistent) or equal to a finite union of intervals and each endpoint is a sum of distinct elements from $S(N)$[11]. This claim can be proved by induction on the number of iterations of the algorithm. Now there are only a finite number of possible sums that can be produced this way and therefore only a finite number of possible intervals that can occur on an edge at any stage of the algorithm. The relations on each edge are never increasing so each edge can be placed in the queue a finite number of times and therefore the algorithm must terminate. $\Box$

[11] Why must the elements in the sum be distinct? Because if the same element occurred twice it would correspond to a constraint on the edge (a, c) created by a looping path. However, either a loop produces an inconsistency (so the algorithm terminates) or an equally tight constraint is produced from the path with the loop deleted.

7.5 Examples

1. From $i\{\text{during}\}j$ and $j\{\text{overlaps}\}k$ we should deduce that $i\{<, \text{meets, overlaps, starts, during}\}k$.
   $i \begin{bmatrix} (-\infty,0) & (0,\infty) \\ (-\infty,0) & ((0,\infty) \end{bmatrix} j$ and $j \begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix} k$ gives
   $i \begin{bmatrix} (-\infty,0) & (0,\infty) \\ (-\infty,0) & ((0,\infty) \end{bmatrix} \begin{bmatrix} (0,\infty) & (0,\infty) \\ (-\infty,0) & (0,\infty) \end{bmatrix} k$ or
   $i \begin{bmatrix} (-\infty,\infty) \cap (-\infty,\infty) & (-\infty,\infty) \cap (0,\infty) \\ (-\infty,\infty) \cap (-\infty,\infty) & (-\infty,\infty) \cap (0,\infty) \end{bmatrix} k$ i.e.
   $i \begin{bmatrix} (-\infty,\infty) & (0,\infty) \\ (-\infty,\infty) & (0,\infty) \end{bmatrix} k$ which says that both endpoints of i must lie before the end of k, as required.

2. Let j start at least 5 second after i finishes and let j finish less than 10 seconds after i starts.

   It should be possible to deduce that the duration of i is less than 5 seconds. Well, $i \begin{bmatrix} (0,\infty) & (0,10) \\ (5,\infty) & (5,\infty) \end{bmatrix} j$ and $j \begin{bmatrix} (-\infty,0) & (-\infty,-5) \\ (-10,0) & (-\infty,-5) \end{bmatrix} i$ so $i \begin{bmatrix} (-10,10) & (-\infty,5) \\ (-5,\infty) & (-\infty,\infty) \end{bmatrix} i$. Intersecting this with the initial constraint on i, gives $i \begin{bmatrix} [0,0] & (0,5) \\ (-5,0) & (0,0) \end{bmatrix} i$ as required.

7.6 Comparisons

A number of other attempts have been made to combine qualitative and quantitative reasoning ([KL91, Lad89, Lig90, Mei91]). The language $\mathcal{M}^2$ of this paper has two main advantages. Firstly, it uses the same uniform representation for all relations. There is no need to refer to a special table when dealing with an interval constraint and a separate table for metrics. All constraints are represented as matrices and compositions are calculated by matrix multiplication. By contrast, [KL91, Lad89, Mei91] are all essentially hybrid systems which handle metric and interval information separately and translate from one to the other.

The other advantage of $\mathcal{M}^2$ is its expressive power. When disjunctive relations are allowed it is possible to express constraints which are neither point-based metric nor qualitative interval relations. For example to assert that interval i either starts more than 5 seconds after interval j ends or ends more than 10 seconds before j ends, we use the disjunction

$$\begin{bmatrix} (-\infty,5) & (-\infty,5) \\ (-\infty,5) & (-\infty,5) \end{bmatrix} \vee \begin{bmatrix} (-\infty,\infty) & (10,\infty) \\ (-\infty,10) & (-\infty,10) \end{bmatrix}.$$

Note that this could not be represented directly in any of the competing systems. It would be necessary to construct additional intervals and put constraints on these. This expressive power is achieved without additional complexity cost (the complexity of checking the consistency of a network in a sub-language of $\mathcal{M}^2$ is the same as that in competing systems).

More tentatively, there is one further advantage. When dealing with the disjunctive case, the simple algorithm for combining matching disjuncts (see page 17, Definition of $\mathcal{M}^2$, 6) is very straightforward and will improve the efficiency considerably. Disjuncts like $before, meets$ in Allen's language translate to

$$\begin{bmatrix} (0,\infty) & (0,\infty) \\ (0,\infty) & (0,\infty) \end{bmatrix} \vee \begin{bmatrix} (0,\infty) & (0,\infty) \\ [0,0] & (0,\infty) \end{bmatrix}.$$

which gets rewritten as $\begin{bmatrix} (0,\infty) & (0,\infty) \\ [0,\infty) & (0,\infty) \end{bmatrix}$ thus eliminating a disjunct which could improve the efficiency. Theoretical results about average case performance are hard to provide in this area so this is most likely to be judged, eventually, by empirical results.

8 Conclusion

An interval relation algebra can be constructed from a point relation algebra provided it has a normal representation. This allows us to construct a metric interval algebra from the metric system $\mathcal{M}$. This representation permits the expressing of Allen type disjuncts like $i\{<,>\}j$. It is thus more expressive than other systems that allow quantitative, metric information. As with these systems [DMP91] the propagation algorithm will be complete and of cubic complexity if there are no disjuncts but its performance in general is intractable.

We have shown that a large class of relation algebras, the pair algebras with infinite representations, are all intractable. The general problem of deciding which relation algebras have a tractable network satisfaction problem remains to be solved.

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