Relation algebra reducts of cylindric algebras and an application to proof theory

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Abstract

We confirm a conjecture, about neat embeddings of cylindric algebras, made in 1969 by J. D. Monk, and a later conjecture by Maddux about relation algebras obtained from cylindric algebras. These results in algebraic logic have the following consequence for predicate logic: for every finite cardinal $\alpha \geq 3$ there is a logically valid sentence X, in a first-order language $\mathcal{L}$ with equality and exactly one nonlogical binary relation symbol $\mathbf{E}$, such that X contains only 3 variables (each of which may occur arbitrarily many times), X has a proof containing exactly $\alpha + 1$ variables, but X has no proof containing only α variables. This solves a problem posed by Tarski and Givant in 1987.

1 Introduction

The completeness theorem of first-order logic says that every valid formula has a proof. However, results of Henkin and Monk showed that the proof of a formula may need more variables than are used in the formula itself. Establishing exactly how many variables are needed to prove a given valid formula can be rather delicate. To establish provability or non-provability with α variables, the methods of algebraic logic — cylindric algebras and relation algebras — are useful. α-dimensional cylindric algebras can be regarded, approximately, as algebras of α-ary relations and relation algebras are an algebraic approximation to algebras of binary relations. From an α-dimensional cylindric algebra $\mathfrak{C}$ it is possible to obtain the *relation algebra reduct* $\mathfrak{Ra}\,\mathfrak{C}$, and if $\alpha \geq 4$ this will be a relation algebra. The central part of this paper is the construction of some relation algebras $\mathfrak{N}_\alpha^\beta$, for $4 \leq \alpha \leq \beta < \omega$, and the proof, for sufficiently large β, that $\mathfrak{N}_\alpha^\beta$ is a subalgebra of $\mathfrak{Ra}\,\mathfrak{C}$ for some α-dimensional cylindric algebra $\mathfrak{C}$, but not a subalgebra of $\mathfrak{Ra}\,\mathfrak{C}'$ for any $(\alpha+1)$-dimensional cylindric algebra $\mathfrak{C}'$. In symbols, $\mathfrak{N}_\alpha^\beta \in \mathsf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha \setminus \mathsf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$. This confirms a conjecture of Maddux, and is used to confirm a related conjecture of Monk about neat reducts of cylindric algebras. We apply this result to logic by showing, for each $\alpha \geq 3$, that there are valid formulas that can be proved with $\alpha + 1$ variables but not with only α variables in a proof system taken from [31].

Here in the introduction we discuss these classes of algebras, some of the history of this investigation, and the proof-theoretic consequences. In the second section we present the algebras and their properties, and the last section we apply them to logic.

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Cylindric algebras and relation algebras. We assume a basic knowledge of relation algebras and cylindric algebras. See [22] for an introduction, and [12, 13] for a comprehensive study. We use the notation of [12, 13]. In particular, for any ordinal α, CA_α is the class of α-dimensional cylindric algebras. Let $\mathfrak{C} \in \mathsf{CA}_\alpha$. Then $\mathfrak{C}$ is an algebra of the form

$$\mathfrak{C} = \langle C, +, \cdot, ^-, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j<\alpha}$$

where $\langle C, +, \cdot, ^-, 0, 1 \rangle$ is a Boolean algebra and, for all $i, j < \alpha$, c_i is a unary operation on C and $\mathsf{d}_{ij} \in C$. For all $i, j < \alpha$, the *j-for-i substitution,* s^i_j, is defined by

$$\mathsf{s}^i_j x = \begin{cases} x & \text{if } i = j, \\ \mathsf{c}_i(x \cdot \mathsf{d}_{ij}) & \text{if } i \neq j. \end{cases}$$

For all $\beta < \alpha$, the set of β-dimensional elements is defined by

$$Nr_\beta \mathfrak{C} = \{x : x \in C,\ x = \mathsf{c}_\gamma x \text{ whenever } \beta \leq \gamma < \alpha\}.$$

The *neat reduct* $\mathfrak{Nr}_\beta \mathfrak{C} = \langle Nr_\beta \mathfrak{C}, +, \cdot, ^-, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j<\beta}$ is a β-dimensional cylindric algebra. An algebra $\mathfrak{Ra}\,\mathfrak{C} = \langle Nr_2 \mathfrak{C}, +, \cdot, ^-, 0, 1, ;, ^\smile, \mathsf{d}_{01} \rangle$, similar to relation algebras, is constructed by restricting the Boolean operations to the set $Nr_2\mathfrak{C}$ of two-dimensional elements of $\mathfrak{C}$ and using the third dimension to define conversion and composition [13, Def. 5.3.7]: $\breve{a} = \mathsf{s}^2_0 \mathsf{s}^0_1 \mathsf{s}^1_2 a$ and $a\,; b = \mathsf{c}_2(\mathsf{s}^1_2 a \cdot \mathsf{s}^0_2 b)$ for all $a, b \in Nr_2\mathfrak{C}$. $\mathfrak{Ra}\,\mathfrak{C}$ is called the *relation algebra reduct* of $\mathfrak{C}$. $\mathsf{Nr}_\beta\mathsf{CA}_\alpha$ denotes the class $\{\mathfrak{Nr}_\beta\mathfrak{C} : \mathfrak{C} \in \mathsf{CA}_\alpha\}$ and $\mathsf{Ra}\,\mathsf{CA}_\alpha$ denotes the class $\{\mathfrak{Ra}\,\mathfrak{C} : \mathfrak{C} \in \mathsf{CA}_\alpha\}$.

It is evident from the definitions that if we preface a relation algebra reduct with a neat β-reduct for $3 \leq \beta \leq \alpha$, the outcome is unchanged: $\mathfrak{Ra}\,\mathfrak{Nr}_\beta\mathfrak{C} = \mathfrak{Ra}\,\mathfrak{C}$. It follows that $\mathsf{Ra}\,\mathsf{Nr}_\beta\mathsf{CA}_\alpha = \mathsf{Ra}\,\mathsf{CA}_\alpha$. We have $\mathsf{Nr}_3\mathsf{CA}_\alpha \supseteq \mathsf{Nr}_3\mathsf{CA}_{\alpha+1}$ by [12, Th. 2.6.31], so $\mathsf{Ra}\,\mathsf{CA}_\alpha = \mathsf{Ra}\,\mathsf{Nr}_3\mathsf{CA}_\alpha \supseteq \mathsf{Ra}\,\mathsf{Nr}_3\mathsf{CA}_{\alpha+1} = \mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ whenever $\alpha \geq 3$. Of course, the same holds for the closure of these classes under subalgebras. For any class K of algebras, we let $\mathbf{S}\mathsf{K}$ be the class of subalgebras of algebras in K, so we obtain:

$$\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_3 \supseteq \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_4 \supseteq \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_5 \supseteq \cdots \supseteq \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha \supseteq \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1} \supseteq \ldots. \tag{1}$$

The study of which inclusions in (1) are proper has a rather long history. Henkin and Tarski proved that every algebra in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_4$ is a relation algebra (see [25, Th. 9.2], [22, pp. 377–378], and [13, Th. 5.3.8]). J. C. C. McKinsey found an algebra that satisfies all the axioms for relation algebras except the associative law. Tarski used McKinsey's algebra to construct a nonrepresentable 3-dimensional cylindric algebra in which composition is not associative [11, footnote 26]. McKinsey's algebra (the relation algebra reduct of Tarski's algebra) is not a relation algebra, so it lies in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_3 \setminus \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_4$. This shows that the first inclusion in (1) is proper.

The second inclusion is also proper. R. Lyndon [17] found a relation algebra that is nonrepresentable because it fails to satisfy the following condition:

$$\text{if} \quad 0 = (\breve{a};c) \cdot (b;\breve{d}) \cdot \big([(\breve{a};e) \cdot (b;\breve{f})];[(\breve{e};c) \cdot (f;\breve{d})]\big) \quad \text{then} \quad 0 = (a;b) \cdot (c;d) \cdot (e;f).$$

Monk showed that this condition is valid in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_5$ [25, Lem. 9.15] and concluded that Lyndon's relation algebra is not in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_5$ [25, Th. 9.16]. Monk also proved that every relation algebra is in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_3$ [25, Th. 9.10], but Maddux improved this to show that every relation algebra is in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_4$ [19, Ch. 10,Th. (21)], [13, Th. 5.3.17]. It follows that Lyndon's relation algebra is in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_4$ but not $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_5$. For more on the connections between relation algebras and cylindric algebras see [13, Sect. 5.3] and [14, 19, 21, 22, 23, 25, 28, 29].

It was known from the work of Henkin, Johnson, and Monk [8, 9, 15, 16, 26, 27] that infinitely many inclusions in (1) are strict. Therefore, if $3 \leq \alpha < \omega$, we may define $\pi(\alpha)$ to be the least $\beta > \alpha$ such that $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha \neq \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\beta$. Results in [15, 26, 27] imply that $\pi(\alpha) \leq 3\alpha!$. Maddux [23] improved this upper bound from factorial to linear by proving that $\pi(\alpha) \leq 3\alpha - 7$ for every finite $\alpha > 3$. When $\alpha = 4$, this yields the strictness of the second inclusion again. The third inclusion was also known to be strict by an unpublished construction; see [23, p. 195–6].

Maddux's proofs use the notion of an *α-dimensional cylindric basis*. He showed that if an atomic relation algebra $\mathfrak{A}$ has such a basis then $\mathfrak{A} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha$ [19, Ch. 10, Th. (13)], [21, Th. 10]. We will use this fact later to show that the algebras we construct can be embedded into relation algebra reducts of cylindric algebras. However, the converse fails: not having a cylindric basis is not enough to show that an algebra cannot be embedded. Indeed, every representable relation algebra $\mathfrak{A}$ must belong to $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha$ for every $\alpha \geq 3$, and $\mathfrak{A}$ can even be embedded in a relation algebra that has an α-dimensional cylindric basis. But $\mathfrak{A}$ itself may not even have a 5-dimensional cylindric basis: examples can be found among the relation algebras constructed from projective geometries by Lyndon [18]. It is also noteworthy that for all finite $\alpha \geq 5$ there is a representable atomic relation algebra $\mathfrak{A}$ with an α-dimensional cylindric basis (so that $\mathfrak{A} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha$) but with no $(\alpha+1)$-dimensional cylindric basis (see [21, p. 954] and [19, Ch. 10, Ex. (23)]). But as we said, the latter does not imply that $\mathfrak{A} \notin \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$, so this does not help in showing that the inclusions in (1) are proper.

In this paper, we confirm a conjecture of Maddux [4, Prob. 17, p. 734], that

THEOREM 1 *For each finite $\alpha \geq 3$, the inclusion $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha \supset \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ is strict.*

Thus, $\pi(\alpha) = \alpha + 1$ for $3 \leq \alpha < \omega$. Theorem 1 is proved by constructing certain relation algebras $\mathfrak{N}_\alpha^\beta$ (for $4 \leq \alpha \leq \beta < \omega$) that belong to $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha$, but do not belong to $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ whenever β is sufficiently large compared to α. See theorems 5 and 8 below. The first of these is easily seen, using cylindric bases; the second involves some combinatorial argument (cf. [7]). The algebras $\mathfrak{N}_\alpha^\beta$ themselves are not too complicated. They have the further property of being generated by a single atom.[1] This is enough to establish theorem 1, since we have already seen that the first few inclusions in (1) are strict.

The neat embedding problem. Monk [26] formulated the following conjecture in 1969. Let R be the set of triples of finite cardinals $\langle \alpha, \kappa, \gamma \rangle$ having the property that there exists a cylindric algebra $\mathfrak{C}$ of dimension α that can be neatly embedded in a cylindric algebra of dimension $\alpha + \kappa$ but not in one of dimension $\alpha + \gamma$, that is, $\mathfrak{C} \in \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} \setminus \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\gamma}$. Various results about cylindric algebras imply that this situation is interesting only if $\alpha \geq 3$ and possible only if $\kappa < \gamma$. Monk conjectured [26, p. 342, l. 28] that $\langle \alpha, \kappa, \kappa + 1 \rangle \in R$ for all $\kappa < \omega$ and $\alpha \geq 3$. This condition is equivalent to $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} \supset \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$, and it implies $\langle \alpha, \kappa, \gamma \rangle \in R$ for all possible and interesting $\langle \alpha, \kappa, \gamma \rangle$. Monk's conjecture was restated as the *neat embedding problem* [12, Prob. 2.12, p. 464]: For $2 < \alpha < \omega$, is there a $\kappa < \omega$ such that $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} = \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$? A simple calculation (replace Ra by Nr_3 in the proof of corollary 2 below) shows Monk's conjecture holds for all $\alpha \geq 3$ if it holds for

[1] A construction without this property appeared in an early draft of this paper (Oct. 1997) by the first two authors.

$\alpha = 3$. It was known that $\mathsf{SNr}_3\mathsf{CA}_\kappa \supset \mathsf{SNr}_3\mathsf{CA}_{\kappa+1}$ for $\kappa = 3, 4, 5$ [23]. Andréka [1] proved $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} \supset \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$ for $\alpha \geq 3$ and $\kappa < \alpha$. This yields the known results in case $\alpha = 3$. See [2, rem. 3], [23, pp. 195–6], and [12, p. 464] for more information concerning this problem.

Theorem 1 provides a negative solution to the neat embedding problem, confirming Monk's conjecture.

Corollary 2 *If* $2 < \alpha < \omega$ *and* $\kappa < \omega$ *then* $\mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa} \supset \mathsf{SNr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$.

Proof:

Observe that $\mathbf{S}\,\mathsf{Ra}\,\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa} = \mathbf{S}\,\mathsf{Ra}\,\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$, and the same for $\alpha + \kappa + 1$. The inclusion '$\subseteq$' is trivial; for '$\supseteq$', note that if $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{B}$ for $\mathfrak{B} \subseteq \mathfrak{C} \in \mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$, then $\mathfrak{A} \subseteq \mathfrak{Ra}\,\mathfrak{C}$, so that $\mathfrak{A} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$. Here, '$\subseteq$' denotes 'subalgebra'. As mentioned above, it is clear that $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+\kappa} = \mathbf{S}\,\mathsf{Ra}\,\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$.

Now assume for contradiction that $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa} = \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$. Then

$$\begin{aligned}
\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+\kappa} &= \mathbf{S}\,\mathsf{Ra}\,\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa} \\
&= \mathbf{S}\,\mathsf{Ra}\,\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa} \\
&= \mathbf{S}\,\mathsf{Ra}\,\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa+1} \\
&= \mathbf{S}\,\mathsf{Ra}\,\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa+1} \\
&= \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+\kappa+1},
\end{aligned}$$

contradicting theorem 1. ∎

Provability with α variables. Each algebra $\mathfrak{N}^\beta_\alpha$ of the proof of theorem 1 is generated by a single atom. This has interesting consequences for proof theory. [31] gives a first-order language $\mathcal{L}$ with a single binary predicate symbol $\mathbf{E}$ and the equality symbol, together with axioms and rules for this language, and a second language, $\mathcal{L}^+$, obtained by extending $\mathcal{L}$ with operators corresponding to the relation algebra operators. The two formalisms are shown to be equipollent in means of expression and equipollent in means of proof [31, section 2.3, parts (vii) and (ix)]. Properties of the algebra $\mathfrak{N}^\beta_\alpha$ translate to properties of $\mathcal{L}$. A consequence of $\mathfrak{N}^\beta_\alpha \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_\alpha \setminus \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ is that there is a valid three-variable sentence X_α in $\mathcal{L}$ with a proof (in either formalism) using $\alpha+1$ variables but with no proof containing only α variables. See theorem 13. The underlying signature of X_α consists of a single binary relation symbol (corresponding to the atom generating $\mathfrak{N}^\beta_\alpha$), so is independent of α.

Corollary 2 has similar proof-theoretic implications for another provability relation $\vdash_{\alpha,\alpha+\kappa}$, described in [3, Section 7]. This provides $(\alpha+\kappa)$-variable proofs of α-variable formulas in a signature consisting of α-ary relation symbols. It is closely connected to $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$. Corollary 2 implies that $\vdash_{\alpha,\alpha+\kappa}$ is strictly weaker than $\vdash_{\alpha,\alpha+\kappa+1}$ for all finite α, κ with $\alpha \geq 3$. In [14, theorem 27], it is shown that the class $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa+1}$ cannot be defined by finitely many axioms within $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\kappa}$, for finite α, κ with $\alpha \geq 3$ and $\kappa \geq 1$. It follows that there is no finite set of α-variable axiom schemata whose set Σ of α-variable instances satisfies $\Sigma \vdash_{\alpha,\alpha+\kappa} \phi \leftrightarrow\ \vdash_{\alpha,\alpha+\kappa+1} \phi$ for all α-variable formulas ϕ. See [3, 14] for details.

Notation We generally use the same notation for an algebra as for its domain. So, for example, $X \subseteq \mathfrak{A}$ will indicate that X is a set of elements of the algebra $\mathfrak{A}$ (but note that we also write $\mathfrak{B} \subseteq \mathfrak{A}$ to denote that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$). As usual, an ordinal is the set of all smaller ordinals. If $\mathfrak{A}$ is a Boolean algebra with operators, $At\mathfrak{A}$ denotes the set of atoms, or minimal non-zero elements, of the Boolean part of $\mathfrak{A}$.

2 The algebras $\mathfrak{N}_\alpha^\beta$

Let $4 \le \alpha \le \beta < \omega$ and let $B_\alpha^\beta = \{1', e, p_1, \ldots, p_{\alpha-2}, q_0, \ldots, q_{\beta-1}\}$, a set with $\alpha+\beta$ elements. We will define a finite symmetric integral nonrepresentable relation algebra, called $\mathfrak{N}_\alpha^\beta$, whose Boolean part is the Boolean algebra of all subsets of B_α^β. For any $a, b, c \in B_\alpha^\beta$, let $[a, b, c]$ be the closure of $\langle a, b, c\rangle$ under permutations:

$$[a,b,c] = \{\, \langle a,b,c\rangle \,, \langle a,c,b\rangle \,, \langle b,c,a\rangle \,, \langle b,a,c\rangle \,, \langle c,a,b\rangle \,, \langle c,b,a\rangle \,\}\,.$$

Such a set is called a **cycle**.[2] The following cycles are *forbidden:*

$$\begin{aligned}
&[1', x, y] && \text{if } x \ne y && (2)\\
&[q_i, q_j, q_k] && \text{for all } i, j, k < \beta && (3)\\
&[r, s, t] && \text{if } r, s, t \in \{p_i, q_i\} \text{ for some } i \text{ with } 1 \le i < \alpha - 1 && (4)\\
&[r, s, t] && \text{for all } r, s, t \in \{e, q_0\} && (5)\\
&[e, q_i, q_j] && \text{if } i, j < \beta \text{ and } |i - j| > 1. && (6)
\end{aligned}$$

Let C be the union of the set of cycles that are *not* forbidden. C determines the binary operation ; as follows. For any $x, y \subseteq B_\alpha^\beta$, let

$$x;y = \sum\{c : \langle a,b,c\rangle \in C, a \in x, b \in y\}.$$

The converse $\breve{x}$ of any $x \subseteq B_\alpha^\beta$ is defined in the trivial way, namely, $\breve{x} = x$. The identity element of $\mathfrak{N}_\alpha^\beta$ is $\{1'\}$. We have, in effect, defined $\mathfrak{N}_\alpha^\beta$ as the complex algebra of the relational structure $\langle B_\alpha^\beta, C, f, \{1'\}\rangle$, where f is the identity relation on B_α^β, but we will use ordinary algebraic notation for $\mathfrak{N}_\alpha^\beta$ and ignore the distinction between $\{a\}$ and a for $a \in B_\alpha^\beta$. Thus, for example, we regard B_α^β as the set of atoms of $\mathfrak{N}_\alpha^\beta$.

The conditions on α and β are sufficient for our purposes, but not entirely necessary. The definitions apply to any α and β, but we assume $\alpha \le \beta$ for simplicity and to avoid exploring fringe cases. The algebras constructed here were obtained from algebras in [21] by the addition of all the cycles of the form $[e, q_\kappa, q_\kappa]$.

THEOREM 3 *For* $4 \le \alpha \le \beta < \omega$, $\mathfrak{N}_\alpha^\beta$ *is a finite symmetric integral relation algebra with cardinality* $2^{\alpha+\beta}$.

PROOF:

By [20, Ths. 1, 3], $\mathfrak{N}_\alpha^\beta$ is a relation algebra if, for all $v, w, x, y, z \in B_\alpha^\beta$, (a) either $[x, y, z] \subseteq C$ or $[x, y, z] \cap C = \emptyset$, (b) $x = y$ iff $\langle x, 1', y\rangle \in C$, and (c) if $\langle v, w, x\rangle \,, \langle x, y, z\rangle \in C$ then there is some $u \in B_\alpha^\beta$ such that $\langle w, y, u\rangle \,, \langle v, u, z\rangle \in C$. Now (a) is obvious from the definition of C, (b) is easy to check using the definition of forbidden cycles, and for (c), one of $1'$, e, p_1, or p_2 will work as u. Here we use $4 \le \alpha \le \beta$. $\mathfrak{N}_\alpha^\beta$ is obviously finite, and it is integral since $1'$ is an atom. ■

[2] This definition of 'cycle' is only appropriate for symmetric algebras.

We recall some notions from [21, 22]. If $\mathfrak{A}$ is an atomic relation algebra, then an α-dimensional *basic matrix* is a function $M : \alpha \times \alpha \to At(\mathfrak{A})$ such that, for all $i, j, k < \alpha$, $M_{i,i} \leq 1'$, $M_{i,j} = M_{j,i}\breve{}$ and $M_{i,j} ; M_{j,k} \geq M_{i,k}$. $B_\alpha \mathfrak{A}$ is the set of all α-dimensional basic matrices over $\mathfrak{A}$. An α-dimensional *cylindric basis* for $\mathfrak{A}$ is a set of S of α-dimensional basic matrices over $\mathfrak{A}$ such that

(C_0) for all atoms $a, b, c \in At\mathfrak{A}$ with $a \leq b; c$ there is $M \in S$ with $M_{0,1} = a$, $M_{0,2} = b$, and $M_{2,1} = c$,

(C_1) if $M, N \in S$, $i, j < \alpha$, $i \neq j$, and $M_{k,l} = N_{k,l}$ for all $k, l < \alpha$ with $i, j \notin \{k, l\}$, then there is $L \in S$ such that $M_{k,l} = L_{k,l}$ for all k, l with $i \notin \{k, l\}$ and $L_{k,l} = N_{k,l}$ for all k, l with $j \notin \{k, l\}$,

(C_2) if $M \in S$ and $i, j < \alpha$ then $M[i/j] \in S$, where $M[i/j] : \alpha \times \alpha \to At\mathfrak{A}$ is defined by $M[i/j]_{k,l} = M_{[i/j](k),[i/j](l)}$, for $k, l < \alpha$, and $[i/j] : \alpha \to \alpha$ is defined by $[i/j](i) = j$ and $[i/j](k) = k$ whenever $i \neq k < \alpha$.

LEMMA 4 *For $4 \leq \alpha \leq \beta < \omega$, $B_\alpha \mathfrak{N}^\beta_\alpha$ is an α-dimensional cylindric basis for $\mathfrak{N}^\beta_\alpha$.*

PROOF:

Since $\mathfrak{N}^\beta_\alpha$ is a relation algebra (by theorem 3), it suffices (by [21, Th. 7]) to show that $0 \neq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$ whenever

$$x_0, \ldots, x_{\alpha-3}, y_0, \ldots, y_{\alpha-3} \in \{e, p_1, p_2, \ldots, p_{\alpha-2}, q_0, q_1, q_2, \ldots, q_{\beta-1}\}.$$

Suppose that there is some γ such that $1 \leq \gamma < \alpha - 1$ and for every $\kappa < \alpha - 2$, $\{x_\kappa, y_\kappa\} \not\subseteq \{p_\gamma, q_\gamma\}$. Then $[p_\gamma, x_\kappa, y_\kappa]$ is not a forbidden cycle (all $\kappa < \alpha - 2$), hence $p_\gamma \leq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$.

If there is no such γ, then for each of the $\alpha - 2$ possible values of γ there is a κ ($= \kappa(\gamma)$) $< \alpha - 2$ with $\{x_\kappa, y_\kappa\} \subseteq \{p_\gamma, q_\gamma\}$. Clearly, the $\kappa(\gamma)$ are distinct. By cardinalities, every possible $\kappa < \alpha - 2$ must arise in this way: i.e., for every $\kappa < \alpha - 2$ there is some γ such that $\{x_\kappa, y_\kappa\} \subseteq \{p_\gamma, q_\gamma\}$ and $1 \leq \gamma < \alpha - 1$. Then $[e, x_\kappa, y_\kappa]$ is not a forbidden cycle so in this case we have $e \leq \prod_{\kappa<\alpha-2} x_\kappa ; y_\kappa$. ■

THEOREM 5 $\mathfrak{N}^\beta_\alpha \in \mathsf{Ra}\,\mathsf{CA}_\alpha$.

PROOF:

$B_\alpha \mathfrak{N}^\beta_\alpha$ is an α-dimensional cylindric basis by Lemma 4, so, by [21, Th. 10], we may apply the complex-algebra operator $\mathfrak{Ca}$ and obtain a cylindric algebra $\mathfrak{C} = \mathfrak{Ca} B_\alpha \mathfrak{N}^\beta_\alpha$ with $\mathfrak{C} \in \mathsf{CA}_\alpha$ and $\mathfrak{N}^\beta_\alpha \cong \mathfrak{Ra}\,\mathfrak{C}$. ■

THEOREM 6 $\mathfrak{N}^\beta_\alpha$ *is generated by e without using $1'$.*

PROOF:

It suffices to verify the following equations:

$$
\begin{aligned}
q_0 &= \overline{e;e}\cdot\overline{e}, \\
1' &= q_0;q_0\cdot\overline{e;q_0}, \\
q_1 &= e;q_0\cdot\overline{q_0;q_0}, \\
q_2 &= e;q_1\cdot\overline{e;q_0}\cdot\overline{e+q_0}, \\
q_\kappa &= e;q_{\kappa-1}\cdot\overline{e;q_{\kappa-2}} && \text{for } 3\leq\kappa<\beta, \\
p_\kappa &= q_0;q_0\cdot\overline{q_\kappa;q_\kappa} && \text{for } 1\leq\kappa\leq\alpha-2.
\end{aligned}
$$

■

DEFINITION 7 Define $f:\omega\to\omega$ by:

$$
\begin{aligned}
f(0) &= 0, \\
f(n+1) &= 1+n\cdot f(n), \quad \text{for } n<\omega.
\end{aligned}
$$

THEOREM 8 *If* $4\leq\alpha<\omega$ *and* $2(f(\alpha-1)+1)\leq\beta<\omega$ *then* $\mathfrak{N}^\beta_\alpha\notin \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$.

By a simple induction on α using the definition of f, the hypothesis of the theorem implies $\beta\geq\alpha$, as required for the definition of $\mathfrak{N}^\beta_\alpha$.

Because the proof is rather technical, it may be worth sketching some of the ideas behind the argument, before giving the detailed proof, which is an encoding of the standard proof of a special case of Ramsey's Theorem [7].[3] We could extend these results to cover the case $\beta=\omega$, though the algebras are no longer finite in this case. In some ways the argument is simplified when $\beta=\omega$ in that we do not have to worry about the size of 'large' sets in the construction — they are simply infinite. However here we use some finite value for β greater than $2f(\alpha-1)+1$ (definition 7), and this makes $\mathfrak{N}^\beta_\alpha$ into a finite relation algebra.

We saw that $\mathfrak{N}^\beta_\alpha\in\mathsf{Ra}\,\mathsf{CA}_\alpha$ (theorem 5) was easily proved using the following property: with only $\alpha-2$ pairs of diversity atoms $x_\kappa,y_\kappa\in\mathfrak{N}^\beta_\alpha$, $(\kappa<\alpha-2)$, it was impossible to force $\prod_{\kappa<\alpha-2}x_\kappa;y_\kappa=0$. However, moving up one dimension, it is quite easy to find diversity atoms $x_\kappa,y_\kappa\in\mathfrak{N}^\beta_\alpha$ for each $\kappa<(\alpha+1)-2$ with $\prod_{\kappa<\alpha-1}x_\kappa;y_\kappa=0$. For example, take $x_0=q_0$, $y_0=q_2$, and $x_\kappa=y_\kappa=p_\kappa$ for $1\leq\kappa<\alpha-1$, and consider the forbidden cycles. Of course, this does not prove that $\mathfrak{N}^\beta_\alpha\notin\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$; it merely shows that the proof of theorem 5 does not go through at this higher dimension.

In the formal proof (which follows this outline), we will assume, for contradiction, that $\mathfrak{N}^\beta_\alpha\subseteq\mathfrak{Ra}\,\mathfrak{C}$ for some cylindric algebra $\mathfrak{C}\in\mathsf{CA}_{\alpha+1}$. We derive a contradiction by proving the existence of certain 'large' subsets S_t of $\mathfrak{C}$ $(1\leq t\leq\alpha-1)$. (The notion of 'large' depends on t and weakens as t rises, so that $S_{\alpha-1}$ merely has to have two distinct elements. This is formalized by requiring that $|S_t|>f(\alpha-t)$—see definition 7.) Each $x\in S_t$ will determine a 'network' of 'colors'—atoms $x(i,t)$ of $\mathfrak{N}^\beta_\alpha$ (for $i<t$) such that $x\leq\mathsf{s}^0_i\mathsf{s}^1_t x(i,t)$. The chief properties that S_t will have is that (i) for each $x\in S_t$, $x(0,t)$ will be a q_γ for γ even and unique to x, and (ii) for each i with $1\leq i<t$ there is a single color $p_{\kappa(i)}$ for some $1\leq\kappa(i)<\alpha-1$, such that the (i,t)th color $x(i,t)$ of any $x\in S_t$ is $p_{\kappa(i)}$. The color $p_{\kappa(i)}$ is independent of $x\in S_t$ (and even independent of t).

[3]Indeed, [7, Sect. 5] shows that any value of β greater than the integer part of $(\alpha-2)!\,2e$ would suffice for our result.

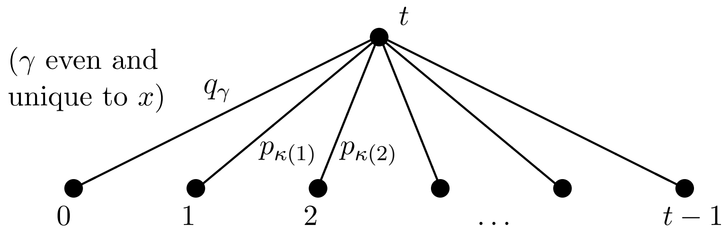

Figure 1: Colors of an element $x \in S_t$

To start ($t = 1$), as there are no values of i to consider, S_1 is built in a straightforward way, as $\{q_\gamma : \gamma < \beta \text{ even}\}$. It is large, as there is a very large number ($f(\alpha - 1)$) of atoms of $\mathfrak{N}^\beta_\alpha$ of the form q_γ for $\gamma < \beta$, γ even. To build S_{t+1} from S_t, we begin by 'gluing together' any two elements x, z of S_t, after first 'moving' the tth coordinate of x to $t + 1$ by taking $\mathsf{s}^t_{t+1}x$. See figure 2. By fixing z and varying x, a large number of different elements can be obtained

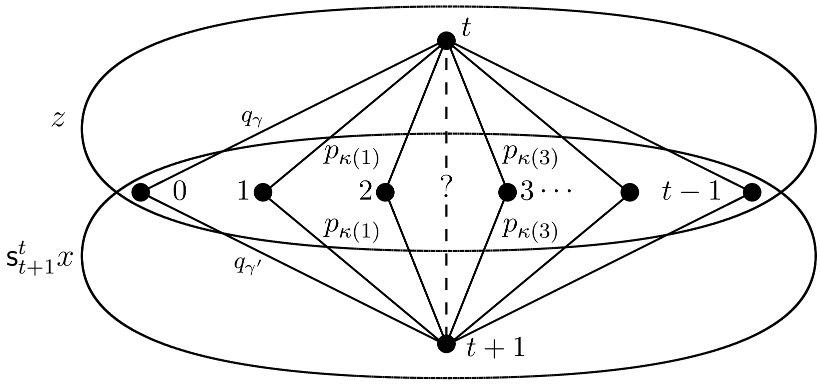

Figure 2: Amalgamation

in this way. Because they lie beneath elements obtained from S_t, their $(i, t+1)$th color is the same as the (i, t)th color for S_t, if $i < t$; but their $(t, t + 1)$th colors are unpredictable and may not all be the same. So we use the pigeon-hole principle to pick out a large subset S_{t+1} of them in which the $(t, t + 1)$th color is fixed; the notion of 'large' is chosen so that this is possible.

It follows from the definition of composition in $\mathfrak{N}^\beta_\alpha$, that for each t, the colors $p_{\kappa(i)}$ ($1 \leq i < t$) are all distinct. This gives a contradiction when $t = \alpha - 1$, because if $x, z \in S_{\alpha-1}$ are distinct, we can 'glue them together' to form a non-zero element whose (i, α)th color is $p_{\kappa(i)}$, for all $1 \leq i < \alpha$. Again, the colors $p_{\kappa(i)}$ must all be distinct—but now there are not enough colors to permit this, as there are $\alpha - 1$ values of i, but only $\alpha - 2$ available colors.

Proof of theorem 8 Now let us prove our result formally. We will freely use the facts about substitutions given below. If one thinks of the elements of an $(\alpha + 1)$-dimensional cylindric algebra $\mathfrak{C}$ as $(\alpha+1)$-ary relations R on some set X, then for $x_0,\ldots,x_\alpha \in X$, $(x_0,\ldots,x_\alpha) \in \mathsf{s}^i_j R$ iff $(x_0,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_\alpha) \in R$. In this light, the facts seem intuitively natural. However, the reader should beware: $\mathfrak{C}$ may not be representable as an algebra of $(\alpha+1)$-ary relations, and not all 'natural facts' about substitutions are valid in cylindric algebras. In fact, the study of substitution is very extensive; see, for example, [12, Sect. 1.5] or [32].

FACT 9 The following equations are valid in $\mathsf{CA}_{\alpha+1}$, for all $i,j,k,l \leq \alpha$.

1. $\mathsf{s}^i_j(x\cdot y) = \mathsf{s}^i_j x \cdot \mathsf{s}^i_j y$, $\mathsf{s}^i_j(x+y) = \mathsf{s}^i_j x + \mathsf{s}^i_j y$, $\mathsf{s}^i_j 0 = 0$, and $\mathsf{s}^i_j 1 = 1$ [12, 1.5.3].
2. If $x = \mathsf{c}_i x$ then $\mathsf{s}^i_j x = x$ [12, 1.5.8(i)].
3. $\mathsf{c}_i \mathsf{s}^j_i x = \mathsf{c}_j \mathsf{s}^i_j x$ [12, 1.5.9(i)].
4. $\mathsf{s}^i_j \mathsf{c}_k x = \mathsf{c}_k \mathsf{s}^i_j x$ if $k \neq i,j$ [12, 1.5.8(ii)].
5. $\mathsf{s}^i_j \mathsf{s}^k_l x = \mathsf{s}^k_l \mathsf{s}^i_j x$ if $\{i,j\}\cap\{k,l\} = \emptyset$ [12, 1.5.10(iii)].
6. $\mathsf{s}^i_j x = \mathsf{s}^i_k \mathsf{s}^i_j x$ if $i \neq j$ [12, 1.5.10(i)].
7. $\mathsf{s}^i_k \mathsf{s}^j_i x = \mathsf{s}^j_k \mathsf{s}^i_j x$ [12, 1.5.10(vi)].
8. $\mathsf{c}_i(x+y) = \mathsf{c}_i x + \mathsf{c}_i y$ and $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$. [12, Th. 1.2.6 and definition 1.1.1 C3].

Recall that we are assuming for contradiction that $\mathfrak{N}^\beta_\alpha \subseteq \mathfrak{Ra}\,\mathfrak{C}$ for some $\mathfrak{C} \in \mathsf{CA}_{\alpha+1}$. The domain of $\mathfrak{N}^\beta_\alpha$ is therefore a subset of the domain of $\mathfrak{C}$, and in fact, each $a \in \mathfrak{N}^\beta_\alpha$ is a two-dimensional element of $\mathfrak{C}$ ($a = \mathsf{c}_i a$ whenever $2 \leq i \leq \alpha$). Of course, $\breve{a} = a$ in $\mathfrak{N}^\beta_\alpha$.

We need two observations on the way elements of $\mathfrak{N}^\beta_\alpha$ behave in $\mathfrak{C}$. First, $a = \mathsf{s}^i_j a$ whenever $a \in \mathfrak{N}^\beta_\alpha$, $i,j < \alpha+1$, and $i \geq 2$. Second, composition uses only the first three dimensions of $\mathfrak{C}$: $a\,;b = \mathsf{c}_2(\mathsf{s}^1_2 a \cdot \mathsf{s}^0_2 b)$ for $a,b \in \mathfrak{Ra}\,\mathfrak{C}$. These dimensions can be 'moved', using substitutions. In the proof of [13, Th. 5.3.8], it is shown that $(*)$ $a;b = \mathsf{c}_j(\mathsf{s}^1_j a \cdot \mathsf{s}^0_j b)$ for $j = 3$, and the proof works whenever $3 \leq j < \alpha+1$. So

$$a;b = \mathsf{c}_j(\mathsf{s}^1_j a \cdot \mathsf{s}^0_j b) \text{ for } 2 \leq j < \alpha+1. \tag{7}$$

In terms of the discussion after the statement of theorem 8, this allows us to relate the (i,j)th color of an element of $\mathfrak{C}$, for varying i,j, to composition in $\mathfrak{N}^\beta_\alpha$, using the following lemma.

LEMMA 10 *Let $a,b \in \mathfrak{N}^\beta_\alpha$ and $i<j<k<\alpha+1$. Then* $\mathsf{s}^0_i \mathsf{s}^1_j a \cdot \mathsf{s}^0_j \mathsf{s}^1_k b \leq \mathsf{s}^0_i \mathsf{s}^1_k (a\,;b)$.

PROOF:

We show that $\mathsf{s}^0_i \mathsf{s}^1_k(a;b) = \mathsf{c}_j(\mathsf{s}^0_i \mathsf{s}^1_j a \cdot \mathsf{s}^0_j \mathsf{s}^1_k b)$, from which the lemma follows immediately. It follows from $i<j$ that $1 \leq j$, so there are two cases, $1 = j$ and $1 < j$. First consider the case $1<j$.

$$\begin{aligned}
\mathsf{s}^0_i \mathsf{s}^1_k(a;b) &= \mathsf{s}^0_i \mathsf{s}^1_k \mathsf{c}_j(\mathsf{s}^1_j a \cdot \mathsf{s}^0_j b) && (7)\\
&= \mathsf{c}_j \mathsf{s}^0_i \mathsf{s}^1_k(\mathsf{s}^1_j a \cdot \mathsf{s}^0_j b) && \text{fact 9.4}\\
&= \mathsf{c}_j(\mathsf{s}^0_i \mathsf{s}^1_k \mathsf{s}^1_j a \cdot \mathsf{s}^0_i \mathsf{s}^1_k \mathsf{s}^0_j b) && \text{fact 9.1}\\
&= \mathsf{c}_j(\mathsf{s}^0_i \mathsf{s}^1_j a \cdot \mathsf{s}^0_j \mathsf{s}^1_k b) && \text{facts 9.5, 9.6}
\end{aligned}$$

In case $j = 1$, we have $0 = i$ and $k \geq 2$. Choose m so that $2 \leq m < \alpha + 1$ and $k \neq m$.

$$
\begin{aligned}
\mathsf{s}_i^0 \mathsf{s}_k^1 (a ; b) &= \mathsf{s}_k^1 \mathsf{c}_m (\mathsf{s}_m^1 a \cdot \mathsf{s}_m^0 b) && i = 0, (7) \\
&= \mathsf{c}_m \mathsf{s}_k^1 (\mathsf{s}_m^1 a \cdot \mathsf{s}_m^0 b) && \text{fact 9.4} \\
&= \mathsf{c}_m (\mathsf{s}_k^1 \mathsf{s}_m^1 a \cdot \mathsf{s}_k^1 \mathsf{s}_m^0 b) && \text{fact 9.1} \\
&= \mathsf{c}_m (\mathsf{s}_m^1 a \cdot \mathsf{s}_m^1 \mathsf{s}_k^1 \mathsf{s}_m^0 b) && \text{fact 9.6} \\
&= \mathsf{c}_m \mathsf{s}_m^1 (a \cdot \mathsf{s}_k^1 \mathsf{s}_m^0 b) && \text{fact 9.1} \\
&= \mathsf{c}_1 \mathsf{s}_1^m (a \cdot \mathsf{s}_m^0 \mathsf{s}_k^1 b) && \text{facts 9.3, 9.5} \\
&= \mathsf{c}_1 (a \cdot \mathsf{s}_1^m \mathsf{s}_m^0 \mathsf{s}_k^1 b) && \text{facts 9.1, 9.2} \\
&= \mathsf{c}_1 (a \cdot \mathsf{s}_1^0 \mathsf{s}_0^m \mathsf{s}_k^1 b) && \text{fact 9.7} \\
&= \mathsf{c}_1 (a \cdot \mathsf{s}_1^0 \mathsf{s}_k^1 b) && \text{facts 9.5, 9.2} \\
&= \mathsf{c}_j (\mathsf{s}_i^0 \mathsf{s}_j^1 a \cdot \mathsf{s}_j^0 \mathsf{s}_k^1 b) && i = 0, j = 1.
\end{aligned}
$$

■

We construct by induction on t, with $1 \leq t < \alpha$, a set $S_t \subseteq \mathfrak{Nr}_{t+1}\mathfrak{C} \setminus \{0\}$ with the following properties:

1. $|S_t| > f(\alpha - t)$.
2. If $x, y \in S_t$ then $\mathsf{c}_t x = \mathsf{c}_t y$.
3. If $x \in S_t$, there is an even number $k < \beta$ with $x \leq \mathsf{s}_t^1 q_k$.
4. If $x, y \in S_t$, $k < \beta$, and $x, y \leq \mathsf{s}_t^1 q_k$, then $x = y$.
5. There is a one-one map $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ such that for each i with $1 \leq i < t$ and each $x \in S_t$, we have $x \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_{\kappa(i)}$.

We let $S_1 = \{q_k : k < \beta,\ k \text{ is even}\}$. Note that $S_1 \subseteq \mathfrak{Nr}_2\mathfrak{C} \setminus \{0\}$. Let us check that S_1 has the required properties. We use fact 9 freely. Clearly, $q_k ; 1 = 1$ for any k. That is, $1 = \mathsf{c}_2(\mathsf{s}_2^1 q_k \cdot \mathsf{s}_2^0 1) = \mathsf{c}_2 \mathsf{s}_2^1 q_k = \mathsf{c}_1 \mathsf{s}_1^2 q_k = \mathsf{c}_1 q_k$. It follows that property 2 above holds for S_1. We have $q_k = \mathsf{s}_1^1 q_k$ for each k, so property 3 holds. Property 4 holds because the q_k (for $k < \beta$) are pairwise disjoint. This gives property 1 because $|S_1| = |\{k < \beta : k \text{ is even}\}| = \lceil \frac{\beta}{2} \rceil \geq f(\alpha - 1) + 1 > f(\alpha - 1)$. Since $t = 1$, the last condition is vacuously true with $\kappa = \emptyset$.

Assume now that S_t is constructed, for some t with $1 \leq t < \alpha - 1$.

LEMMA 11 *Suppose that $x \in S_t$, $i < t$, and $a \in \mathfrak{N}_\alpha^\beta$ satisfy $x \leq \mathsf{s}_i^0 \mathsf{s}_t^1 a$. Then $\mathsf{s}_{t+1}^t x \leq \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 a$.*

PROOF:

By fact 9, $\mathsf{s}_{t+1}^t x \leq \mathsf{s}_{t+1}^t \mathsf{s}_i^0 \mathsf{s}_t^1 a = \mathsf{s}_i^0 \mathsf{s}_{t+1}^t \mathsf{s}_t^1 a = \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 \mathsf{s}_1^t a = \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 a$, the last equality holding trivially if $t = 1$, and because $a \in \mathfrak{Nr}_2\mathfrak{C}$ if $t \geq 2$. ■

Now we show how to obtain S_{t+1}. Fix some $z \in S_t$. For each atom $p \in \{p_1, \ldots, p_{\alpha-2}\}$ and $x \in S_t \setminus \{z\}$, define

$$\tau_x^p = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p).$$

Let $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ be the one-one function given by property 5 for S_t, and let $D = \{p_k : 1 \leq k < \alpha - 1, k \notin rng(\kappa)\}$. Note that $|D| = \alpha - 2 - |rng(\kappa)| = \alpha - 1 - t$.

LEMMA 12 *If $x \in S_t \setminus \{z\}$, then $z \leq \sum\{\tau_x^p : p \in D\}$.*

PROOF:

Fix $x \in S_t \setminus \{z\}$.
CLAIM. $z \cdot \mathsf{s}_{t+1}^t x \leq \sum_{p\in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p$.
PROOF OF CLAIM. Using fact 9.1, we have

$$z \cdot \mathsf{s}_{t+1}^t x \leq 1 = \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 1 = \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 \Big(\sum_{a \in At\mathfrak{N}_\alpha^\beta} a \Big) = \sum_{a \in At\mathfrak{N}_\alpha^\beta} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a.$$

So it suffices to show that if $a \in At\mathfrak{N}_\alpha^\beta$ and $(z \cdot \mathsf{s}_{t+1}^t x) \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$, then $a \notin \{1', e, q_k : k < \beta\} \cup \{p_l : l \in rng(\kappa)\}$.

Therefore let $w = z \cdot \mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a$, for some atom a of $\mathfrak{N}_\alpha^\beta$, and assume that $w \neq 0$. Properties 3 and 4 for S_t give us distinct even numbers $l, m < \beta$ such that $z \leq \mathsf{s}_t^1 q_l$ and $x \leq \mathsf{s}_t^1 q_m$. Observe that by lemma 11, $\mathsf{s}_{t+1}^t x \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m$, so that $w \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m$. So w satisfies:

$$w \leq \mathsf{s}_0^0 \mathsf{s}_t^1 q_l, \quad w \leq \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a, \quad w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 q_m.$$

Hence, $w \leq \mathsf{s}_0^0 \mathsf{s}_t^1 q_l \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a$, and by lemma 10, $w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 (q_l \,; a)$. We now see that $w \leq \mathsf{s}_0^0 \mathsf{s}_{t+1}^1 ((q_l \,; a) \cdot q_m)$, whence $(q_l \,; a) \cdot q_m \neq 0$—i.e., $[a, q_l, q_m]$ is not a forbidden cycle.

Now, from the definition of composition in $\mathfrak{N}_\alpha^\beta$, $a = 1'$ is impossible, since as $l \neq m$, this would give a forbidden cycle of type (2), $a = q_k$ is impossible, for $k < \beta$, since this gives a forbidden cycle of type (3), and $a = e$ is impossible, since this gives a forbidden cycle of type (6) (using the fact that l and m are distinct even numbers, so $|l - m| > 1$).

To rule out p_l for $l \in rng(\kappa)$ when $t > 1$, suppose that $i \in dom(\kappa)$ is such that $\kappa(i) = l$, so that $x, z \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_l$ by property 5. So $w \leq z \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_l$, and by lemma 11, $w \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_l$. As $w \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$, we have $\mathsf{s}_i^0 \mathsf{s}_t^1 p_l \cdot \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_l \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 a \neq 0$. By lemma 10, this implies that $\mathsf{s}_i^0 \mathsf{s}_{t+1}^1 (p_l \,; a \cdot p_l) \neq 0$. Thus, $a \neq p_l$, since $[p_l, p_l, p_l]$ is a forbidden cycle of type (4). This proves the claim.

Now, to prove the lemma, observe that

$$\begin{array}{rll}
z & = \; z \cdot \mathsf{c}_t z \; = \; z \cdot \mathsf{c}_t x & \text{by property 2 of } S_t, \\
& = \; z \cdot \mathsf{c}_t \mathsf{s}_t^{t+1} x \; = \; z \cdot \mathsf{c}_{t+1} \mathsf{s}_{t+1}^t x & \text{by } x \in \mathfrak{Nr}_{t+1}\mathfrak{C} \text{ and fact 9.3}, \\
& = \; \mathsf{c}_{t+1}(z \cdot \mathsf{s}_{t+1}^t x) & \text{since } z \in \mathfrak{Nr}_{t+1}\mathfrak{C}, \\
& = \; \mathsf{c}_{t+1}(z \cdot \mathsf{s}_{t+1}^t x \cdot \sum_{p\in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by the claim}, \\
& \leq \; \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \sum_{p\in D} \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by fact 9.8}, \\
& = \; \mathsf{c}_{t+1} \sum_{p\in D} (\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by distributing}, \\
& = \; \sum_{p\in D} \mathsf{c}_{t+1} (\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) & \text{by fact 9.8} \\
& = \; \sum_{p\in D} \tau_x^p & \text{by the definition.}
\end{array}$$

■

Our path is now clear. Since the lemma holds for all x, by Boolean manipulation we get

$$z \;\leq\; \prod_{x \in S_t \setminus \{z\}} \sum_{p \in D} \tau_x^p \;=\; \sum_{g : S_t \setminus \{z\} \to D} \; \prod_{x \in S_t \setminus \{z\}} \tau_x^{g(x)}.$$

So as $z \neq 0$, there is $g : S_t \setminus \{z\} \to D$ with $z \cdot \prod_{x \in S_t \setminus \{z\}} \tau_x^{g(x)} \neq 0$. Choose $X \subseteq S_t \setminus \{z\}$ with $|X| > f(\alpha - (t+1))$ on which g is constant. This is possible, for if $|g^{-1}(p)| \leq f(\alpha - t - 1)$ for all $p \in D$, then $|S_t \setminus \{z\}| \leq |D| \cdot f(\alpha - t - 1) = (\alpha - t - 1) \cdot f(\alpha - t - 1) = f(\alpha - t) - 1$, so that $|S_t| \leq f(\alpha - t)$, contradicting property 1 for S_t. Let p be the constant value of g on X, and define:

$$\begin{array}{rcl} \zeta & = & z \cdot \prod_{x \in X} \tau_x^p \neq 0, \\ x' & = & \mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p \cdot \zeta, \quad \text{for each } x \in X, \\ S_{t+1} & = & \{x' : x \in X\}. \end{array}$$

We check the required properties for S_{t+1}. Recall that $\tau_x^p = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p)$. To begin, note that $\mathsf{c}_i y = y$ for every $y \in S_{t+1}$ and $i \geq t+2$, so that $S_{t+1} \subseteq \mathfrak{Nr}_{t+2}\mathfrak{C}$. For property 2, observe that $\mathsf{c}_{t+1}\zeta = \zeta$, so for all $x \in X$,

$$\mathsf{c}_{t+1} x' = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p \cdot \zeta) = \mathsf{c}_{t+1}(\mathsf{s}_{t+1}^t x \cdot \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p) \cdot \zeta = \tau_x^p \cdot \zeta = \zeta.$$

This also shows that the $x' \in S_{t+1}$ are non-zero, since $\zeta \neq 0$.

Consider property 3. Let $x \in X$. By property 3 for S_t, $x \leq \mathsf{s}_t^1 q_k$ for some even k. By definition of x' and lemma 11 (with $i = 0$), $x' \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_{t+1}^1 q_k$. This proves property 3.

Now we deal with properties 4 and 1. For property 4, assume that $x, y \in X$ and $x', y' \leq \mathsf{s}_{t+1}^1 q_b$ for some $b < \beta$. We will prove that $x = y$. Choose even $k, l < \beta$ such that $x \leq \mathsf{s}_t^1 q_k$ and $y \leq \mathsf{s}_t^1 q_l$. By the argument for property 3, $x' \leq \mathsf{s}_{t+1}^1 q_k$ and $y' \leq \mathsf{s}_{t+1}^1 q_l$. So $x' \leq \mathsf{s}_{t+1}^1 q_k \cdot \mathsf{s}_{t+1}^1 q_b = \mathsf{s}_{t+1}^1 (q_k \cdot q_b)$. Since $x' \neq 0$, we must have $q_k = q_b$ and $k = b$. Similarly, $l = b$. So $k = l$, and by property 4 for S_t, we obtain $x = y$, as required. Hence, $x' = y'$, which yields property 4 for S_{t+1}. This argument also shows that $x' = y' \Rightarrow x = y$, since if $x' = y'$ then by property 3 there does exist $b < \beta$ with $x', y' \leq \mathsf{s}_{t+1}^1 q_b$. So the map $x \mapsto x'$ is a bijection from X to S_{t+1}, whence $|S_{t+1}| = |X| > f(\alpha - (t+1))$. This proves property 1 for S_{t+1}.

Finally, we arrange property 5. Recall that $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ is the existing (possibly empty) one-one function for S_t. Let $1 \leq i < t$ and $x \in X$. Then $x \leq \mathsf{s}_i^0 \mathsf{s}_t^1 p_{\kappa(i)}$, so by lemma 11, $x' \leq \mathsf{s}_{t+1}^t x \leq \mathsf{s}_i^0 \mathsf{s}_{t+1}^1 p_{\kappa(i)}$. Finally, for the case $i = t$, we have $x' \leq \mathsf{s}_t^0 \mathsf{s}_{t+1}^1 p$ by definition of x'. So we may define $\kappa' : \{1, \ldots, t\} \to \{1, \ldots, \alpha - 2\}$ suitable for S_{t+1} by: $\kappa' \restriction_{\{1, \ldots, t-1\}} = \kappa$, and $p_{\kappa'(t)} = p$. Then κ' is one-one, because $p \in D$. This finishes the inductive construction.

In the case $t = \alpha - 1$, the construction yields a set S_t with $|S_t| > f(\alpha - t) = 1$. So there are distinct $z, x \in S_t$. In this case, the one-one map $\kappa : \{1, \ldots, t-1\} \to \{1, \ldots, \alpha - 2\}$ is surjective, so that $D = \emptyset$. Applying lemma 12 to x now shows that $z \leq \sum\{\tau_x^p : p \in D\} = 0$. This is a contradiction, and proves theorem 8. ■

3 Application to logic

We apply the main result to the logical systems of [31] and solve a problem raised by Tarski and Givant[4] by showing in theorem 13 that the first displayed inequality on page 93 of [31] does hold for every $n > 4$. It was previously known to fail for 3, to hold for 4, and to hold for infinitely many n [21, Th. 24(i), Th. 25], [31, pp. 92–3]. We review here the concepts we need from [31]. $\mathcal{L}$ is a language of first order logic that has the equality symbol $\mathring{\mathbf{1}}$ and one additional binary predicate $\mathbf{E}$ (for "elementhood"). The set of variables of $\mathcal{L}$ is $\mathbf{\Upsilon} = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots\}$. The only connectives used in [31] are the implication and negation symbols. The only quantifier is

[4] [31, p. 93], [6, p. 208, problem 15], and [4, p. 735, problem 23].

the universal quantifier. The other connectives and the existential quantifier are introduced as abbreviations. Atomic formulas are formed as usual; $\mathbf{v}_0\mathbf{E}\mathbf{v}_1$ and $\mathbf{v}_2\overset{\circ}{\mathbf{1}}\mathbf{v}_3$ are examples of atomic formulas. Formulas are formed in the standard way from atomic formulas using propositional connectives and quantifiers. A sentence is a formula with no free variables. The set of formulas of $\mathcal{L}$ is $\mathbf{\Phi}$, and the set of sentences of $\mathcal{L}$ is $\mathbf{\Sigma}$. For any formula $X \in \mathbf{\Phi}$, the universal closure $[\,X\,]$ of X is formed by universally quantifying X with respect to all its free variables, where the order of quantification is determined by the indices of the variables. There are nine logical axiom schemata for $\mathcal{L}$, adopted from [30], namely,

(AI) $[\,(X \to Y) \to ((Y \to Z) \to (X \to Z))\,]$,

(AII) $[\,(\neg X \to X) \to X\,]$,

(AIII) $[\,X \to (\neg X \to Y)\,]$,

(AIV) $[\,\forall x \forall y X \to \forall y \forall x X\,]$,

(AV) $[\,\forall x(X \to Y) \to (\forall x X \to \forall x Y)\,]$,

(AVI) $[\,\forall x X \to X\,]$,

(AVII) $[\,X \to \forall x X\,]$, where x is not free in X,

(AVIII) $[\,\exists x(x\overset{\circ}{\mathbf{1}}y)\,]$, where $x \neq y$.

(AIX) $[\,x\overset{\circ}{\mathbf{1}}y \to (X \to Y)\,]$, where X is atomic and Y is obtained from X by replacing a single occurrence of x by y,

for $X, Y, Z \in \mathbf{\Phi}$ and $x, y \in \mathbf{\Upsilon}$. All these axioms are sentences, and *modus ponens* is the only rule of inference. The rule of "generalization" is not needed. If a sentence $X \in \mathbf{\Sigma}$ can be proved in $\mathcal{L}$ we write $\vdash X$. By the completeness theorem, for any sentence $X \in \mathbf{\Sigma}$ we have $\vdash X$ iff X is logically valid (true in all models).

Tarski's extension of $\mathcal{L}$, called $\mathcal{L}^+$, is obtained by adding operators on predicates and another equality symbol, namely $=$ (which is why $\overset{\circ}{\mathbf{1}}$ is used earlier). There are two binary and two unary predicate operators, corresponding to the basic operations of relation algebras. For example, if P and Q are binary predicates of $\mathcal{L}^+$, then so are $P{+}Q$, $P{\odot}Q$, $\overline{P}$, and $\breve{P}$. The set of predicates so obtained, starting from $\mathbf{E}$ and $\overset{\circ}{\mathbf{1}}$, is $\mathbf{\Pi}$. The set of variables of $\mathcal{L}^+$ is $\mathbf{\Upsilon}$ (as it is for $\mathcal{L}$). The atomic formulas of $\mathcal{L}^+$ are formed using $\mathbf{\Pi}$ instead of $\{\mathbf{E}, \overset{\circ}{\mathbf{1}}\}$. For example, $\mathbf{v}_0\mathbf{E} \odot \mathbf{E}\mathbf{v}_1$ is an atomic formula of $\mathcal{L}^+$. Additionally, $\mathcal{L}^+$ contains other atomic formulas, namely the variable-free equations $P = Q$ for all predicates $P, Q \in \mathbf{\Pi}$. The formulas and sentences of $\mathcal{L}^+$ are obtained from this larger set of atomic formulas in the same way as in $\mathcal{L}$. $\mathbf{\Phi}^+$ and $\mathbf{\Sigma}^+$ are the sets of formulas and sentences of $\mathcal{L}^+$, respectively. The logical axioms (AI)–(AIX) of $\mathcal{L}$ are included among the logical axioms of $\mathcal{L}^+$. In addition, some axioms are added that serve as definitions of the predicate operators, namely,

(DI) $\forall \mathbf{v}_0 \forall \mathbf{v}_1(\mathbf{v}_0 P + Q \mathbf{v}_1 \leftrightarrow (\mathbf{v}_0 P \mathbf{v}_1 \vee \mathbf{v}_0 Q \mathbf{v}_1))$,

(DII) $\forall \mathbf{v}_0 \forall \mathbf{v}_1(\mathbf{v}_0 \overline{P} \mathbf{v}_1 \leftrightarrow \neg \mathbf{v}_0 P \mathbf{v}_1)$,

(DIII) $\forall \mathbf{v}_0 \forall \mathbf{v}_1(\mathbf{v}_0 P{\odot}Q \mathbf{v}_1 \leftrightarrow \exists \mathbf{v}_2(\mathbf{v}_0 P \mathbf{v}_2 \wedge \mathbf{v}_2 Q \mathbf{v}_1))$,

(DIV) $\forall \mathbf{v}_0 \forall \mathbf{v}_1(\mathbf{v}_0 \breve{P} \mathbf{v}_1 \leftrightarrow \mathbf{v}_1 P \mathbf{v}_0)$,

(DV) $P = Q \leftrightarrow \forall \mathbf{v}_0 \forall \mathbf{v}_1 (\mathbf{v}_0 P \mathbf{v}_1 \leftrightarrow \mathbf{v}_0 Q \mathbf{v}_1)$.

where $P, Q \in \mathbf{\Pi}$. For an $\mathcal{L}^+$-sentence X, we write $\vdash^+ X$ if there is a proof of X in $\mathcal{L}^+$. $\mathcal{L}^+$ is an extension of $\mathcal{L}$. Indeed, an immediate consequence of the definitions is that $\mathbf{\Phi} \subseteq \mathbf{\Phi}^+$ and $\mathbf{\Sigma} \subseteq \mathbf{\Sigma}^+$. In [31] it is shown that $\mathcal{L}^+$ is actually equipollent with $\mathcal{L}$ in means of both expression and proof; there is a recursive function $\mathbf{G}$ that eliminates occurrences of the predicate operators in accordance with (DI)–(DV), preserving provability and meaning; see [31, Ch. 2] for details.

For $3 \leq \alpha < \omega$, the formalisms $\mathcal{L}_\alpha$ and $\mathcal{L}^+_\alpha$ are obtained from $\mathcal{L}$ and $\mathcal{L}^+$ by restricting the set of variables to $\mathbf{\Upsilon}_\alpha = \{\mathbf{v}_0, \ldots, \mathbf{v}_{\alpha-1}\}$ [31, p. 91]. The sets of formulas and sentences of $\mathcal{L}_\alpha$ and $\mathcal{L}^+_\alpha$ are $\mathbf{\Phi}_\alpha$, $\mathbf{\Phi}^+_\alpha$, $\mathbf{\Sigma}_\alpha$, and $\mathbf{\Sigma}^+_\alpha$. In accordance with [31, p. 70], we may take the logical axiom schemata of $\mathcal{L}_\alpha$ to be (AI)–(AIX), restricted to formulas in $\mathbf{\Phi}_\alpha$, together with

(AIX″) $[x \mathring{\mathbf{1}} y \to (X \to X_{xy})]$,

where $x, y \in \mathbf{\Upsilon}$, $X \in \mathbf{\Phi}_\alpha$, and X_{xy} is obtained from X by transposing x and y, that is, simultaneously replacing *every* occurrence (whether free or bound) of x with y and y with x, and, in case $\alpha = 3$,

(AX′) $[\exists \mathbf{v}_2 (\exists \mathbf{v}_1 (X \wedge Y_{\mathbf{v}_0\mathbf{v}_2}) \wedge Z_{\mathbf{v}_0\mathbf{v}_2}) \leftrightarrow \exists \mathbf{v}_2 (X_{\mathbf{v}_1\mathbf{v}_2} \wedge \exists \mathbf{v}_0 (Y_{\mathbf{v}_1\mathbf{v}_2} \wedge Z)]$,

where X, Y, Z are any formulas in which $\mathbf{v}_0$ and $\mathbf{v}_1$ are the only free variables. Axiom (AIX″) accompanies (AIX) for various technical reasons discussed in [31, pp. 70–1]; it makes it possible to respell bound variables by allowing a proof that "alphabetic variants" are equivalent. Axiom (AX′), which expresses the associative law for relative multiplication, is not included when $\alpha > 3$ because in that case it is derivable on the basis of the remaining axioms. The logical axioms of $\mathcal{L}^+_\alpha$ are those of $\mathcal{L}_\alpha$ together with (DI)–(DV). We write $\vdash_\alpha X$ if there is a proof of X in $\mathcal{L}_\alpha$ and we write $\vdash^+_\alpha X$ if there is a proof of X in $\mathcal{L}^+_\alpha$. Note that the next theorem fails when $\alpha = 3$ [21, Th. 24(i)].

THEOREM 13 *Let $4 \leq \alpha < \omega$. There is an equation $P = Q \in \mathbf{\Sigma}^+_3$ with*

$$\vdash^+_{\alpha+1} P = Q \text{ but } \nvdash^+_\alpha P = Q$$

and a sentence $X \in \mathbf{\Sigma}_3$ such that

$$\vdash_{\alpha+1} X \text{ but } \nvdash_\alpha X.$$

PROOF:

The proof is based on the proof of [21, Th. 25]. We concentrate on the first part of the theorem; the second part then follows easily using the elimination mapping $\mathbf{G}$.

Suppose $\beta > \alpha$. Later we will choose $\beta \geq 2(f(\alpha - 1) + 1)$. We use the algebra $\mathfrak{N}^\beta_\alpha$ to form an algebraic realization of $\mathcal{L}^+_\alpha$. By Lemma 4, the set $B_\alpha \mathfrak{N}^\beta_\alpha$ is an α-dimensional cylindric basis for $\mathfrak{N}^\beta_\alpha$. It is obvious that $B_\alpha \mathfrak{N}^\beta_\alpha$ is 'permutation closed': if $M \in B_\alpha \mathfrak{N}^\beta_\alpha$ and τ is a transposition on α then $B_\alpha \mathfrak{N}^\beta_\alpha$ also contains the matrix $M\tau$, which is defined by $(M\tau)_{ij} = M_{\tau(i),\tau(j)}$ for all $i, j < \alpha$. So $\mathfrak{M} = \langle \mathfrak{N}^\beta_\alpha, e, B_\alpha \mathfrak{N}^\beta_\alpha \rangle$ is an α-dimensional algebraic realization according to [21, Def. 16].

$$\begin{tikzcd}
\mathfrak{P} \arrow[r, "h"] \arrow[dr, "D^{\mathfrak{M}}"'] & \mathfrak{A} \arrow[d, dashed, "g"] \\
& \mathfrak{N}^{\beta}_{\alpha}
\end{tikzcd}$$

Figure 3: The homomorphism g

The set $\mathbf{\Pi}$ of predicates forms an algebra under the predicate operations, namely

$$\mathfrak{P} = \left\langle \mathbf{\Pi}, +, {}^{-}, \odot, {}^{\smile}, \mathring{\mathbf{1}} \right\rangle .$$

The algebra $\mathfrak{P}$ is an absolutely free algebra that is freely generated by $\mathbf{E}$ (see [31, p. 238]). Treating the connectives and quantifiers as operations on formulas, we construct another absolutely free algebra, this time similar to $(\alpha+1)$-dimensional cylindric algebras, namely,

$$\mathfrak{C} = \left\langle \mathbf{\Phi}^{+}_{\alpha+1}, \wedge, \vee, \neg, \exists \mathbf{v}_{\kappa}, \mathbf{v}_{\kappa} \mathring{\mathbf{1}} \mathbf{v}_{\lambda} \right\rangle_{\kappa,\lambda<\alpha+1} .$$

Define a binary relation $\simeq^{+}_{\alpha+1}$ on $\mathbf{\Phi}^{+}_{\alpha+1}$ as follows. If $X, Y \in \mathbf{\Phi}^{+}_{\alpha+1}$, then

$$X \simeq^{+}_{\alpha+1} Y \quad \text{iff} \quad \vdash^{+}_{\alpha+1} [X \leftrightarrow Y].$$

It is easy (but somewhat tedious) to show, using axioms (AI)–(AIX) and (AIX$''$), that $\simeq^{+}_{\alpha+1}$ is a congruence relation on $\mathfrak{C}$ and that $\mathfrak{C}/\simeq^{+}_{\alpha+1} \in \mathsf{CA}_{\alpha+1}$. Note that $\mathfrak{Ra}\left(\mathfrak{C}/\simeq^{+}_{\alpha+1}\right)$ is a relation algebra since $\alpha + 1 \geq 4$.

Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{Ra}\left(\mathfrak{C}/\simeq^{+}_{\alpha+1}\right)$ generated by $\mathbf{v}_0\mathbf{E}\mathbf{v}_1/\simeq^{+}_{\alpha+1}$. Observe that $\mathfrak{A} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$. Let h be the homomorphism from $\mathfrak{P}$ onto $\mathfrak{A}$ determined by the condition that $h(\mathbf{E}) = \mathbf{v}_0\mathbf{E}\mathbf{v}_1/\simeq^{+}_{\alpha+1}$. With the help of (DI)–(DV), it is easy to prove that $h(P) = \mathbf{v}_0 P \mathbf{v}_1/\simeq^{+}_{\alpha+1}$ for every $P \in \mathbf{\Pi}$. (Alternatively, one can define the mapping h by this formula, and then use (DI)–(DV) to prove that h is a homomorphism.)

The denotation function $D^{\mathfrak{M}}$ [21, Def. 17] maps predicates in $\mathbf{\Pi}$ to elements of $\mathfrak{N}^{\beta}_{\alpha}$ and formulas in $\mathbf{\Phi}^{+}_{\alpha}$ to subsets of $B_{\alpha}\mathfrak{N}^{\beta}_{\alpha}$. Restricted to $\mathbf{\Pi}$, $D^{\mathfrak{M}}$ is a homomorphism from $\mathfrak{P}$ onto $\mathfrak{N}^{\beta}_{\alpha}$. Since $\mathfrak{P}$ is absolutely freely generated by $\mathbf{E}$, this homomorphism is completely determined by the condition that $D^{\mathfrak{M}}(\mathbf{E}) = e$. It follows from [21, Th. 22(iii)] that

$$\vdash^{+}_{\alpha} X \text{ implies } D^{\mathfrak{M}}(X) = B_{\alpha}\mathfrak{N}^{\beta}_{\alpha} \tag{8}$$

Next we will derive a contradiction by assuming that $D^{\mathfrak{M}}(P) = D^{\mathfrak{M}}(Q)$ whenever $h(P) = h(Q)$ and $P, Q \in \mathbf{\Pi}$. This assumption implies that $D^{\mathfrak{M}}$ can be factored through h, yielding a homomorphism g from $\mathfrak{A}$ onto $\mathfrak{N}^{\beta}_{\alpha}$ where $g(h(P)) = D^{\mathfrak{M}}(P)$ for every $P \in \mathbf{\Pi}$. See figure 3. Since $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ is a variety, it is closed under forming homomorphic images, and since $\mathfrak{A} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ we must have $\mathfrak{N}^{\beta}_{\alpha} \in \mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ too. But $\mathfrak{N}^{\beta}_{\alpha}$ is not in $\mathbf{S}\,\mathsf{Ra}\,\mathsf{CA}_{\alpha+1}$ if we choose $\beta \geq 2(f(\alpha-1)+1)$.

From this contradiction, it follows that there must be $P, Q \in \mathbf{\Pi}$ with $h(P) = h(Q)$ and $D^{\mathfrak{M}}(P) \neq D^{\mathfrak{M}}(Q)$. From $h(P) = h(Q)$ we get $\vdash^{+}_{\alpha+1} [\, \mathbf{v}_0 P \mathbf{v}_1 \leftrightarrow \mathbf{v}_0 Q \mathbf{v}_1 \,]$, which implies $\vdash^{+}_{\alpha+1} P = Q$. On the other hand, $D^{\mathfrak{M}}(P) \neq D^{\mathfrak{M}}(Q)$ implies $D^{\mathfrak{M}}(P = Q) \neq B_\alpha \mathfrak{N}^{\beta}_{\alpha}$ by [21, Lem. 21(x)], hence by (8), $\nvdash^{+}_{\alpha} P = Q$. Note that $P = Q$ is in $\mathbf{\Sigma}^{+}_{3}$ since it is a sentence with no variables at all. Thus, $P = Q$ witnesses the first part of the theorem. To get a sentence witnessing the second part, just apply the elimination mapping $\mathbf{G}$ [31, 2.3(iii)] to $P = Q$, obtaining a sentence $\mathbf{G}(P = Q) \in \mathbf{\Sigma}_3$. By the equipollence results [31, Sect. 3.8] this gives $\nvdash_{\alpha} \mathbf{G}(P = Q)$ and $\vdash_{\alpha+1} \mathbf{G}(P = Q)$. ■

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