The Finite Representation Property for reducts of Relation Algebra

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Abstract

We survey the finite representation property for signatures definable by the signature of relation algebra. We show that signatures excluding composition have the finite representation property but signatures containing either \{·, 1′; \} or \{·, ; \} do not. We show that the class of representable structures in the signature \{0, ≤, 1′, ; dom, ran\} is finitely axiomatisable, furthermore if \(A\) is a representable structure in this signature then it has a representation of size at most \(2^{|A|}\). There are 567 definitionally complete subsignatures of \{0, 1, −, +, ≤, 1′, ; dom, ran\} of which 302 are shown to have the finite representation property and 43 are shown to fail the finite representation property.

A Set Relation Algebra (SRA) over a base set \(X\) is a set of binary relations over \(X\), including the empty relation, a maximal relation, the identity relation on \(X\) and closed under set union, complementation relative to the maximal relation, conversion and composition. A Relation Algebra (RA) is a structure in the signature \{0, 1, +, −, 1′, ; \} obeying a certain finite set of equations [CT51, JT52]. RRA denotes the class of all algebras in this signature isomorphic to an SRA, certainly the relation algebra axioms are valid over RRA, so RRA \(\subseteq\) RA. When Tarski proposed relation algebra as a basis for set theory and mathematics he may have hoped that his axiomatic system would also be complete, i.e. RA = RRA or equivalently every true property of SRAs could be derived from the RA-axioms, but his axioms turned out not to be complete [Lyn50] nor could any finite set of axioms be complete [Mon64]. Most of the results concerning the logical and computational behaviour of relation algebras turned out to be negative: although RRA is a canonical variety, it cannot be axiomatised by any equational theory using only finitely many variables [Jon91], nor by any theory of canonical equations [HV05], the equational theory is undecidable [Tar41], and the problem of determining whether a finite relation algebra is isomorphic to an SRA is undecidable [HH01] (but note that it is not known whether the problem of determining whether a finite relation algebra is isomorphic to an SRA over a finite base is decidable). Consequently, there has been much interest in finding a signature weaker than the full relation algebra signature whose representation class can be handled more easily. In this article we study the finite representation...
property for subsignatures of the relation algebra signature and we discover a fairly expressive signature whose representation class is finitely axiomatisable and where finite representable structures always have finite representations.

**DEFINITION 1** All signatures considered in this paper will include the equality predicate $=$ and are subsets of

$$\{=, 0, 1, -, \cdot, \leq, 1', \cdots, \text{dom}, \text{ran}\}$$

$0, 1, 1'$ are constants, $-$, $\cdot$, $\text{dom}, \text{ran}$ are unary functions, $+,$ $\cdot$ are binary functions, $=,$ $\leq$ are binary predicates. Since equality is always included we will not explicitly mention it in the signatures below. Let $S$ be any subset of $(\text{Sig})$ and let $A$ be an $S$-structure. An interpretation $R$ of $A$ over base $X$ assigns a binary relation $a_R \subseteq X \times X$ to each element $a \in A$. An interpretation $R$ of $A$ over base $X$ is called an $S$-representation if each element of $S$ is properly represented:

$$(a = b) \iff a_R = b_R$$

$$(0) = \emptyset$$

$$(a - b)_R = \{(x, y) : \exists b \in A (x, y) \in h(b) \setminus h(a)\}$$

$$(a + b)_R = a_R \cup b_R$$

$$(a \cdot b)_R = a_R \cap b_R$$

$$(a \leq b) \iff a_R \subseteq b_R$$

$$(1')_R = \{(x, x) : x \in X\}$$

$$(a^-)_R = \{(x, y) : (y, x) \in a^R\}$$

$$(a; b)_R = \{(x, y) : \exists z (x, z) \in a^R \land (z, y) \in b^R\}$$

$$\text{dom}(a))_R = \{(x, x) : \exists y (x, y) \in a^R\}$$

$$\text{ran}(a))_R = \{(y, y) : \exists x (x, y) \in a^R\}$$

whenever the relevant constant, function or predicate belongs to $S$.

We write $R(S)$ for the class of all $S$-structures possessing representations. $S$ has the finite representation property if every finite representable $S$-structure possesses a representation over a finite base.

If $S$ is includes the relevant symbols then

$$R(S) \models (a \leq b \iff a + b = b)$$

$$R(S) \models (a \cdot b = -(a + -b))$$

$$R(S) \models (\text{dom}(a) = 1' \cdot a; a^-)$$

$$R(S) \models (\text{ran}(a) = \text{dom}(a^-))$$

so $\leq, \cdot, \text{dom}, \text{ran}$ are term-definable by the rest of the signature, but we may wish to include these as primitive symbols when some of the other symbols are missing from our signature. The full list of term-definitions within $(\text{Sig})$ is given in figure 1. Other functions are definable by the signature of relation algebra.

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and are of interest, for example the antidomain operator \(1' \cdot \text{dom}(x)\), the residuals \(-\left(x;(-y)\right), -((x);y^-)\), relational addition \(-((x);(-y))\), symmetric difference \((x \cdot (-y)) + ((-x) \cdot y)\), etc., but we do not cover them here.

We will survey the finite representation property for subsignatures \(S\) of \((\text{Sig})\) and we will also be interested to see if \(R(S)\) is definable by a finite set of axioms. Our investigation touches on two other problems that have received some attention recently: (i) is membership of \(R(S)\) decidable, for finite \(S\)-structures and (ii) is the class of all finite \(S\)-structures with representations over finite bases recursive? It is known that membership of \(R(S)\) is undecidable, for finite \(S\)-structures, when \(S \supseteq \{1', \cdot, -\}\) [HH01, HJ11], so problem (i) has a negative answer for such signatures. Very little is known about problem (ii) (see [HH02, Problem 18.18]). We can use the results of the current paper to find signatures where (i) and (ii) have positive answers by finding signatures \(S\) where \(R(S)\) is finitely axiomatisable and has the finite representation property.

Unfortunately, \((\text{Sig})\) has eleven elements, not counting equality, so there are \(2^{11}\) subsets to consider. In many cases two distinct subsignatures will be ‘interdefinable’ i.e. each symbol in one signature may be term-defined by symbols in the other in the manner of (1), so we restrict our attention to subsets of \(S\) closed under definability and for the rest of this paper we will refer to such a signature as complete.

**Definition 2** Let \(S\) be a subset of \((\text{Sig})\). \(S\) is complete if it satisfies the following conditions: if \(+\in S\) or \(\cdot\) in \(S\) then \(\le\in S\); for all \(n\)-ary functions \(f \in S\) (where \(n = 0, 1\) or \(2\)) if there is a term \(\tau\) using only symbols \(\bar{s}\) from \(S\) such that \(R(S) \models f(\bar{x}) \leftrightarrow \tau(\bar{s})(\bar{x})\) (where \(\bar{x}\) is an \(n\)-tuple of variables) then \(f \in S\). The symbols definable by others are listed in figure 1.

Each subset of \((\text{Sig})\) generates a complete set, the intersection of all the complete sets containing it, e.g. \(\{+, -\}\) generates the complete set \(\{0, 1, -, \cdot, +, \le\}\).

**Lemma 3** The number of complete subsignatures of \((\text{Sig})\) is 567. This set of complete subsignatures may be divided into eight parts, according to which of \(1', \cdot\) and \(;\) is included, the number of complete subsignatures in each of the eight parts is shown in a three-variable Karnaugh map in figure 2.

**Proof:**

Let \(B = \{0, 1, -, \cdot, \le\}\) and \(C = \{1', -, \cdot, \text{dom}, \text{ran}\}\) so \((\text{Sig})\) is the disjoint union of \(B\) and \(C\). In figure 1, in all but the last two cases we define a symbol in either \(B\) or \(C\) using only symbols from the same set (the last two cases define symbols from \(C\) using symbols from both \(B\) and \(C\)). For any complete subset \(S\) of \((\text{Sig})\), observe
that \(S \cap B\) and \(S \cap C\) are also complete (from figure 1, symbols from \(C\) alone cannot define symbols form \(B\) nor can symbols of \(B\) alone define symbols of \(C\)). So we start by considering complete subsets of the boolean signature \(B\) and complete subsets of \(C\).

**Claim 1:** There are 29 complete subsets of \(B\), of these 10 are disjoint from \(\{1, \cdot\}\), 4 include \(\cdot\) but not 1, 10 include 1 but not \(\cdot\), and 5 contain \(\{1, \cdot\}\). To prove the claim note first that there are four complete subsets \(A\) of \(\{0, 1\}\) (two of which include 1) and there are five complete subsets \(A'\) of \(\{+, \cdot, \leq\}\), namely \(\emptyset, \{\leq\}, \{\leq, +\}, \{\leq, \cdot\}, \{\leq, +, \cdot\}\). This gives \(4 \times 5 = 20\) distinct subsets \(A \cup A'\) of \(B\), all complete, and of these 6 are disjoint from \(\{1, \cdot\}\), 4 include \(\cdot\) but not the unit 1, 6 include 1 but not \(\cdot\), and 4 contain \(\{1, \cdot\}\). We also count the complete subsets \(S \subseteq B\) with \(\sim \in S\). The eight subsets of \(\{\sim, \leq, +, \cdot\}\) including negation generate three distinct complete sets namely \(\{\sim\}, \{\sim, \leq\}, \{\sim, \leq, +, \cdot, 0, 1\}\). In the first two cases we may add any of the four complete subsets of \(\{0, 1\}\) and produce four distinct complete subsets, in the last case \(\{0, 1\}\) is already contained, therefore there are \(2 \times 4 + 1 = 9\) complete subsets of \(B\) including negation, of which 4 are disjoint from \(\{1, \cdot\}\), 4 include 1 but not \(\cdot\), and one contains \(\{1, \cdot\}\) (none contain \(\cdot\) but not 1). Altogether, there are 29 complete subsets of \(B\), partitioned into four subcategories as in the claim 1.

**Claim 2:** There are 24 complete subsets of \(C\), of these 12 include \(\upharpoonright\) and 12 exclude it. For this claim note that any complete set containing \(\{\neg, \text{dom}\}\) also includes \(\text{ran}\) and each complete subset containing \(\{\neg, \text{ran}\}\) also includes \(\text{dom}\) but there are no other constraints. There are six complete subsets of \(\{\neg, \text{dom}, \text{ran}\}\) (four without \(\neg\) and two with \(\neg\)) and we may take the union of each of these six with each of the four subsets of \(\{1', \upharpoonright\}\) to get 24 complete subsets altogether, 12 with \(\upharpoonright\) and 12 without, as claimed. However, it is not always true that the union of a complete subset
of \( B \) with a complete subset of \( C \) is complete, because it may violate either of the last two definition rules in figure 1. If a complete subset \( S \) of \( \text{Sig} \) includes 1 then \( \text{dom} \in S \) or \( \text{ran} \in S \) implies \( 1' \in S \). Eight of the 24 complete subsets of \( C \) include \( \text{dom} \) or \( \text{ran} \) but not \( 1' \), so if we take the union of a complete subset of \( B \) including 1 with one of these eight complete subsets of \( C \) then the result will not be complete. This situation splits equally into two halves, with and without composition: of the 12 complete subsets of \( C \) including ;, 4 include \( \text{dom} \) or \( \text{ran} \) but not \( 1' \), and of the 12 complete subsets of \( C \) excluding ;, 4 include \( \text{dom} \) or \( \text{ran} \) but not \( 1' \).

Similarly, if a complete subset \( S \) of \( \text{Sig} \) contains \( \{.,1',\;\;.;\;\}\) then it contains \( \{\text{dom, ran}\} \). There is just one complete subset of \( C \) containing \( \{1',\;\;.;\;\}\) but not containing \( \{\text{dom, ran}\} \), namely \( \{1',\;\;.;\;\}\). If we take the union of a complete subset of \( B \) including \( . \) with the complete subset \( \{1',\;\;.;\;\} \) of \( C \) the result is not complete.

We can now compute the values in figure 2, starting with the first column, i.e. sets not including ;. For the first entry (complete sets disjoint from \( \{1,.;\}\)) we can take the union of any of the 10 complete subsets of \( B \) disjoint from \( \{1,;\} \) with any of the 12 complete subsets of \( C \) without ; to get 120 complete subsets of \( C \). Below that (\( 1 \in S,.;\not\in S \)) we may take the union of any of the 10 complete subsets of \( B \) including 1 but not \( . \) with 8 of the 12 complete subsets of \( C \) without composition (recall that 4 yield incomplete sets when 1 is added) to get 80 complete subsets of \( \text{Sig} \). Next, the number of complete sets containing \( \{1,.;\} \) but not ; is 5 \( \times \) 8, because we may take the union of any of the five complete subsets of \( B \) containing \( \{1,.;\} \) with 8 of the 12 complete subsets of \( C \) to get 40 complete sets. Finally, for complete sets with \( . \) but not 1 and not including ; we may take the union of any of the 4 complete subsets of \( B \) with \( . \) but not 1 with any of the 12 complete subsets of \( C \) to get 4 \( \times \) 12 complete subsets of \( \text{Sig} \). Thus the numbers in the first column of figure 2 are correct.

For the second column (complete sets including ;) the numbers may be calculated similarly as 10 \( \times \) 12, 10 \( \times \) 8, 5 \( \times \) 7, 4 \( \times \) 12, noting

<table>
<thead>
<tr>
<th>( \subseteq ) ( B )</th>
<th>Signatures ( S )</th>
<th>( ; \not\in S )</th>
<th>( ; \in S )</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1 ( \not\in S ), ( . \not\in S )</td>
<td>120</td>
<td>120</td>
<td>240</td>
</tr>
<tr>
<td>10</td>
<td>1 ( \in S ), ( . \not\in S )</td>
<td>80</td>
<td>80</td>
<td>160</td>
</tr>
<tr>
<td>5</td>
<td>1 ( \in S ), ( . \in S )</td>
<td>40</td>
<td>35</td>
<td>75</td>
</tr>
<tr>
<td>4</td>
<td>1 ( \not\in S ), ( . \in S )</td>
<td>48</td>
<td>44</td>
<td>92</td>
</tr>
<tr>
<td>29</td>
<td>Total</td>
<td>288</td>
<td>279</td>
<td>567</td>
</tr>
</tbody>
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in the third case (complete sets containing \{1, \cdot , 1\}) that four of the
twelve complete subsets of \(C\) with \; may not be used because they
include \(\text{dom}\) or \(\text{ran}\) but not \(1’\) and the set \{1', \; \} may not be used
because it does not contain \{\text{dom, ran}\}, hence we may take the union
of any of the 5 complete subsets of \(B\) containing \{1, \cdot \} with 7 of the
complete subsets of \(C\), giving 35 complete subsets of \(\text{Sig}\). The grand
total is 567.

That’s quite a lot, so when proving the finite representation property, or its
failure, we aim to make our proofs cover many of the 567 complete signatures
in one go.

Our main results are summarised by the next theorem.

**THEOREM 4** Let \(S \subseteq \{0,1, -, +, \leq, 1', \; \} \cup \text{dom, ran}\).

1. If composition does not belong to \(S\) and \(A \in R(S)\) is finite then \(A\) has a
representation whose base has at most \(2|A|\) elements.

2. If \(S \supseteq \{\cdot, 1'; \}\) or \(S \supseteq \{\cdot, ;\}\) then \(S\) does not have the finite represen-
tation property.

3. If \(\{\leq, \; , \text{dom, ran}\} \subset S \subseteq \{0,1, \leq, 1', \; ; \} \cup \text{dom, ran}\) and \(A \in R(S)\) then
\(A\) has a representation of size at most \(2^{|A|}\).

We will prove the first part in lemma 5, the second part in lemma 7 and the
third part in lemma 11 below.

**Signatures without composition:** An obvious example of a signature with
the finite representation property is the signature of boolean algebra \{0, 1, \-, +\}.A boolean algebra \(B\) may be represented over the set of ultrafilters \(\text{uf}(B)\) by
mapping \(b \in B\) to \(((\gamma, \gamma) : b \in \gamma \in \text{uf}(B))\), a trivial modification of the Stone
representation. If \(B\) is a finite boolean algebra then each ultrafilter is generated
by an atom of \(B\), hence the size of the base of the representation is the number
of atoms of \(B\), the logarithm (base two) of \(|B|\).

Similar representations can be constructed for weaker signatures. For ex-
ample, consider \(S = \{\leq\}\). An \(S\)-structure \(A\) is representable iff \(\leq\) is a partial
order. A representation \(R\) whose base is the set \(\text{up}(A)\) of all upwardly closed
subsets of \(A\) may be defined by

\[ a^R = \{(\gamma, \gamma) : a \in \gamma \in \text{up}(A)\} \] (2)

Rather than attempting to prove finite axiomatisability for each signature with-
out composition and proving the finite representation property for each one,
we prove the finite representation property in one go, for arbitrary signatures
excluding composition.
LEMMA 5 Let $S$ be a subsignature of $(\mathbb{S})$ with $:\not\in S$. If $A \in \mathcal{R}(S)$ then $A$ has a representation of size at most $2|A|$.

PROOF: Let $S$ be a signature without composition, let $A \in \mathcal{R}(S)$ be finite and let $R$ be a representation of $A$ with base $X$. By considering the complete set generated by $S$, we may assume that $S$ is complete.

For each distinct pair $a, b \in A$ there are $x, y \in X$ (possibly equal) such that either $(x, y) \in a^R \setminus b^R$ or $(x, y) \in b^R \setminus a^R$, we assume the former. Pick a map $h_{a, b} : \{x, y\} \times \{x, y\} \rightarrow \wp(A)$, such that $h_{a, b}(x, y) \subseteq \{c : (x, y) \in c^R\}$ as follows. $h_{a, b}(x, y)$ is a minimal set such that it includes $a$, always, includes 1 if $1 \in S$, it includes $1'$ if $x = y$ and $1' \in S$, it is closed upwards if $\leq \subseteq S$, i.e. $c \in h_{a, b}(x, y) \wedge c \leq d \rightarrow d \in h_{a, b}(x, y)$, it is closed under $\cdot$ if meet belongs to $S$ and it is join prime if $+ \in S$. The last requirement may be satisfied by repeatedly picking an already included element $c + d$ where $(x, y) \in (c + d)^R$ and also including $c$ if $(x, y) \in c^R$ else $(x, y) \in d^R$ and we include $d$, then by Zorn’s lemma there is a prime extension contained in $\{c : (x, y) \in c^R\}$. If $x = y$, $h_{a, b}$ is now completely defined, so suppose $x \neq y$. We let $h_{a, b}(y, x)$ be empty unless $\neg \in S$ in which case $h_{a, b}(y, x) = \{c^- : c \in h_{a, b}(x, y)\}$. $h_{a}(x, x)$ includes $1'$ if $1' \in S$ and it contains $\{\text{dom}(c) : c \in h_{a, b}(x, y)\}$ if $\text{dom} \in S$ and it is empty when $\{1', \text{dom}\} \cap S = \emptyset$, and where $h_{a, b}(x, x)$ includes $1'$ if $1' \in S$ and contains $\{\text{ran}(c) : c \in h_{a, b}(x, y)\}$ if $\text{ran} \in S$ and is empty otherwise. Define an interpretation $H_{a, b}$ of $A$ over the base $X_{a, b} = \{x, y\}$ by

$$e^{H_{a, b}} = \{(u, v) : u, v \in \{x, y\}, c \in h_{a, b}(u, v)\}$$

By construction of $h_{a, b}$. $H_{a, b}$ respects every symbol in $S$ except perhaps equality. But we do know that $(x, y) \in a^{H_{a, b}} \setminus b^{H_{a, b}}$. Next, we rename the elements $x, y$ of $X_{a, b}$ so that $(a, b) \neq (a', b')$ implies $X_{a, b} \cap X_{a', b'} = \emptyset$. Let $X$ be the disjoint union of the $X_{a, b}$. Now define an interpretation $H$ of $A$ over the base $X$ by

$$e^H = \bigcup_{a \neq b \in A} e^{H_{a, b}}.$$  

This now respects the equality predicate $(c \neq d \Rightarrow e^H \neq d^H)$ as well as all the other elements of $S$, hence it is a representation of $A$ over a base of size $2|A|$. □

Relational Signatures: Let $S = \{1', ;\}$. An $S$-structure $A$ is $S$-representable iff $(A, 1', ;)$ is a monoid. A Cayley representation $R$ of such an algebra may be obtained by letting the base be $A$ itself and letting

$$a^R = \{(b, b; a) : b \in A\}$$

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for each $a \in \mathcal{A}$, hence if $\mathcal{A}$ is representable then it has a representation of size $|\mathcal{A}|$, so the finite representation property holds for this signature. The same representation works if the identity is dropped from the signature. Signatures without composition have the finite representation property by lemma 5. Hence:

**LEMMA 6** If $S \subseteq \{1, \cdot, \rrigg\}$ then $S$ has the finite representation property.

If we have a signature including other symbols, e.g. $\leq, \dom, \ran$ then there are difficulties in making the Cayley representation work. One problem is that $a;\dom(b)$ is not in general equal to $a$, but in any representation $\dom(b)$ should be represented as a subidentity relation. We will prove the finite representation property for a large signature including $\leq, \rrigg, \dom, \ran$ in lemma 11.

Signatures containing $\{\cdot, \cdot\}$

**LEMMA 7** Suppose $\{\cdot, 1, \cdot\} \subseteq S$ or $\{\cdot, \cdot, \cdot\} \subseteq S$. There is a finite, representable $S$-structure possessing only representations over infinite bases.

**PROOF:**

Consider the finite relation algebra $P$ whose atoms are $\{1, <, >\}$ (so $P$ has eight elements), $1'$ is the identity, the converse of $<$ is $>$, and the following compositions hold: $<;=\llleq, <;: = 1'+ < +>$ (this suffices to determine all the operators, using boolean algebra axioms, additivity the identity and involution laws). A representation $Q$ may be defined with base $\mathcal{Q}$ by letting $(1')^Q$ be the identity relation, $<Q = \{(p,q) : p < q\}$, $>Q = \{(p,q) : p > q\}$. Hence the reduct of $P$ to any subsignature is necessarily representable.

Now let $S \supseteq \{\cdot, 1, \cdot\}$ or $S \supseteq \{\cdot, \cdot, \cdot\}$ and consider the reduct $P|_S$. Let $R$ be any $S$-representation of $P|_S$ over the base $X$. Since $\cdot$ is included in $S$ we know that $\leq$ is definable and 0 is below $<$, hence $0^R \subseteq <^R$ (note that $0^R$ need not be the empty set, as 0 may not belong to $S$). Since $<$ is not equal to 0, there must be $x, y \in X$ with $(x, y) \in <^R \setminus 0^R$. Whenever $(u, v) \in <^R \setminus 0^R$ we must have $u \neq v$ for the following reason. If $1' \in S$ then $(u, u) \in (1')^R$ so $u = v$ implies $(u, v) \in <^R \cap (1')^R = (\llleq \cdot 1')^R = 0^R$. Alternatively, if $\rrigg \in S$ then $u = v$ implies $(u, v) \in (\cdot >)^R = 0^R$. Also, since $<;=\llleq$, whenever $(u, v) \in <^R \setminus 0^R$ there is $w \in X$ such that $(u, w) \in <^R$, $(w, v) \in <^R$. We cannot have $(u, w) \in 0^R$ (else $(u, v) \in (0; <) = 0^R$ nor can we have $(w, v) \in R$, hence $(u, w), (w, v) \in <^R \setminus 0^R$. Hence $a^R \setminus 0^R$ is irreflexive and dense, though it is not necessarily transitive. Still, we can prove inductively that for any $n \in \mathbb{N}$ there are $x_0, x_1, \ldots x_{n-1} \in X$ such that for $i < j < n$ we have $(x_i, x_j) \in <^R \setminus 0^R$, hence $|\{x_0, \ldots, x_{n-1}\}| = n$. It follows that $|X| \geq n$ (all $n \in \mathbb{N}$), so $X$ is infinite. \( \Box \)

These signatures therefore fail to have the finite representation property. An example is the full relation algebra signature ($\text{Sig}$). We do not know if $\{\cdot, \cdot\}$ or
\{\neg,.;\} has the finite representation property, but clearly all other subsignatures of \{.,1';.;\} and of \{.,\neg,.;\} have the finite representation property, by lemmas 5 and 6. See the remark at the end for an even smaller signature which fails to have the finite representation property.

The remaining cases are signatures including composition and excluding the boolean meet operator. In general, we do not know which of these signatures has the finite representation property. The main new result of the current paper is the identification of a large signature \(S\) such that \(S\) has the finite representation property.

Signatures containing \(\{\leq,\neg,.;\dom,\ran\}\). We prove the finite representation property for the signature \(S = \{0,1,\leq,\neg,.;\dom,\ran\}\) (the whole signature \(\text{Sig}\) with \(\{-,+,;\}\) subtracted), then we extend this result to some subsignatures. We define a finite set of axioms, sound and complete for \(R(S)\) and we prove the finite representation property for \(S\). Our completeness proof has elements similar to the representation of ordered sets (2) in that the base of the representation consists of upward closed sets (subject to further constraints see (11)) but it has much in common with the Cayley representation because the interpretation of an element is based on composing by that element (see (13)).

Let \(\text{Ax}\) denote the following formulas.

**partial order** \(\leq\) is reflexive, transitive and antisymmetric, the bounds are 0, 1,

**monotonicity and normalcy** the operators \(\neg,.;\) \(\dom,\ran\) are monotonic, e.g.

\[
a \leq b \rightarrow a; c \leq b; c \text{ etc. and normal, } 0^- = 0; a = a; 0 = \dom(0) = \ran(0) = 0.
\]

**involuted monoid** \(\Bullet\) is associative, 1' is left and right identity for \(.;\) \(\neg\) is an involution: \((a^-)^- = a\), \((a;b)^- = b^-\cdot a^-\),

**domain/range axioms**

\[
\begin{align*}
\dom(a) = (\dom(a))^- & \leq 1' \quad (3) \\
\dom(a) & \leq a; a^- \quad (4) \\
\dom(a^-) & = \ran(a) \quad (5) \\
\dom(\dom(a)) & = \dom(a) \quad (6) \\
\dom(a); a & = a \quad (7) \\
\dom(a;b) & = \dom(a;\dom(b)) \quad (8) \\
\dom(\dom(a);\dom(b)) & = \dom(a);\dom(b) = \dom(b);\dom(a) \quad (9)
\end{align*}
\]

A model of these axioms is called a **domain algebra**.

A consequence of axioms (6) and (7) is

\[
\dom(a);\dom(a) = \dom(a) \quad (10)
\]
Each of the axioms (3)–(9) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions and we denote the dual axiom by appending a ‘D’, thus for example (8D) is \( \text{ran}(b; a) = \text{ran}(\text{ran}(b); a) \). The dual axioms can be obtained from the axioms above, using the involution axioms and (5).

**Lemma 8** If \( \mathcal{A} \in \mathbf{R}(S) \) then \( \mathcal{A} \models \text{Ax}. \)

We will prove the converse to this in lemma 11, below.

**Definition 9** Let \( \mathcal{A} \) be a domain algebra.

1. Write \( D(\mathcal{A}) \) for the set of domain elements of \( \mathcal{A} \) — those elements \( d \in \mathcal{A} \) such that \( \text{dom}(d) = d \). Observe that \( (D(\mathcal{A})); ) \) forms a lower semilattice ordered by \( \leq \).

2. For \( a \in \mathcal{A} \) let \( a^\dagger = \{b \in \mathcal{A} : a \leq b\} \) and more generally, for \( x \subseteq \mathcal{A} \) let \( x^\dagger = \{b \in \mathcal{A} : \exists a \in x, a \leq b\} \).

3. We extend the operators so as to apply to sets, if \( x, y \subseteq \mathcal{A} \), \( a \in \mathcal{A} \) then
   \[
   \begin{align*}
   x^- &= \{a^- : a \in x\}^\dagger \\
   x; a &= \{b; a : b \in x\}^\dagger, \quad a; x = \{a; b : b \in x\}^\dagger, \quad x; y = \{a; b : a \in x, b \in y\}^\dagger \\
   \text{dom}(x) &= \{\text{dom}(b) : b \in x\}^\dagger \\
   \text{ran}(x) &= \{\text{ran}(b) : b \in x\}^\dagger
   \end{align*}
   
   Note that these sets are all ‘closed upwards’, this is to reduce notational clutter later on.

4. A non-empty subset \( x \) of \( \mathcal{A} \) is closed if
   \[ \text{dom}(x); x; \text{ran}(x) = x \] (11)

Observe, from definition 9(3) and the transitivity of \( \leq \), that \( (\text{dom}(x); x; \text{ran}(x))^\dagger = \text{dom}(x); x; \text{ran}(x) \), for any set \( x \subseteq \mathcal{A} \), as we noted, so every closed set is upwards closed. Also, the inclusion \( x \subseteq \text{dom}(x); x; \text{ran}(x) \) holds (by (3) and monotonicity) so to prove that \( x \) is closed it suffices to show that
   \[ \text{dom}(x); x \subseteq x \text{ and } x; \text{ran}(x) \subseteq x \] (12)

**Lemma 10**

1. For any \( a \in \mathcal{A} \), \( a^\dagger \) is closed.

2. If \( x \) is closed, \( a \in \mathcal{A} \) and \( \text{dom}(a) \in \text{ran}(x) \) then \( x; a \) is closed and \( \text{dom}(x; a) = \text{dom}(x) \).

3. For any index set \( I \), if \( x_i \) is closed for \( i \in I \) and \( \text{dom}(x_i) = \text{dom}(x_j) = \text{ran}(x_i) = \text{ran}(x_j) \), for all \( i, j \in I \), then \( \bigcup_{i \in I} x_i \) is closed. In particular \( (|I| = 2) \) if \( x, y \) are closed and \( \text{dom}(x) = \text{dom}(y) \), \( \text{ran}(x) = \text{ran}(y) \) then \( x \cup y \) is closed.
PROOF:

1. By monotonicity and (7).

2. To prove that \( x; a \) is closed we must check (12), that \( \text{dom}(x; a); (x; a) \subseteq x; a \) and \( (x; a); \text{ran}(x; a) \subseteq x; a \). These two cases are not symmetrical. For the first, \( \text{dom}(x; a) = \text{dom}(x; \text{dom}(a)) = \text{dom}(x; \text{ran}(x); \text{dom}(a)) = \text{dom}(x) \) by (8), the closure of \( x \) and since \( \text{dom}(a) \in \text{ran}(x) \) so \( \text{dom}(x;a); x; a = \text{dom}(x); x; a = x; a \). On the other side, typical elements of \( x; a \) and \( \text{ran}(x; a) \) are above elements \( b; a \) and \( \text{ran}(b'; a) \), respectively, for some \( b, b' \in x \). Then \( (b; a); \text{ran}(b'; a) \geq (b; \text{ran}(b'); a); \text{ran}(b; \text{ran}(b'); a) = b; \text{ran}(b'); a \in x; a \), using (8D) and (7D), thus \( x; a; \text{ran}(x; a) \subseteq x; a \), as required.

3. Immediate from the definition of closed.

Let \( X \) be the set of all closed subsets of \( A \). Since \( X \subseteq \wp(A) \) we have \( |X| \leq 2^{2^{|A|}} \). Define an \( A \)-structure \( F \) with base \( X \) by

\[
(x, y) \in a^F \iff x; a \subseteq y \text{ and } y; a^\ominus \subseteq x. \tag{13}
\]

**LEMMA 11** \( F \) is a representation of \( A \).

PROOF:

Let \( a \not\subseteq b \). By lemma 10(1), \( (\text{dom}(a))^\uparrow, a^1 \) are closed. By monotonicity, (7) and (4), \( (\text{dom}(a))^\uparrow; a \subseteq a^\uparrow \text{ and } a^\downarrow \subseteq (\text{dom}(a))^\uparrow \), so \( ((\text{dom}(a))^\uparrow; a^1) \in a^F \). Also, we cannot have \( \text{dom}(a); b \geq a \), by transitivity, monotonicity and (3), since \( a \not\subseteq b \), so \( ((\text{dom}(a))^\uparrow; a^1) \not\in b^F \), hence \( F \) is faithful.

\( 0^F = \emptyset \), by normalcy and the partial order axioms. \( \leq \) is correctly represented by the partial order axioms and monotonicity. \( \ominus \) is correctly represented, by the involution axioms. We check composition. If \( (x, y) \in a^F \) and \( (y, z) \in b^F \) then \( x; a \subseteq y, y; a^\ominus \subseteq x, y; b \subseteq z \) and \( z; b^\ominus \subseteq y \). Then \( x; (a; b) \subseteq z \) and \( z; (a; b)^\ominus = z; b^\ominus; a^\ominus \subseteq x \) by associativity and the involution axioms, so \( (x, z) \in (a; b)^F \).

Conversely, if \( (x, z) \in (a; b)^F \) then

\[
x; a; b \subseteq z, z; b^\ominus; a^\ominus \subseteq x \tag{14}
\]

Since \( x; a; b \subseteq z \), by (8), \( \text{dom}(x) \subseteq \text{dom}(x; a; b) \subseteq \text{dom}(z) \) and from the other condition \( \text{dom}(z) \subseteq \text{dom}(x) \), so

\[
\text{dom}(x) = \text{dom}(z) \tag{15}
\]

**Claim 1:** The sets

\[
\alpha = x; a; \text{ran}(z; b^\ominus) \text{ and } \beta = z; b^\ominus; \text{ran}(x; a)
\]

11
are closed. To prove claim 1,

\[
\text{dom}(\alpha); \alpha = \text{dom}(x; a; \text{ran}(z; b^-)); \alpha
\]

\[
= \text{dom}(x; a; \text{dom}(b; z^-)); \alpha \quad \text{by (5), Inv.}
\]

\[
= \text{dom}(x; \text{dom}(a; b; z^-)); \alpha \quad \text{by (8)}
\]

\[
= \text{dom}(x; \text{ran}(z; b^-; a^-)); \alpha \quad \text{by (5), Inv.}
\]

\[
\subseteq \text{dom}(x; \text{ran}(x)); \alpha \quad \text{by (14)}
\]

\[
= \text{dom}(x); x; a; \text{ran}(z; b^-) \quad \text{def. of } \alpha
\]

\[
= x; a; \text{ran}(z; b^-) = \alpha
\]

as \( x \) is closed

and

\[
\alpha; \text{ran}(\alpha) = \alpha; \text{ran}(x; a; \text{ran}(z; b^-))
\]

\[
= (x; a; \text{ran}(z; b^-); \text{ran}(x; a); \text{ran}(z; b^-)) \quad \text{by (8D), (9D)}
\]

\[
= x; a; \text{ran}(z; b^-) \quad \text{by (10D), (7D), (9D)}
\]

By (12), this proves claim 1 for \( \alpha \), and \( \beta \) is similar. Observe that
dom(\alpha) = \text{dom}(\beta) = \text{dom}(x) and ran(\alpha) = \text{ran}(\beta) = \text{ran}(x; a; \text{ran}(z; b^-)).

By lemma 10(3), \( y = \alpha \cup \beta \) is closed.

Claim 2: \((x, y) \in a^F\). To prove claim 2 we must show that

\( x; a \subseteq y \) and \( y; a^- \subseteq x \). Since \( x; a \subseteq \alpha \), the first inclusion holds. For the other inclusion, note first that

\[
\text{ran}(x) \supseteq \text{ran}(z; b^-; a^-) = \text{dom}(a; \text{ran}(z; b^-))
\]

by (14), (5), (8) and the involution axioms, and therefore

\[
x; \text{dom}(a; \text{ran}(z; b^-)) = x \quad (16)
\]

Also,

\[
z; b^-; \text{ran}(x; a); a^- = z; b^-; \text{ran}(\text{ran}(x); a); a^-
\]

\[
= z; b^-; \text{dom}(a^-; \text{ran}(x)); a^-
\]

\[
\subseteq z; b^-; \text{dom}(a^-; \text{ran}(x)); (a^-; \text{ran}(x))
\]

\[
= z; b^-; a^-; \text{ran}(x) \subseteq x; \text{ran}(x)
\]

\[
= x
\]

Now we prove that \( y; a^- \subseteq x \).

\[
y; a^-
\]

\[
= (x; a; \text{ran}(z; b^-) \cup z; b^-; \text{ran}(x; a)); a^-
\]

\[
\subseteq x; (a; \text{ran}(z; b^-)); (a; \text{ran}(z; b^-)) \cup x \quad \text{by Inv., Mon., (10), (17)}
\]

\[
\subseteq x; \text{dom}(a; \text{ran}(z; b^-)) \cup x \quad \text{by (4)}
\]

\[
= x
\]

hence \((x, y) \in a^F\) as claimed, and similarly \((y, z) \in b^F\). So \( ; \) is properly represented.
THEOREM 12 Let \( \{\leq, \sim, \text{dom}, \text{ran}\} \subseteq S \subseteq \{0, 1, \leq, 1', \sim; , \text{dom}, \text{ran}\} \). Let \( \mathcal{A} \) be an \( S \)-structure. Then \( \mathcal{A} \) satisfies these axioms in \( \text{Ax} \) involving only symbols from \( S \) if and only if \( \mathcal{A} \) has a representation on a base of size at most \( 2^{|\mathcal{A}|} \).

PROOF:

Without loss, we assume that \( S \) is complete. Soundness is clear (every representable \( S \)-structure satisfies the axioms), we have to prove completeness. The case \( S = \{0, 1, \leq, 1', \sim; , \text{dom}, \text{ran}\} \) is lemma 11. The proof is not affected if 0, 1 or 1’ are dropped from the signature. □

COROLLARY 13 If \( \{\sim; , \text{dom}, \text{ran}\} \subseteq S \subseteq \{0, 1, \leq, 1', \sim; , \text{dom}, \text{ran}\} \) then if \( \mathcal{A} \in R(S) \) then \( \mathcal{A} \) has a representation on a base of size at most \( 2^{|\mathcal{A}|} \).

PROOF:

We have already established this result for signatures including \( \leq \). Let \( S \) be a signature satisfying the condition in the corollary but with \( \leq \notin S \) and let \( \mathcal{A} \in R(S) \). Let \( R \) be a representation of \( \mathcal{A} \). The set \( \{a^R : a \in \mathcal{A}\} \) forms an \( (S \cup \{\leq\}) \) structure (with \( \leq \) interpreted as set inclusion) hence we may add \( \leq \) to the signature and define a representable expansion \( \mathcal{A}^+ \) of \( \mathcal{A} \), with the same universe but with \( \leq \) added to the signature. By the previous theorem \( \mathcal{A}^+ \) has a representation of size at most \( 2^{|\mathcal{A}^+|} = 2^{|\mathcal{A}|} \), and the reduct to \( S \) is an \( S \)-representation of \( \mathcal{A} \). □

Theorem 12 contrasts strikingly with the following theorem, where the signatures do not include \( \leq \) or \( \sim \). We are all familiar with signatures \( S \) where \( R(S) \) is finitely axiomatisable but where the representation class of an expanded signature is not finitely axiomatisable, e.g. boolean algebra and relation algebra, but here it is the other way round: any of the signatures below (except those involving \( \text{anti} \)) may be expanded to include \( \leq \) and \( \sim \) and becomes finitely axiomatisable.

THEOREM 14 ([HM10]) If \( \{\text{dom}, \} \subseteq S \subseteq \{0, 1', ; , \text{dom}, \text{ran}, \text{anti}\} \) (where \( \text{anti}(x) = 1' \cdot (\neg \text{dom}(x)) \)) then \( R(S) \) is not finitely axiomatisable.
**THEOREM 15** Of the 567 complete subsignatures of \((\text{Sig})\), 302 definitely have the finite representation property, 43 definitely do not have the finite representation property. It is not currently known which of the remaining 224 complete subsignatures have the finite representation property.

**PROOF:**

The 288 complete signatures without composition all have the finite representation property by lemma 5. Also, signatures \{;\}, \{1';;\} have the finite representation property, by lemma 6. Further, each of the twelve complete signature \(S\) with \{\sim ; ; \text{dom ran}\} \subseteq S \subseteq \{0,1,\leq,1',\sim ; ; \text{dom ran}\} has the finite representation property by corollary 13 (there are three subsets \(S\) of \(\{1,1'\}\) satisfying \(1 \in S \rightarrow 1' \in S\) and we may take the union of each of these three with \{\sim ; ; \text{dom ran}\} and any of the four subsets of \(\{0,\leq\}\) to obtain the twelve required complete subsets). Hence there are \(288+2+12 = 302\) complete subsignatures which definitely have the finite representation property.

From figure 2 there are \(44 + 35 = 79\) complete subsignatures containing \{\cdot ; ;\}. Of these, 36 are disjoint from \(\{1',\sim\}\) (the complete sets disjoint from \(\{1',\sim\}\) containing \{\leq ; ;\} and including negation are of the form \(\{0,1,\sim,\cdot,+,\leq ; ;\}\cup S\) for \(S \subseteq \{\text{dom ran}\}\) and those without negation are of the form \(\{\leq ; ;\}\cup S\) for \(S \subseteq \{0,1,+,\text{dom ran}\}\) so there are \(2^2 + 2^5 = 36\) of them) hence none of the 43 complete subsignatures containing \{\cdot ; ;\} and not disjoint from \(\{1',\sim\}\) has the finite representation property, by lemma 7.

For any signature \(S\) where \(R(S)\) is recursively enumerable (all the signatures mentioned have recursively enumerable representation classes) if \(S\) has the finite representation property then the following decision problems are decidable.

1. Is \(A \in R(S)\)?
2. Does \(A\) have an \(S\)-representation over a finite base?

If \(S\) has the finite representation property but also there is a recursive bound to the size of a representation of a finite representable \(S\)-structure then the requirement that \(R(S)\) is recursively enumerable may be dropped. Further, if we know that a representable, finite \(S\)-structure \(A\) has a representation of a given polynomial size then both problems belong to \(\text{NP}\), one non-deterministically guesses a representation of the given size and then tests whether it is a representation in polynomial time. Similarly, if the size of the representation is bounded by an exponential function then both decision problems belong to \(\text{NEXPTIME}\).

This covers all the signatures above where the finite representation property is known to hold. In many cases the complexity is lower. For example, in those cases where we have found sound and complete axioms for \(R(S)\), since the axioms involve at most three variables, these decision problems may be solved in cubic time.
PROBLEM 16 Determine which of the 224 remaining complete subsignatures have the finite representation property. The unknown cases may be divided into the following disjoint subcases.

- $S \supseteq \{\cdot,\} \text{ with } \{1',\} \cap S = \emptyset$ (there are 36 of these),
- $: \in S, \not\in \emptyset, S \supseteq \{\cdot,\} \text{ dom ran}$ (there are 42 of these)
- $: \in S, \not\in \emptyset$ and either $+ \in S$ or $- \in S$ (there are 120 of these as there are 200 complete signatures with ; but not $\cdot$, by figure 2, but 80 of them are disjoint from $\{\cdot,+,\}$),
- $\{\cdot,\} \subseteq S, \{\cdot,+,\} \cap S = \emptyset$ and $\{\cdot,\} \text{ dom ran} \not\subseteq S$ (there are 26).

Marcel Jackson makes the following interesting observation. Let fix be the unary operator defined by $\text{fix}(a) = 1' \cdot a$, observe that this term definition involves $\cdot$ and is fairly similar to the term $1' \cdot a$ defining $\text{dom}$, yet it has a rather different effect on finite representability. Let $S$ be a signatures with $\{\cdot,\} \subseteq S$. Then $S$ fails to have the finite representation property. To see why, let $\mathcal{A}$ be the $\{\cdot,\}$-structure with two elements $a, 0$ and $\text{fix}(a) = \text{fix}(0) = 0, a=a$ and $0=0; a=a; 0=0$. A representation $Q$ of $\mathcal{A}$ may be defined over the rational numbers by letting $a^Q = \{(p,q) : p < q\}$ and $0^Q = \emptyset$.

Using this representation $Q$ we may expand the signature $\{\cdot,\}$ to any $S$ with $\{\cdot,\} \subseteq S$ and expand $\mathcal{A}$ to the representable $S$-structure $\mathcal{A}^+$. If $R$ is an arbitrary representation of $\mathcal{A}^+$ then it is impossible for $(x,y) \in R \setminus a^R$ as $0^R$ is a subidentity relation ($\text{fix}(0) = 0$) and $0 = 0; a$ hence there must be points $x, y$ with $(x,y) \in a^R \setminus 0^R$. We cannot have $x = y$ else $(x,y) \in \text{fix}(a)^R = 0^R$. Also if $(x,y) \in a^R \setminus 0^R$ there is $z$ with $(x,z), (z,y) \in a^R \setminus 0^R$ (use $a = a; a$ and $0 = a; 0 = a; a$). As in the proof of lemma 7, for any $n \in \mathbb{N}$ we may construct a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $i < j \leq n \rightarrow (x_i, x_j) \in a^R \setminus 0^R$, thus the points in the sequence are distinct. Hence there can be no finite bound to the number of distinct points in the base, so the base must be infinite.

Thus, signatures including $\text{fix}$ are easily categorised: those including $\cdot$ do not have the finite representation property, while those without $\cdot$ do have the finite representation property. Note that although $\text{fix}$ may be defined by $\{\cdot,1'\}$ or $\{\cdot,\text{dom}\}$ among other sets, the only symbol of $\text{Sig}$ that may be defined using $\text{fix}$ is $1' = \text{fix}(1)$. It can be shown from this that there are 258 complete subsets of $\text{Sig} \cup \{\text{fix}\}$ including $\text{fix}$ but not $\cdot$ (all with finite representation property) and there are 249 complete subsets of $\text{Sig} \cup \{\text{fix}\}$ containing $\{\cdot,\}$ (and none has the finite representation property).

References


