Every logic between $\mathbf{K}^3$ and $\mathbf{S5}^3$ is undecidable and non-finitely axiomatizable

R. Hirsch, I. Hodkinson and A. Kurucz *
Draft

Abstract

We prove, using the result of [5], that every 3-modal logic between $\mathbf{K}^3$ and $\mathbf{S5}^3$ is undecidable. We also show that each of these logics is non-finitely axiomatizable, lacks the product finite model property, and there is no algorithm deciding whether a finite frame validates the logic.

1 Introduction and results

Let L be any 3-modal logic with $\mathbf{K}^3 \subseteq L \subseteq \mathbf{S5}^3$. Then the following hold.

Theorem 1. *L is undecidable.*

Theorem 2. *It is undecidable whether a finite frame validates L.*

Theorem 3. *L is not finitely axiomatizable.*

Theorem 4. *L does not have the product finite model property in the following sense: there is some formula which does not belong to L but cannot be refuted in any finite product frame.*

Theorem 5. *Theorems 1–4 also hold for any n-modal logic between $\mathbf{K}^n$ and $\mathbf{S5}^n$, if $n \geq 3$.*

Theorems 1 and 3 answer questions 22 and 24 of Gabbay–Shehtman [2] (cf. also Q16.163 of Gabbay [1]): $\mathbf{K}^3$ is undecidable, and all the logics of the form $L \times \mathbf{S5}^2$ are undecidable and non-finitely axiomatizable, if $\mathbf{K} \subseteq L \subseteq \mathbf{S5}$. $\mathbf{K}^3$ is a natural example of an undecidable, recursively enumerable, Kripke complete logic which has the finite model property (see [2]). Note that undecidability of $\mathbf{S5}^3$ follows from Maddux [8]. Non-finite axiomatizability of the two logics at the end of the above interval was also known: see Johnson [6] for $\mathbf{S5}^3$ and [7] for $\mathbf{K}^3$.

In the proofs we will use the following result of Hirsch–Hodkinson [5]:

$$\textit{It is undecidable whether a finite relation algebra is representable.} \tag{1}$$

For any finite simple relation algebra $\mathfrak{A}$, we define (in a recursive way) a finite 3-modal frame $\mathcal{F}_{\mathfrak{A}}$ and a 3-modal formula $\varphi_{\mathfrak{A}}$, and prove the following lemmas.

*Research supported by UK EPSRC grants GR/L85961 and GR/L85978, and by Hungarian National Foundation for Scientific Research grant T30314.

Lemma 6. *$\mathfrak{A}$ is representable iff $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of some universal product* $\mathbf{S5}^3$*-frame*[1]*. Further, $\mathfrak{A}$ is representable with a finite base iff $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of some finite universal product* $\mathbf{S5}^3$*-frame.*

Lemma 7. *Let L be any 3-modal logic with $\mathbf{K}^3 \subseteq L \subseteq \mathbf{S5}^3$. Then the following are equivalent:*

(i) The frame $\mathcal{F}_{\mathfrak{A}}$ validates L.

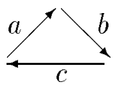
(ii) The formula $\neg\varphi_{\mathfrak{A}}$ does not belong to L.

(iii) $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of some universal product $\mathbf{S5}^3$*-frame.*

Now Theorems 1 and 2 follow straightforwardly from (1) and Lemmas 6, 7. Theorem 3 follows from Theorem 2, since if L were finitely axiomatizable then there would be a recursive test for finite frames. We return to the proof of Theorem 4 after proving Lemmas 6 and 7.

2 Definitions and proofs

Fix some finite simple relation algebra $\mathfrak{A} = (A, +, \cdot, -, 1, 0, ;, \breve{}, 1')$. Call a triplet (a, b, c) of atoms of $\mathfrak{A}$ *consistent* if $\breve{c} \le a\,;b$ holds.



Definition of the frame $\mathcal{F}_{\mathfrak{A}}$: Introduce a possible world $t = abc$ for each consistent triplet (a, b, c) of atoms of $\mathfrak{A}$. We write t_0 for a, t_1 for b, t_2 for c. Write $\mathcal{T}$ for the set of all such worlds. For $t, t' \in \mathcal{T}$ and $i < 3$ define $tR_i t'$ iff $t_i = t'_i$, and let $\mathcal{F}_{\mathfrak{A}} = (\mathcal{T}, R_0, R_1, R_2)$. Since $1\,;a\,;1 = 1$ holds for each atom a of $\mathfrak{A}$, it is not hard to see the following property of $\mathcal{F}_{\mathfrak{A}}$:

$$(\forall t, t' \in \mathcal{T})(\exists r, s \in \mathcal{T})\; tR_0 r,\; rR_1 s \text{ and } sR_2 t'. \tag{2}$$

Proof of Lemma 6. This follows from a chain of known results in algebraic logic and duality between Kripke frames and Boolean algebras with operators. Here we only list these results, but in Appendix A below we give the proofs in a modal logic setting.

The frame $\mathcal{F}_{\mathfrak{A}}$ is in fact the atom-structure of the diagonal-free reduct $\mathsf{Df}_3\mathfrak{A}$ of the (finite, simple) 3-dimensional cylindric algebra $\mathsf{Ca}_3\mathfrak{A}$ constructed from the relation algebra $\mathfrak{A}$. Such a construction was first considered in Monk [9], who also proved that

> $\mathfrak{A}$ is representable (as a relation algebra) iff $\mathsf{Ca}_3\mathfrak{A}$ is representable (as a cylindric algebra). Further, $\mathfrak{A}$ is representable with a finite base iff $\mathsf{Ca}_3\mathfrak{A}$ is representable with a finite base.

Since $\mathsf{Df}_3\mathfrak{A}$ is a reduct of a cylindric algebra and generated by binary elements, the following statement holds (see Johnson [6], Halmos [3], cf. also [4, Thm.5.1.51]):

> $\mathsf{Ca}_3\mathfrak{A}$ is representable (as a cylindric algebra) iff $\mathsf{Df}_3\mathfrak{A}$ is representable (as a diagonal-free cylindric algebra). Further, $\mathsf{Ca}_3\mathfrak{A}$ is representable with a finite base iff $\mathsf{Df}_3\mathfrak{A}$ is representable with a finite base.

[1] A universal product $\mathbf{S5}^3$-frame is a product of three Kripke frames $(U, U\times U)$, $(V, V\times V)$ and $(W, W\times W)$, for some sets U, V, W. We write (U, V, W) for such a frame, and call it the *universal product frame on* $U\times V\times W$.

Finally, since $\mathsf{Df}_3\mathfrak{A}$ is finite and simple, from basic duality theory we have:

> $\mathsf{Df}_3\mathfrak{A}$ is representable with base $U \times V \times W$ (i.e., embeddable into the diagonal-free cylindric set algebra $\mathfrak{B}$ of all subsets of $U \times V \times W$) iff its atom-structure $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of the universal product $\mathbf{S5}^3$-frame on $U \times V \times W$.

Now the lemma clearly follows. □

Definition of the formula $\varphi_{\mathfrak{A}}$: For each world $t \in \mathcal{T}$, that is, for each consistent triplet of atoms of $\mathfrak{A}$, introduce a propositional variable t. Define $\varphi_{\mathfrak{A}}$ as the Yankov-Fine frame description formula of $\mathcal{F}_{\mathfrak{A}}$:

$$\Box^+ \bigvee_{t \in \mathcal{T}} (t \wedge \neg \bigvee_{t' \neq t} t') \ \wedge \tag{3}$$

$$\wedge \ \Box^+ \bigwedge_{\substack{i<3, t, t' \in \mathcal{T} \\ tR_i t'}} t \to \Diamond_i t' \ \wedge \tag{4}$$

$$\wedge \ \Box^+ \bigwedge_{\substack{i<3, t, t' \in \mathcal{T} \\ \neg(tR_i t')}} t \to \neg\Diamond_i t'. \tag{5}$$

Here, $\Box_i^+\psi$ abbreviates $\psi \wedge \Box_i\psi$, and $\Box^+\psi$ abbreviates $\Box_0^+\Box_1^+\Box_2^+\psi$.

Proof of Lemma 7.
(i) implies (ii): This is obvious, since $\mathcal{F}_{\mathfrak{A}} \not\models \neg\varphi_{\mathfrak{A}}$.

(iii) implies (i): This is also obvious, by $L \subseteq \mathbf{S5}^3$.

(ii) implies (iii): If $\neg\varphi_{\mathfrak{A}}$ does not belong to L, then by $\mathbf{K}^3 \subseteq L$, it does not belong to $\mathbf{K}^3$. Thus $\varphi_{\mathfrak{A}}$ is satisfiable in a product $\mathbf{K}^3$ model. Let $\mathfrak{M}$ be a model on the product of three $\mathbf{K}$-frames (U, R_U), (V, R_V) and (W, R_W), and let $M = U \times V \times W$. We write uvw rather than (u, v, w) for worlds of $\mathfrak{M}$. Let $u_0 \in U$, $v_0 \in V$, $w_0 \in W$ be such that $\mathfrak{M}, u_0v_0w_0 \models \varphi_{\mathfrak{A}}$.

Now let

$$\begin{aligned} U_0 &= \{u \in U \ : \ u = u_0 \text{ or } u_0 R_U u\} \\ V_0 &= \{v \in V \ : \ v = v_0 \text{ or } v_0 R_V v\} \\ W_0 &= \{w \in W \ : \ w = w_0 \text{ or } w_0 R_W w\}. \end{aligned}$$

Define a function $h : U_0 \times V_0 \times W_0 \to \mathcal{T}$ as follows:

$$h(uvw) = t \quad \text{iff} \quad \mathfrak{M}, uvw \models t.$$

We claim that h is well-defined, and a p-morphism from the universal frame (U_0, V_0, W_0) onto $\mathcal{F}_{\mathfrak{A}}$. We will use that $\mathfrak{M}, u_0v_0w_0 \models \varphi_{\mathfrak{A}}$ all the time. h is well-defined by (3), and onto by (3), (4) and (2).

Next we show that if $p_0p_1p_2$, $q_0q_1q_2 \in U_0 \times V_0 \times W_0$, $i < 3$, $p_j = q_j$ for $j \neq i$, $j < 3$, $\mathfrak{M}, p_0p_1p_2 \models t$ and $\mathfrak{M}, q_0q_1q_2 \models t'$, then $t_i = t'_i$ (or equivalently, tR_it'). We may assume without loss of generality that $i = 0$. If p_0 and q_0 are R_U-connected, say $p_0R_Uq_0$, then $\mathfrak{M}, p_0p_1p_2 \models \Diamond_0 t'$ which, by (5), implies that tR_0t'. If p_0 and q_0 are not R_U-connected then either $p_0 = q_0$ (when $t_0 = t'_0$ follows by (3)) or $u_0 R_U p_0$ and $u_0 R_U q_0$. Then, repeating the above argument for $u_0 p_1 p_2$, $p_0 p_1 p_2$ and for $u_0 p_1 p_2$, $q_0 q_1 q_2$, we get $t_0 = t'_0$, by transitivity of equality.

Finally, we show that if $p_0 p_1 p_2 \in U_0 \times V_0 \times W_0$, $\mathfrak{M}, p_0 p_1 p_2 \models t$ and $t R_0 t'$ then there is some $u \in U_0$ such that $\mathfrak{M}, u p_1 p_2 \models t'$. Similar statements hold for 1 and V_0, and 2 and W_0, respectively. Indeed, by (3), there is a unique $t'' \in \mathcal{T}$ with $\mathfrak{M}, u_0 p_1 p_2 \models t''$. By the previous paragraph, $t''_0 = t_0$ holds implying $t'' R_0 t'$, by $t R_0 t'$ and definition of R_0. Then, by (4), there is some $u \in U_0$ with $u_0 R_U u$ and $\mathfrak{M}, u p_1 p_2 \models t'$. Thus we proved that $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of a universal product $\mathbf{S5}^3$-frame, which completes the proof of Lemma 7. $\square$

Proof of Theorem 4. Take some finite representable relation algebra $\mathfrak{A}$ which is representable only with an infinite base, and consider the formula $\varphi_{\mathfrak{A}}$. Then, by Lemmas 6 and 7, $\neg\varphi_{\mathfrak{A}}$ is not in L. Suppose that there is some finite product frame $\mathcal{G}$ refuting $\neg\varphi_{\mathfrak{A}}$. Then, as we have seen in the proof of '(ii) implies (iii)' of Lemma 7, $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of some universal product $\mathbf{S5}^3$-frame which is now — being a subframe of $\mathcal{G}$ — finite. This contradicts Lemma 6, since $\mathfrak{A}$ is representable only with an infinite base. Note that in Appendix B we demonstrate how such a formula forces an infinite product frame. $\square$

3 Appendix A

Here we prove Lemma 6 in a modal logic setting, via Claims 1–4.

Claim 1. *If $\mathfrak{A}$ is representable with base $U \times U$ (i.e., it is embeddable into the set relation algebra of all subsets of some set $U \times U$) then $\mathcal{F}_{\mathfrak{A}}$ is a p-morphic image of the universal product $\mathbf{S5}^3$-frame on $U \times U \times U$.*

Proof. We may regard the atoms of $\mathfrak{A}$ as binary relations forming a partition of $U \times U$ and composition in $\mathfrak{A}$ as the usual composition (relative product) of binary relations. Define the following function h from $U \times U \times U$ to the set $\mathcal{T}$ of consistent triplets of atoms of $\mathfrak{A}$:

$$h(u_0 u_1 u_2) = abc \text{ iff } (u_0, u_1) \in c,\ (u_2, u_0) \in b,\ (u_1, u_2) \in a.$$

It is easy to check that h is well-defined, onto, and a p-morphism. $\square$

Define, for each $i < j < 3$, a subset E_{ij} of the set $\mathcal{T}$ of worlds of $\mathcal{F}_{\mathfrak{A}}$ as follows. Let $k < 3$ be different from both i and j. Then take

$$E_{ij} = \{t \in \mathcal{T} \ : \ t_k \leq 1'\}.$$

(Recall that $1'$ denotes the identity element of $\mathfrak{A}$.) It is not hard to see that the following properties hold, for all $i < j < 3$:

$$(\forall t \in \mathcal{T})(\exists t', t'' \in E_{ij})\ t R_i t' \text{ and } t R_j t'' \tag{6}$$

$$(\forall t, t' \in \mathcal{T})\ t \in E_{ij} \text{ and } t R_k t' \text{ implies } t' \in E_{ij} \quad (k < 3, k \neq i, j) \tag{7}$$

$$E_{01} \cap E_{02} \subseteq E_{12}, \quad E_{01} \cap E_{12} \subseteq E_{02}, \quad E_{02} \cap E_{12} \subseteq E_{01} \tag{8}$$

$$(\forall t, t' \in E_{ij})\ t R_i t' \text{ or } t R_j t' \text{ implies } t = t'. \tag{9}$$

Claim 2. *Assume that there is a p-morphism h from some universal product frame (U_0, U_1, U_2) onto $\mathcal{F}_{\mathfrak{A}}$. Let U be the disjoint union of the U_is, $i < 3$. Then there is a p-morphism f from the universal product frame (U, U, U) onto $\mathcal{F}_{\mathfrak{A}}$ such that*

$$\text{for all } u_0u_1u_2 \in U \times U \times U,\ i < j < 3,\ \text{ if } u_i = u_j \text{ then } f(u_0u_1u_2) \in E_{ij}. \tag{10}$$

Proof. Note first that for any triplet of surjections $f_i : U \to U_i$ $(i < 3)$, the map f defined by

$$f(u_0u_1u_2) = h(f_0(u_0)f_1(u_1)f_2(u_2))$$

is a p-morphism from (U, U, U) onto $\mathcal{F}_{\mathfrak{A}}$. We will define surjections $f_i : U \to U_i$ $(i < 3)$ such a way that (10) holds.

We claim that for every $u_0 \in U_0$ there is a world $g^{u_0} = u_0u_1u_2$ in $U_0 \times U_1 \times U_2$ such that $h(g^{u_0}) \in E_{01} \cap E_{02} \cap E_{12}$. Indeed, take any $u_0xy \in U_0 \times U_1 \times U_2$. By (6), there is a $u_1 \in U_1$ with $h(u_0u_1y) \in E_{01}$, and there is a $u_2 \in U_2$ with $h(u_0u_1u_2) \in E_{12}$. By (7), $h(u_0u_1u_2) \in E_{01}$ also holds, and so, by (8), $h(u_0u_1u_2) \in E_{02}$. In the same way we can show that for every $u_1 \in U_1$ $(u_2 \in U_2)$ there is $g^{u_1} = u_0u_1u_2$ (respectively, $g^{u_2} = u_0u_1u_2$) in $U_0 \times U_1 \times U_2$ such that $h(g^{u_1}) \in E_{01} \cap E_{02} \cap E_{12}$ (and $h(g^{u_2}) \in E_{01} \cap E_{02} \cap E_{12}$).

Define maps f_i from U onto U_i $(i < 3)$ by taking $f_i(u)$ to be the i-th coordinate of g^u, for every $u \in U$. (Since f_i is identical on U_i, f_i is surjective.) Let $f : U \times U \times U \to \mathcal{T}$ be defined as above. We show that f satisfies (10). For any $u \in U$,

$$f(uuu) = h(f_0(u)f_1(u)f_2(u)) = h(g^u) \in E_{01} \cap E_{02} \cap E_{12}.$$

Now, for any $v \in U$, $f(uuv) \in E_{01}$, $f(uvu) \in E_{02}$ and $f(vuu) \in E_{12}$ follow, by (7). $\square$

Claim 3. *Assume there is a p-morphism f from some universal product frame (U, U, U) onto $\mathcal{F}_{\mathfrak{A}}$ such that (10) holds. Then there is some set V with $|V| \leq |U|$ and a p-morphism g from (V, V, V) onto $\mathcal{F}_{\mathfrak{A}}$ such that*

$$\text{for all } v_0v_1v_2 \in V \times V \times V,\ i < j < 3,\ v_i = v_j \text{ iff } g(v_0v_1v_2) \in E_{ij}. \tag{11}$$

Proof. For every $i < j < 3$, we define a relation $R_{ij} \subseteq U \times U$ by taking

$$R_{ij} = \{(x, y) \in U \times U : \exists u_0, u_1, u_2 \in U \text{ with } u_i = x,\ u_j = y \text{ and } f(u_0u_1u_2) \in E_{ij}\}.$$

In fact, these three relations coincide. Let us check, for instance, that we have $R_{01} \subseteq R_{02}$. Suppose $f(xyz) \in E_{01}$. By (7), $f(xyy) \in E_{01}$ and by (10), $f(xyy) \in E_{12}$. It follows then from (8) that $f(xyy) \in E_{02}$.

We denote R_{ij} by R and show that it is an equivalence relation on $U \times U$. By (10) it is reflexive. Let us see that it is symmetric. Suppose $f(xyz) \in E_{01}$. As we showed above, it follows that $f(xyy) \in E_{02}$. By (7), $f(xxy) \in E_{02}$ and by (10), $f(xxy) \in E_{01}$, from which by (8), $f(xxy) \in E_{12}$ follows. Now by (7), we have $f(yxy) \in E_{12}$. Then, as before, by (10), $f(yxy) \in E_{02}$ and by (8), $f(yxy) \in E_{01}$. Thus, $(y, x) \in R_{01}$ holds. To prove transitivity, suppose $xR_{01}y$ and $yR_{12}z$. This means that we have $f(xys) \in E_{01}$ and $f(ryz) \in E_{12}$, for some s and r. It follows from (7) that $f(xyz) \in E_{01} \cap E_{12}$, and by (8) we obtain $f(xyz) \in E_{02}$, i.e., $xR_{02}z$.

Denote by $[u]$ the R-equivalence class containing u. Let $V = \{[u] : u \in U\}$. Define the function $g : V \times V \times V \to \mathcal{T}$ by taking

$$g([u_0][u_1][u_2]) = f(u_0u_1u_2).$$

We show that this g is well-defined, that is, if u_iRv_i, for each $i<3$, then $f(u_0u_1u_2)=f(v_0v_1v_2)$ holds. We prove first that $f(u_0u_1u_2)=f(u_0v_1u_2)$, if u_1Rv_1. We do this by showing that, for each $i<3$, $f(u_0u_1u_2)_i=f(u_0v_1u_2)_i$, i.e., they are the same atom of $\mathfrak{A}$. For $i=1$ it is obvious by the definition of $\mathcal{F}_{\mathfrak{A}}$. Let $i=2$ first. By (6), there is some $x\in U$ with $f(u_0u_1x)\in E_{12}$, thus u_1Rx follows, which implies v_1Rx. Thus there is some $y\in U$ with $f(yv_1x)\in E_{12}$. By (7), $f(u_0v_1x)\in E_{12}$ also holds, thus, by (9), $f(u_0u_1x)=f(u_0v_1x)$. Therefore,

$$f(u_0u_1u_2)_2=f(u_0u_1x)_2=f(u_0v_1x)_2=f(u_0v_1u_2)_2.$$

The case of $i=0$ is analogous. Further, it can be shown similarly, that $f(u_0v_1u_2)=f(v_0v_1u_2)$ and $f(v_0v_1u_2)=f(v_0v_1v_2)$ also hold which proves that g is well-defined.

It is obvious by its definition that g is a p-morphism satisfying (11). $\square$

CLAIM 4. *Assume there is a p-morphism g from some universal product frame (V,V,V) onto $\mathcal{F}_{\mathfrak{A}}$ such that (11) holds. Then the relation algebra $\mathfrak{A}$ is embeddable into the set relation algebra of all subsets of $V\times V$.*

Proof. Define a function *rep* from $\mathcal{T}$ to the subsets of $V\times V$ as follows: For each $c\in\mathcal{T}$, take

$$rep(c)=\{(u_0,u_1)\in V\times V\ :\ \exists u_2\in V\ \exists a,b \text{ atoms with } g(u_0u_1u_2)=abc\}.$$

Extend *rep* to all elements of $\mathfrak{A}$ the usual way. It is not hard to check that *rep* is a relation algebra embedding. $\square$

4 Appendix B

Here we give a 6-element frame $\mathcal{F}$ (inspired by the *point relation algebra*) and demonstrate how the Yankov–Fine formula $\varphi_{\mathcal{F}}$ of $\mathcal{F}$ forces an infinite product frame.

Let F consist of all triplets of 0s and 1s, excluding 000 and 111. For $i<3$, $t,t'\in F$, let tR_it' iff $t_i=t'_i$, and let $\mathcal{F}=(F,R_0,R_1,R_2)$. For each $t\in F$, introduce a propositional variable t, and let $\varphi_{\mathcal{F}}$ be the Yankov–Fine formula of $\mathcal{F}$ with root 011:

$$011\ \wedge\ \Box^+\bigvee_{t\in F}(t\wedge\neg\bigvee_{t'\neq t}t')\ \wedge\ \Box^+\bigwedge_{\substack{i<3,t,t'\in F\\ tR_it'}}(t\to\Diamond_it')\ \wedge\ \Box^+\bigwedge_{\substack{i<3,t,t'\in F\\ \neg(tR_it')}}(t\to\neg\Diamond_it').$$

CLAIM 5. *There is a product frame satisfying $\varphi_{\mathcal{F}}$.*

Proof. Let

$$Q_0=\left\{\frac{p}{2^n}:p,n<\omega,\ 2\not|p\right\},\ Q_1=\left\{\frac{p}{3^n}:p,n<\omega,\ 3\not|p\right\},\ Q_2=\left\{\frac{p}{5^n}:p,n<\omega,\ 5\not|p\right\}.$$

Take the universal product frame (Q_0,Q_1,Q_2) and define a valuation v of the variables as follows.

$$\begin{aligned}
v(001)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ x<z<y\}\\
v(010)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ z<y<x\}\\
v(011)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ z<x<y\}\\
v(101)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ x<y<z\}\\
v(110)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ y<z<x\}\\
v(110)&=\{(x,y,z)\in Q_0\times Q_1\times Q_2\ :\ y<x<z\}.
\end{aligned}$$

Now let $\mathfrak{M} = (Q_0, Q_1, Q_2, v)$ and take any $(x, y, z) \in v(011)$. It is not hard to check that $\mathfrak{M}, (x, y, z) \models \varphi_{\mathcal{F}}$. □

Claim 6. *Any product frame satisfying $\varphi_{\mathcal{F}}$ is infinite.*

Proof. Let $\mathfrak{M}$ be a model on the product of three frames (U, R_U), (V, R_V) and (W, R_W), and let $M = U \times V \times W$. Let $x_0 \in U$, $y_0 \in V$, $z_0 \in W$ be such that

$$\mathfrak{M}, x_0 y_0 z_0 \models \varphi_{\mathcal{F}}. \tag{12}$$

We will show that both U and V must be infinite sets. Let $0 < n < \omega$ and assume inductively that we have defined points $x_i \in U$ and $y_i \in V$ for each $i < n$ satisfying:

$$\begin{aligned} &x_0 R_U x_i \ \text{ and } \ y_0 R_V y_i, \ \text{ for } \ 0 < i < n, \\ &x_i \neq x_j \ \text{ and } \ y_i \neq y_j, \ \text{ for } \ i \neq j < n, \\ &\mathfrak{M}, x_i y_j z_0 \models 011, \ \text{ for } \ i \leq j < n, \\ &\mathfrak{M}, x_i y_j z_0 \models 010, \ \text{ for } \ j < i < n. \end{aligned} \tag{13}$$

We will now define x_n and y_n. First consider x_n. We have $\mathfrak{M}, x_0 y_{n-1} z_0 \models 011$ and $011 R_0 010$. By (12), there is some $x_n \in U$ such that $x_0 R_U x_n$ and $\mathfrak{M}, x_n y_{n-1} z_0 \models 010$. By (12) and (13), $x_n \neq x_i$, for $i < n$. We show that

$$\mathfrak{M}, x_n y_i z_0 \models 010, \ \text{ for all } \ i < n, \tag{14}$$

must hold. We need the following claim:

Claim 6.1. Assume $u_0, u_1 \in U$ and $v_0, v_1 \in V$ are such that, for each $i < 2$, either $u_i = x_0$ or $x_0 R_U u_i$, and either $v_i = y_0$ or $y_0 R_V v_i$. If $\mathfrak{M}, u_0 v_0 z_0 \models 010$, $\mathfrak{M}, u_0 v_1 z_0 \models 011$, and $\mathfrak{M}, u_1 v_1 z_0 \models 010$ then $\mathfrak{M}, u_1 v_0 z_0 \models 010$ must hold.

Proof of Claim 6.1. We will use (12) all the time (cf. Figure 1). We have $\mathfrak{M}, u_0 v_1 z_0 \models 011$ and $011 R_2 001$, thus there is some $z \in W$ such that $z_0 R_W z$ and $\mathfrak{M}, u_0 v_1 z \models 001$. Then $\mathfrak{M}, u_0 y_0 z \models a$, for some $a \in F$ with $a_1 = 0$, which implies $\mathfrak{M}, u_0 v_0 z \models b$, for some $b \in F$ with $b_1 = 0$. On the other hand, $b_2 = 0$ must also hold, thus $b = 100$. Further, $\mathfrak{M}, x_0 v_1 z \models c$, for some $c \in F$ with $c_0 = 0$, thus $\mathfrak{M}, u_1 v_1 z \models d$, for some $d \in F$ with $d_0 = 0$. On the other hand, $d_2 = 0$ must hold, thus $d = 010$. Therefore, $\mathfrak{M}, u_1 y_0 z \models e$, for some $e \in F$ with $e_1 = 1$, thus $\mathfrak{M}, u_1 v_0 z \models f$, for some $f \in F$ with $f_1 = 1$. Also, by $\mathfrak{M}, u_0 v_0 z \models 100$, we have $\mathfrak{M}, x_0 v_0 z \models g$, for some $g \in F$ with $g_0 = 1$, thus $f_0 = 1$ which implies $f = 110$. Finally, by $\mathfrak{M}, u_0 v_0 z_0 \models 010$, we have $\mathfrak{M}, x_0 v_0 z_0 \models h$, for some $h \in F$ with $h_0 = 0$, thus $\mathfrak{M}, u_1 v_0 z_0 \models t$, for some $t \in F$ with $t_0 = 0$. We have $t_1 = 1$ by $\mathfrak{M}, u_1 v_1 z_0 \models 010$, and $t_2 = 0$ by $\mathfrak{M}, u_1 v_0 z \models 110$, proving $\mathfrak{M}, u_1 v_0 z_0 \models 010$. □

Now, for each $i < n-1$, apply Claim 6.1 with $u_0 = x_{n-1}$, $u_1 = x_n$, $v_0 = y_i$ and $v_1 = y_{n-1}$ to obtain $\mathfrak{M}, x_n y_i z_0 \models 010$ (cf. Figure 2).

Next we define y_n. We have $\mathfrak{M}, x_n y_0 z_0 \models 010$ and $010 R_1 011$. By (12), there is some $y_n \in V$ such that $y_0 R_V y_n$ and $\mathfrak{M}, x_n y_n z_0 \models 011$. By (12) and (14), $y_n \neq y_i$, for $i < n$. It remains to show that, for all $i < n$, $\mathfrak{M}, x_i y_n z_0 \models 011$ must hold as well.

Claim 6.2. Assume $u_0, u_1 \in U$ and $v_0, v_1 \in V$ are as in Claim 6.1. If $\mathfrak{M}, u_0 v_0 z_0 \models 011$, $\mathfrak{M}, u_1 v_0 z_0 \models 010$, and $\mathfrak{M}, u_1 v_1 z_0 \models 011$ then $\mathfrak{M}, u_0 v_1 z_0 \models 011$ must hold.

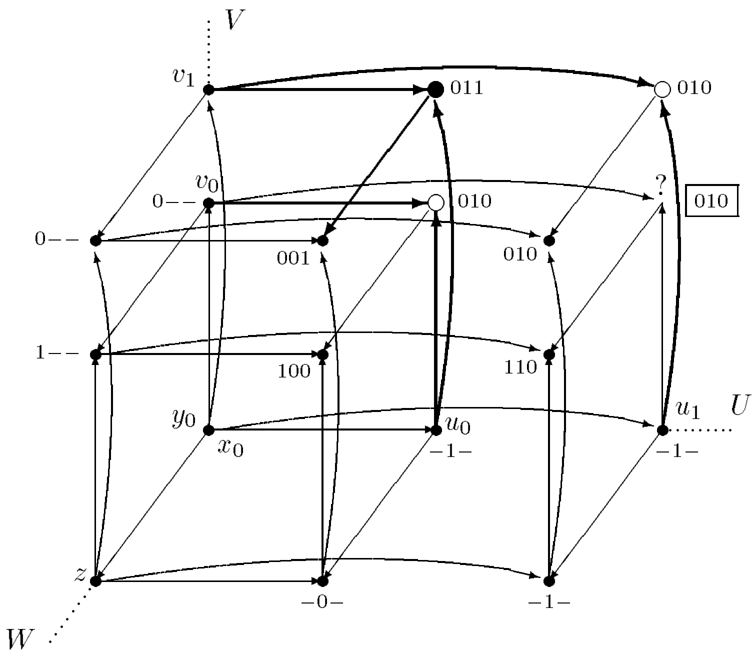


Figure 1: Forcing 010.

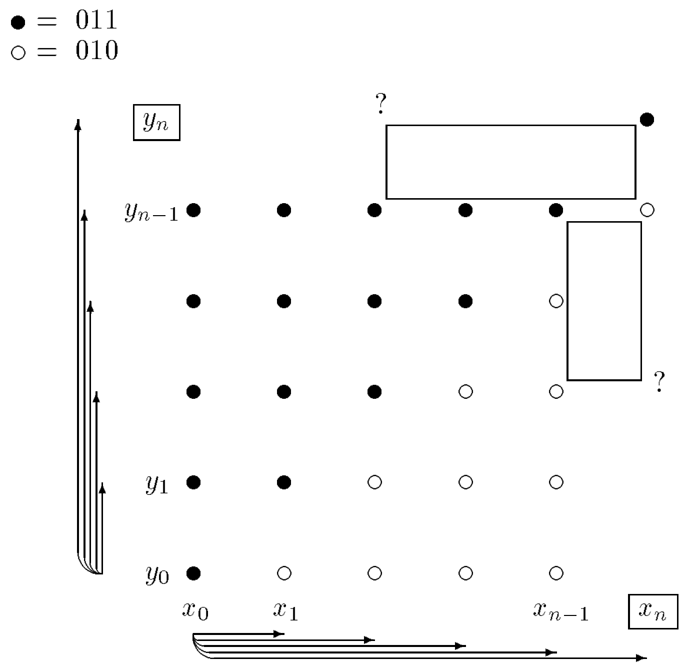


Figure 2: The points x_n and y_n.

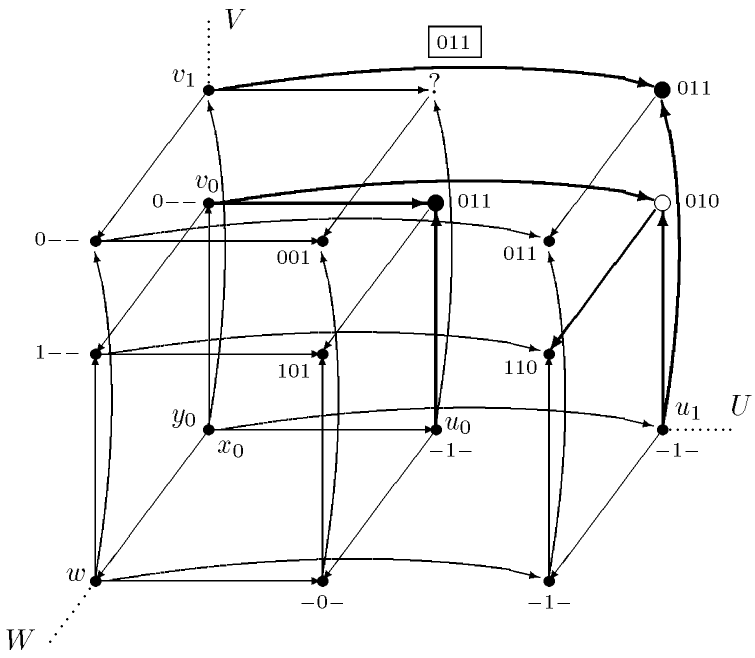


Figure 3: Forcing 011.

Proof of Claim 6.2. We have $\mathfrak{M}, u_1v_0z_0 \models 010$ and $010R_2110$, thus there is some $w \in W$ such that z_0R_Ww and $\mathfrak{M}, u_1v_0w \models 110$. From now on this is similar to the proof of Claim 6.1, see Figure 3. $\square$

For each $i < n$, apply Claim 6.2 with $u_0 = x_i$, $u_1 = x_n$, $v_0 = y_{n-1}$ and $v_1 = y_n$ to obtain $\mathfrak{M}, x_iy_nz_0 \models 011$ (cf. Figure 2 again).

This way we showed that both U and V must be infinite sets, which completes the proof of Claim 6. $\square$

References

[1] D.M. Gabbay. *Fibring logics.* Clarendon Press, Oxford, 1999.

[2] D.M. Gabbay and V.B. Shehtman. Products of modal logics, part I. *Logic Journal of the IGPL*, 6:73–146, 1998.

[3] P. Halmos. Algebraic logic, IV. *Transactions of the AMS*, 86:1–27, 1957.

[4] L. Henkin, J.D. Monk, and A. Tarski. *Cylindric algebras, part II.* North Holland, 1985.

[5] R. Hirsch and I. Hodkinson. Representability is not decidable for finite relation algebras. *Trans. Amer. Math. Soc.*, 1999. To appear.

[6] J.S. Johnson. Nonfinitizability of classes of representable polyadic algebras. *Journal of Symbolic Logic*, 34:344–352, 1969.

[7] A. Kurucz. On axiomatising products of Kripke frames. *Journal of Symbolic Logic*, To appear.

[8] R. Maddux. The equational theory of CA_3 is undecidable. *Journal of Symbolic Logic*, 45:311–316, 1980.

[9] J.D. Monk. Studies in cylindric algebra. Doctoral Dissertation, University of California, Berkeley, 1961.