Axiomatizability of Representable Domain Algebras

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Abstract

The family of domain algebras provide an elegant formal system for automated reasoning about programme verification. Their primary models are algebras of relations, viz. representable domain algebras. We prove that, even for the minimal signature consisting of the domain and composition operations, the class of representable domain algebras is not finitely axiomatizable. Then we show similar results for extended similarity types of domain algebras.
Keywords: domain algebras, relation algebras, finite axiomatizability

1 Introduction

Domain algebras provide an elegant, one-sorted formalism for automated reasoning about program and system verification [DS08a, DS08b]. Traditionally, similar algebraic formalisms (like dynamic algebras [Pr90, Pr91] and Kleene algebras with tests KAT [Ko97]) used a two-sorted approach: there is one sort for states and another sort for actions, and some operations mapping between actions and states. Using the domain operation d, one sort (sort of actions) is enough, since states can be modelled as those actions a for which we have $a = \mathsf{d}(a)$. Such a one-sorted approach is simpler and more suitable for automated reasoning, see [DS08a, DS08b] and the references therein for more details.

Using the domain operation we can express if an action is enabled at a certain state: $\mathsf{d}(a)$ consists of those states at which action a can be taken. Besides the domain operation, domain algebras contain an operation for modelling sequential composition of actions (or processes), we will denote it by ;. Other connectives that can be included are: join + for modelling non-deterministic choice, identity $1'$ for modelling the ineffective action skip, zero 0 for modelling the abortive action, and the reflexive–transitive closure * for modelling iteration. If we want the set of states, or domain elements, to have a more expressive structure than a (semi)lattice, then we can include boolean negation on states. In the one-sorted approach, this can be done by including an antidomain operation a: $\mathsf{a}(a)$ consists of those states where action a is not enabled. The dual of domain is given by the range operation: $\mathsf{r}(a)$ consists of those states that can be reached via action a. Depending

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on the choice of operations, one can define (anti)domain(–range) semigroups/monoids/semirings, etc., see [DS08a, DS08b, DJS09] for axiomatizations of these classes of algebras.

[DMS04] explains in detail how a variant of domain algebras, called modal Kleene algebras, can be applied to partial and total program correctness. Here we focus on the connection to partial correctness. As with Kleene algebras with tests (KAT) [Ko00], domain algebras enable us to formulate the partial correctness assertions and the inference rules of propositional Hoare logic as equations and quasiequations (or universal Horn formulas), respectively. Indeed, consider an assertion $\{b\}p\{c\}$ with formulas b, c and program p expressing that c must hold in the output state whenever program p terminates after an execution such that b was true in the input state (at the beginning of the execution). The above assertion can be translated to $b\,;p = b\,;p\,;c$ where, in KAT, b and c are of boolean sort (tests) and p is of kleenean sort (program), and, in the domain-algebra formalism, b and c are domain elements (elements x such that $\mathsf{d}(x) = x$). Hence we can write this equation as

$$\mathsf{d}(b) \,;\, p = \mathsf{d}(b) \,;\, p \,;\, \mathsf{d}(c) \tag{1}$$

or, using the range operation, as

$$\mathsf{r}(\mathsf{d}(b) \,;\, p) \leq \mathsf{d}(c) \tag{2}$$

see [MS06] for similar translations. Inference rules can be translated similarly. For instance, the composition rule with tests b, c, d and programs p, q translates to the quasiequation

$$(b \,;\, p = b \,;\, p \,;\, c \wedge c \,;\, q = c \,;\, q \,;\, d) \to b \,;\, p \,;\, q = b \,;\, p \,;\, q \,;\, d \tag{3}$$

with domain elements b, c, d. We also recall from [Ko00] that there are intuitively valid inference rules that are not derivable in propositional Hoare logic, e.g.,

$$\{c\}(\text{if } b \text{ then } p \text{ else } p)\{c\} \Rightarrow \{c\}p\{c\} \tag{4}$$

We refer the reader to [Ko00] for more details on propositional Hoare logic and its connection to KAT.

As [Ko06] argues: "In programming language semantics and verification, the relational models are of primary importance, because correctness conditions are often expressed as input/output conditions on the start and final state of the computation". Similarly, [DJS09] writes that the "primary model of interest is the algebra $\mathit{Rel}(X)$ of binary relations R on a set X with composition and unary (anti)domain and (anti)range operations [since it] is a standard semantic model for the input-output relation of nondeterministic programs and specifications, and the domain/range operations can be used to define pre- and postconditions and modal (program) operators on a state space". Hence, in this paper, we will focus on the semantics of domain algebras provided by binary relations. An action is modelled as the binary relation of input–output pairs, states as subidentity relations, and the operations of the domain algebra as "natural" operations on binary relations. For instance,

$$\mathsf{d}(a) = \{(s, s) : (s, t) \in a \text{ for some } t\} \tag{5}$$

$$a \,;\, b = \{(s, t) : (s, u) \in a \text{ and } (u, t) \in b \text{ for some } u\} \tag{6}$$

see Definition 2.2 below for exact details. One of the fundamental questions is whether domain algebras are complete with respect to this semantics, i.e., whether every semantically valid (quasi)equation is a theorem of domain algebras. In fact, the recent publication [DJS09] poses some open problems regarding the completeness of domain algebras. These questions boil down to the problem of whether the axiomatically defined domain algebras are isomorphic to algebras of binary relations, i.e., using the slogan of algebraic logic, whether they are representable. We can formulate the question in a more general setting: are the (quasi)equational theories of representable domain algebras finitely axiomatizable? Besides its theoretical importance, a completeness result would enable us to argue about partial program correctness in a finite, equational derivation system that provides "a simple flexible basis for automated theorem proving in program and system verification" [DJS09]. As we have seen above with the translation of propositional

Hoare logic, inference rules in general translate to quasiequations. For instance, rule (4) translates to the relationally valid quasiequation

$$(\mathsf{d}(c)\,;(\mathsf{d}(b)\,;p+\mathsf{a}(b)\,;p)=\mathsf{d}(c)\,;(\mathsf{d}(b)\,;p+\mathsf{a}(b)\,;p)\,;\mathsf{d}(c))\rightarrow \mathsf{d}(c)\,;p=\mathsf{d}(c)\,;p\,;\mathsf{d}(c) \qquad (7)$$

Hence ideally we would like to have a complete axiomatization of the valid quasiequations.[1] Similar problems have been investigated for Kleene algebras. For instance, [Ko06] notes that the quasiequation $p \leq 1' \rightarrow p\,;p = p$ is valid in relational models but not true in all KATs, and gives sufficient conditions for the representability of KATs and their *-free fragment. [Ko90] states as an open problem whether the quasiequational theory of regular events is finitely based, and [Ko06] writes "Axiomatization of the universal Horn theory of relational models is another interesting open problem".

Representable domain algebras (without the Kleene star) are subreducts of Tarski's representable relation algebras RRAs, see, e.g., [HH02, Ma06]. Finite axiomatizability of various fragments has been extensively investigated, see [Sc91, Mi04, AM10] for surveys. We recall that a representable relation algebra is a boolean set algebra of binary relations expanded with the following extra-boolean operators: the identity constant $1'$ (interpreted as the identity, or diagonal, relation), the unary converse operation $\smile$ (interpreted as the inverse relation) and the binary composition operation ; (see (6) above). In RRAs, the domain, range and antidomain operations are definable as

$$\mathsf{d}(x) = 1' \cdot (x\,;x^{\smile}) \quad \text{and} \quad \mathsf{r}(x) = 1' \cdot (x^{\smile}\,;x) \quad \text{and} \quad \mathsf{a}(x) = -\mathsf{d}(x)$$

The problem is that these definitions of domain and range explicitly use the converse $\smile$ and meet $\cdot$ operations, and the $\{;, \cdot, \smile, 1'\}$-reduct of RRA has non-finitely axiomatizable equational and quasiequational theories, cf. [HM00]. If we take d and r as basic operations, then we can avoid the use of converse and meet and hope for finite axiomatizability. As it turns out, the quasiequational theory is still not finitely based even for the minimal domain algebra signature $\{\mathsf{d}, ;\}$ of domain semigroups and its various extensions with additional operations, cf. Theorem 2.3 and Corollary 5.2. We conjecture that the equational theories of representable *-free domain algebras are finitely based, see [AM10] for similar axiomatizability results for reducts of RRA and [Ho97] for axiomatizing the equational theory of representable antidomain monoids.

The rest of the paper is organized as follows. In the next section, we recall the precise definitions of domain algebras and representable algebras, and state our main result that representable domain algebras are not finitely axiomatizable even for the minimal signature $\{\mathsf{d}, ;\}$ consisting of domain and composition. In Section 3, we define domain algebras and show that they are not representable, and in Section 4, we characterize representability by a two-player game, and show that a non-principal ultraproduct of our non-representable domain algebras is representable, establishing the main result. In Section 5, we look at the problem of expanding the similarity type of domain algebras with extra operations, and in Section 6, we conclude by stating some open problems.

2 Basics and main result

We recall the definition of domain algebras from [DJS09].

Definition 2.1 *A* domain–range monoid $\mathfrak{A} = (A, \mathsf{d}, \mathsf{r}, ;, 1')$ *consists of a monoid* $(A, ;, 1')$ *and*

[1] [Ko06] shows that, in KAT, inference rules of propositional Hoare logic can be translated to equations. In this paper, we will also consider similarity types in which the Kleene star is not expressible, hence Kozen's method may not work.

unary operations $\mathsf{d}, \mathsf{r} : A \to A$ *such that the axioms* $(D1) - (D5)$ *and* $(R1) - (R5)$ *are satisfied:*

$$\begin{array}{llll}
(D1) & \mathsf{d}(x) \,;\, x = x & (R1) & x \,;\, \mathsf{r}(x) = x \\
(D2) & \mathsf{d}(x \,;\, y) = \mathsf{d}(x \,;\, \mathsf{d}(y)) & (R2) & \mathsf{r}(x \,;\, y) = \mathsf{r}(\mathsf{r}(x) \,;\, y) \\
(D3) & \mathsf{d}(\mathsf{d}(x) \,;\, y) = \mathsf{d}(x) \,;\, \mathsf{d}(y) & (R3) & \mathsf{r}(x \,;\, \mathsf{r}(y)) = \mathsf{r}(x) \,;\, \mathsf{r}(y) \\
(D4) & \mathsf{d}(x) \,;\, \mathsf{d}(y) = \mathsf{d}(y) \,;\, \mathsf{d}(x) & (R4) & \mathsf{r}(x) \,;\, \mathsf{r}(y) = \mathsf{r}(y) \,;\, \mathsf{r}(x) \\
(D5) & \mathsf{d}(\mathsf{r}(x)) = \mathsf{r}(x) & (R5) & \mathsf{r}(\mathsf{d}(x)) = \mathsf{d}(x)
\end{array}$$

A domain monoid $\mathfrak{A} = (A, \mathsf{d}, ;, 1')$ *is defined similarly, by dropping all the conditions that involve* r*. A* domain *or* domain–range semigroup *is an* $1'$*-free subreduct of the corresponding monoid.*

An antidomain monoid *is* $\mathfrak{A} = (A, \mathsf{a}, ;, 1')$ *with a monoid* $(A, ;, 1')$ *and unary operation* $\mathsf{a} : A \to A$ *satisfying the following axioms*

$$\begin{array}{ll}
(A1) & \mathsf{a}(x) \,;\, x = 0 \\
(A2) & x \,;\, 0 = 0 \\
(A3) & \mathsf{a}(x) \,;\, \mathsf{a}(y) = \mathsf{a}(y) \,;\, \mathsf{a}(x) \\
(A4) & \mathsf{a}(\mathsf{a}(x)) \,;\, x = x \\
(A5) & \mathsf{a}(x) = \mathsf{a}(x \,;\, y) \,;\, \mathsf{a}(x \,;\, \mathsf{a}(y)) \\
(A6) & \mathsf{a}(x \,;\, y) \,;\, x = \mathsf{a}(x \,;\, y) \,;\, x \,;\, \mathsf{a}(y)
\end{array}$$

where 0 *is defined as* $\mathsf{a}(1')$.

We will use the term *domain algebra* for any algebra $\mathfrak{A}$ of similarity type τ such that $\{;, \mathsf{d}\} \subseteq \tau \subseteq \{;, \mathsf{d}, \mathsf{r}, \mathsf{a}, 1', 0\}$ and $\mathfrak{A}$ satisfies the relevant axioms above. We may use the term *antidomain algebra* when we want to emphasize that a is in the similarity type. The set of *domain elements* D of an (anti)domain algebra is the set of all elements e such that $\mathsf{d}(e) = e$.

We will use the notation $X;Y = \{x;y : x \in X, y \in Y\}$, $\mathsf{d}(X) = \{\mathsf{d}(x) : x \in X\}$, etc. for subsets of elements X and Y. If D is the set of domain elements of $\mathfrak{A}$ then $(D, ;)$ is a lower semilattice. We write $\leq$ for the ordering induced by this operation: $e \leq e'$ iff $e \,;\, e' = e$. In an antidomain monoid, we define $\mathsf{d}(x) = \mathsf{a}(\mathsf{a}(x))$. In an antidomain monoid, the domain elements D form a boolean algebra $\mathfrak{D} = (D, \cdot, -)$ with the operations defined as $x \cdot y = x \,;\, y$ and $-x = \mathsf{a}(x)$.

Definition 2.2 *A* representation $\mathcal{M}$ *of a domain algebra* $\mathfrak{A}$ *consists of a set* M *(the* base *of the representation) and an interpretation* $x^{\mathcal{M}} \subseteq M \times M$ *of each element* $x \in A$ *such that* $(x \,;\, y)^{\mathcal{M}}$ *is the composition of the relations* $x^{\mathcal{M}}$ *and* $y^{\mathcal{M}}$

$$(x \,;\, y)^{\mathcal{M}} = \{(u, v) \in M \times M : (u, w) \in x^{\mathcal{M}} \text{ and } (w, v) \in y^{\mathcal{M}} \text{ for some } w \in M\}$$

and for each operation in the signature of $\mathfrak{A}$,

$$\begin{aligned}
(\mathsf{d}(x))^{\mathcal{M}} &= \{(u, u) \in M \times M : (u, v) \in x^{\mathcal{M}} \text{ for some } v \in M\} \\
(\mathsf{r}(x))^{\mathcal{M}} &= \{(v, v) \in M \times M : (u, v) \in x^{\mathcal{M}} \text{ for some } u \in M\} \\
(\mathsf{a}(x))^{\mathcal{M}} &= \{(u, u) \in M \times M : (u, v) \notin x^{\mathcal{M}} \text{ for all } v \in M\} \\
(1')^{\mathcal{M}} &= \{(u, v) \in M \times M : u = v\}
\end{aligned}$$

and $x \neq y$ *implies* $x^{\mathcal{M}} \neq y^{\mathcal{M}}$.

A domain algebra is representable *if it has a representation. In general, if* τ *is a similarity type, we write* $\mathsf{R}(\tau)$ *for the class of representable* τ*-algebras.*

It is easy to check that the class of representable τ-algebras is closed under subalgebras, direct products and ultraproducts (it is pseudo-axiomatizable). Hence it forms a quasivariety. Our main result is the following.

Theorem 2.3 *Let* τ *be a similarity type such that* $\{\mathsf{d}, ;\} \subseteq \tau \subseteq \{\mathsf{d}, \mathsf{r}, \mathsf{a}, ;, 1', 0\}$*. The class* $\mathsf{R}(\tau)$ *of representable* τ*-algebras is not finitely axiomatizable in first-order logic.*

Proof: We will define domain algebras $\mathfrak{B}_n$ for every $n \in \omega$ of signature $\{\mathsf{d}, \mathsf{r}, \mathsf{a}, ;, 1'\}$ in Definition 3.6, and show that

1. the $\{\mathsf{d}, ;\}$-reduct of $\mathfrak{B}_n$ is not representable, Corollary 3.10,
2. non-principal ultraproducts $\prod_U \mathfrak{B}_n$ of $\mathfrak{B}_n$ over ω are representable, Corollary 4.8.

Now suppose, for contradiction, that a finite set of formulas, without loss of generality a single formula $\sigma(\tau)$, defines $\mathsf{R}(\tau)$. Since $\mathfrak{B}_n$ is not representable, $\mathfrak{B}_n \models \neg\sigma(\tau)$ for $n \in \omega$. Since Łoś theorem states that first-order formulas are preserved under ultraproducts (see [CK90]), we have $\prod_U \mathfrak{B}_n \models \neg\sigma(\tau)$, so $\prod_U \mathfrak{B}_n$ is not representable, a contradiction. ∎

3 Non-representability

In this section, we define domain algebras $\mathfrak{A}_n$ and show that they are not representable. Before we define these algebras, we review some basic results about domain algebras and introduce some additional properties and relations that may be defined.

We refer the reader to [DJS09] for the basics of domain-algebra arithmetic. We list a few easy consequences of the axioms — similar statements hold for r instead of d.

Proposition 3.1 *Let $\mathfrak{A}$ be a domain algebra and let $x, y \in A$.*

1. *If $e = \mathsf{r}(x) ; \mathsf{d}(y)$, then e is a domain element and $x ; y = x ; e ; y$.*
2. *$\mathsf{d}(x ; y) = \mathsf{d}(x) ; \mathsf{d}(x ; y)$ and $\mathsf{d}(\mathsf{d}(x)) = \mathsf{d}(x)$.*
3. *For any domain element e, we have $\mathsf{d}(a ; e ; b) \le \mathsf{d}(a ; b)$.*

Definition 3.2 *A domain algebra $\mathfrak{A}$ with a set of domain elements D is said to be* loose *if it includes an element 0 such that, for all $x, y \in A$,*

1. *$0 ; x = x ; 0 = \mathsf{d}(0) = \mathsf{r}(0) = 0$*
2. *$x ; y \in D \iff (x ; y = 0$ or both $x, y \in D)$.*

Next, we introduce two binary relations $\sqsubseteq, \preccurlyeq$ on the elements of a domain algebra.

Definition 3.3 *Let $\mathfrak{A}$ be a domain algebra, and let D be the set of domain elements of $\mathfrak{A}$. Define a binary relation $\sqsubseteq$ on $\mathfrak{A}$ by*

$$a \sqsubseteq b \quad \text{iff} \quad a = e_0 ; b ; e_1 \ \text{ for some } \ e_0, e_1 \in D.$$

Let $a \sqsubset b$ iff $a \sqsubseteq b$ and $a \neq b$. For any $a \in A$, $a^{\uparrow}$ denotes $\{b \in A : a \sqsubseteq b\}$.

We define another binary relation $\preccurlyeq$ on $\mathfrak{A}$ by letting

$$a \preccurlyeq b \quad \text{iff} \quad a = e_0 ; u_0 ; e_1 ; u_1 ; \ldots ; u_{n-1} ; e_n \ \text{ and } \ b = u_0 ; u_1 ; \ldots ; u_{n-1} \tag{8}$$

for some natural number n, domain elements $e_0, e_1, \ldots, e_n$ and elements $u_0, \ldots, u_{n-1}$. We will write $a \prec b$ if $a \preccurlyeq b$ and $a \neq b$.

*Let $1 \le k < \omega$. A sequence $(\alpha_0, \ldots, \alpha_{k-1})$ of elements, where for each $i < k-1$, we have $\alpha_i \prec \alpha_{i+1}$, is called a k-*chain *from α_0* to *α_{k-1} or simply* a chain. *The length of such a chain is k. For $k \ge 2$, a k-*cycle *is a k-chain $(\alpha_0, \ldots, \alpha_{k-1})$ such that $\alpha_{k-1} \prec \alpha_0$.*

Note that the sequence (a_0) is a 1-chain from a_0 to a_0 (the chain condition holds vacuously), but it is not a cycle, since the length of a cycle has to be at least two.

Lemma 3.4 *If a domain algebra has a cycle, then it is not representable.*

Proof: In any representation $\mathcal{M}$, it is very easy to verify that $a \preccurlyeq b$ implies $a^{\mathcal{M}} \subseteq b^{\mathcal{M}}$ (see the next proposition, below). Hence, if $(a_0, a_1, \ldots, a_{k-1})$ is a cycle, then $a_0^{\mathcal{M}} = a_1^{\mathcal{M}} = \ldots = a_{k-1}^{\mathcal{M}}$. Since $a_0 \neq a_1$, this contradicts the faithfulness of the representation. ∎

The main idea for the construction of the non-representable domain algebra $\mathfrak{A}_n$ (below) is that it includes such an n-cycle.

Clearly $a \sqsubseteq b$ implies $a \preccurlyeq b$, but the converse fails, as we will see. The relation $a \sqsubseteq b$ is equivalent to an equation $a = \mathsf{d}(a)\,;b\,;\mathsf{r}(a)$, but the definition of the relation $a \preccurlyeq b$ seems to make essential use of existential quantification. Note that $\sqsubseteq$ coincides with the boolean ordering $\leq$ on D. If $\mathfrak{A}$ is a loose domain algebra, then $\preccurlyeq$ also coincides with the boolean ordering on D.

All of the parts of the next proposition follow easily from the definitions of $\sqsubseteq$ and $\preccurlyeq$ and from Proposition 3.1.

Proposition 3.5 *Let $\mathfrak{A}$ be a domain algebra.*

1. *The relation $\sqsubseteq$ is reflexive and transitive, but ; generally is not monotone w.r.t. $\sqsubseteq$.*
2. *If e is a domain element and $a \preccurlyeq e$, then a is a domain element.*
3. *If $a \preccurlyeq b$, then $\mathsf{d}(a) \leq \mathsf{d}(b)$ and $\mathsf{r}(a) \leq \mathsf{r}(b)$.*
4. *$\preccurlyeq$ is reflexive and ; is monotone w.r.t. $\preccurlyeq$*

   $$\text{if } a_1 \preccurlyeq a_2 \text{ and } b_1 \preccurlyeq b_2, \text{ then } a_1\,;b_1 \preccurlyeq a_2\,;b_2$$

   but generally is not transitive.
5. *Let $\mathcal{M}$ be a representation of $\mathfrak{A}$. If $a \preccurlyeq b$, then $a^{\mathcal{M}} \subseteq b^{\mathcal{M}}$.*

Now let $\mathfrak{A}$ be a loose domain algebra.

6. *If e is a domain element and $e \preccurlyeq f$, then f is a domain element.*

Next we define, for every $n \in \omega$, a domain algebra $\mathfrak{A}_n$. These algebras are formally defined in Definition 3.6, but first we give a rough outline of how they may be constructed. Start with the representable domain algebra $\mathfrak{C}$ with elements $\{0, e, f, e_*, a, b, c\}$ where the set D of domain elements is $\{0, e, f, e_*\}$. The domain and range operations are defined by $\mathsf{d}(\delta) = \mathsf{r}(\delta) = \delta$ for $\delta \in D$, $\mathsf{d}(a) = \mathsf{d}(c) = e$, $\mathsf{d}(b) = e_* = \mathsf{r}(a)$ and $\mathsf{r}(b) = \mathsf{r}(c) = f$. Composition is defined for domain elements by $\delta\,;\delta' = 0$ if $\delta \neq \delta'$ else δ, $\mathsf{d}(x)\,;x = x = x\,;\mathsf{r}(x)$ (all x) and $a\,;b = c$, and all other compositions are zero. A representation of $\mathfrak{C}$ over the base set $\{0,1,2\}$ maps the non-zero elements of $\mathfrak{C}$ to singleton sets of ordered pairs (i,j) where $i \leq j < 3$ as shown in Figure 1(a). Secondly, we 'split' $\mathfrak{C}$ to obtain a representable domain algebra $\mathfrak{C}_n$, by replacing the elements e_*, a and b by $\{e_i : i < n\}$, $\{a_i : i < n\}$ and $\{b_i : i < n\}$, respectively, leaving the operations unchanged, except $\mathsf{d}(e_i) = \mathsf{r}(e_i) = e_i$, $\mathsf{d}(a_i) = e$, $\mathsf{r}(b_i) = f$, $\mathsf{r}(a_i) = e_i = \mathsf{d}(b_i)$, and $a_i\,;b_j = 0$ if $i \neq j$ and $a_i\,;b_i = c$ for $i < n$. A representation of $\mathfrak{C}_n$ can be obtained from the representation shown in Figure 1(a) of $\mathfrak{C}$ by replacing the base point 1 by n points $1_0, 1_1, \ldots, 1_{n-1}$, as shown in Figure 1(b). Finally, we obtain the non-representable domain algebra $\mathfrak{A}_n$ by further splitting the elements e_i, a_i, b_i so that for $x \in \{e_i, a_i, b_i\}$ we replace x by $\{x^{01}, x^{10}, x^{11}\}$, and we replace c by $\{c_i : i < n\}$. The operations are defined on these new elements in such a way as to make $c_0 \prec c_1 \prec \ldots \prec c_{n-1} \prec c_0$ an n-cycle of $\mathfrak{A}_n$. Again, the domain and range operations are mostly unchanged, but $\mathsf{d}(c_i) = e$, $\mathsf{r}(c_i) = f$, $\mathsf{r}(a_i^{\lambda,\mu}) = e_i^{\lambda,\mu} = \mathsf{d}(b_i^{\lambda,\mu})$, for $(\lambda, \mu) \in \{(0,1), (1,0), (1,1)\}$. For composition, we refine the composition of $\mathfrak{C}_n$ in such a way that $c_i \prec c_{i+1}$, for $i < n$, making $(c_0, c_1, \ldots, c_{n-1})$ into an n-cycle, see (14) below. (In passing we note that, since our signature does not include boolean meet or join, there is no requirement that $c_{i-1} + c_i = a_i^{11}\,;e_i^{01}\,;b_i^{11} + a_i^{11}\,;e_i^{10}\,;b_i^{11} = a_i^{11}\,;e_i^{11}\,;b_i^{11} = c_i$.) Now we give the formal definition.

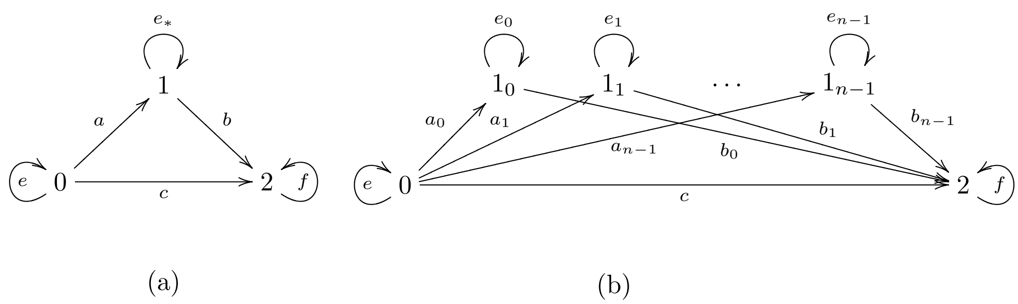

Figure 1: Representations of $\mathfrak{C}$ and $\mathfrak{C}_n$.

Definition 3.6 *Let $n \in \omega$. Let $\mathfrak{D}_n = (D_n, \cdot, -)$ be the boolean algebra generated by the following set of atoms $At(\mathfrak{D}_n) = \{e, f, e_i^{01}, e_i^{10} : i < n\}$. Let $\leq$ denote the usual ordering on $\mathfrak{D}_n$ and $+$ boolean join. We denote the top and bottom elements of $\mathfrak{D}_n$ by $1'$ and 0, respectively, and define $e_i^{11} = e_i^{01} + e_i^{10}$, for $i < n$.*

Let $\mathfrak{A}_n = (A_n, ;, \mathsf{d}, \mathsf{r}, \mathsf{a}, 1')$ be an algebra with elements

$$D_n \cup \{a_i^{01}, a_i^{10}, a_i^{11}, b_i^{01}, b_i^{10}, b_i^{11}, c_i : i < n\}$$

with operations defined below. For $x \in \{e, a, b\}$ and $i < n$ the symbol x_i^{00} denotes 0. The set of domain elements of $\mathfrak{A}_n$ is D_n. The domain and range operators are defined by

$$\mathsf{d}(x) = \mathsf{r}(x) = x \quad \text{if } x \in D_n \tag{9}$$

and for non-domain elements x

$$\mathsf{d}(x) = \begin{cases} e & \text{if } x \in \{a_i^{01}, a_i^{10}, a_i^{11}, c_i : i < n\} \\ e_i^{\lambda,\mu} & \text{if } x = b_i^{\lambda,\mu} \end{cases} \tag{10}$$

$$\mathsf{r}(x) = \begin{cases} f & \text{if } x \in \{b_i^{01}, b_i^{10}, b_i^{11}, c_i : i < n\} \\ e_i^{\lambda,\mu} & \text{if } x = a_i^{\lambda,\mu} \end{cases} \tag{11}$$

for any $i < n$ and $(\lambda, \mu) \in \{(0,1), (1,0), (1,1)\}$. Composition is defined first for domain elements:

$$d ; d' = d \cdot d' \qquad \text{for all domain elements } d \text{ and } d'.$$

Of course, we will have

$$x = \mathsf{d}(x) ; x = x ; \mathsf{r}(x) \text{ and } 0 = 0 ; x = x ; 0 \qquad \text{for all } x.$$

In addition, we require

$$a_i^{\lambda,\mu} ; e_i^{\nu,\pi} = a_i^{\lambda\cdot\nu,\mu\cdot\pi} \tag{12}$$

$$e_i^{\lambda,\mu} ; b_i^{\nu,\pi} = b_i^{\lambda\cdot\nu,\mu\cdot\pi} \tag{13}$$

$$a_i^{\lambda,\mu} ; b_i^{\nu,\pi} = \begin{cases} c_i & \text{if } \lambda\cdot\nu = 1 \\ c_{i-1} & \text{if } \lambda\cdot\nu = 0 \text{ and } \mu\cdot\pi = 1 \\ 0 & \text{if } \lambda\cdot\nu = \mu\cdot\pi = 0 \end{cases} \tag{14}$$

for $\lambda, \mu, \nu, \pi \in \{0, 1\}$ and $i < n$, $-$ is modulo n. All other compositions are defined to be zero.

The antidomain operation a is defined by taking the complement of d in $\mathfrak{D}_n$: $\mathsf{a}(x) = -\mathsf{d}(x)$. The antirange operation is already definable in this signature, the antirange of x is $\mathsf{a}(\mathsf{r}(x))$. The boolean connectives of $\mathfrak{D}_n$ can be recovered by using a as complement and $;$ as meet.

Lemma 3.7 *For each $n \in \omega$, $\mathfrak{A}_n$ is a loose domain algebra. Furthermore, if $x, y \in A_n$ and neither x nor y is a domain element, then there is an atom d of $\mathfrak{D}_n$ such that $x ; d ; y = x ; y$.*

Proof: We check the second sentence of the lemma first. It can easily be checked, from Definition 3.6, that

$$x\,;y=0 \iff \mathsf{r}(x);\mathsf{d}(y)=0 \tag{15}$$

If $x\,;y=0$, then we may pick an arbitrary atom d of $\mathfrak{D}_n$ and we have $x\,;d\,;y=x\,;y=0$. Suppose $x\,;y\neq 0$, so $\mathsf{r}(x)\,;\mathsf{d}(y)\neq 0$. Since x is not a domain element by assumption, we observe that $\mathsf{r}(x)\in\{e,f,e_i^{01},e_i^{10},e_i^{11}:i<n\}$, hence $\mathsf{r}(x)\,;\mathsf{d}(y)$ also belongs to this set. If $\mathsf{r}(x)\,;\mathsf{d}(y)$ is an atom of $\mathfrak{D}_n$, then we may let d equal this atom. The final case is where $\mathsf{r}(x)\,;\mathsf{d}(y)$ is neither zero nor an atom, i.e., it is e_i^{11} for some $i<n$. Since neither x nor y is a domain element by assumption, we must have $x=a_i^{11}$ and $y=b_i^{11}$ by (12) and (13). The required domain atom d is e_i^{10} in this case (but note by (14) that the other atom below e_i^{11}, namely e_i^{01}, would not work).

Looseness of $\mathfrak{A}_n$ is easily checked from Definition 3.6. We must show that $\mathfrak{A}_n$ is a domain algebra. The axiom which is the most difficult to check is associativity of composition:

$$(x\,;y)\,;z=x\,;(y\,;z) \tag{16}$$

If $x\,;y=0$, then, since $\mathsf{d}(y\,;z)\leq\mathsf{d}(y)$, we have $\mathsf{r}(x)\,;\mathsf{d}(y\,;z)\leq\mathsf{r}(x)\,;\mathsf{d}(y)=0$, so both sides of (16) are zero by (15). Hence we may assume that $\mathsf{r}(x)\,;\mathsf{d}(y)\neq 0$ and similarly $\mathsf{r}(y)\,;\mathsf{d}(z)\neq 0$.

If x,y,z are all domain elements, then, since composition of domain elements is defined by boolean meet, (16) is true. So we may assume that at least one of x,y,z is not a domain element. Let

$$E^*=\{e,f,e_i^{01},e_i^{10},e_i^{11}:i<n\}$$

Note that E^* is a proper subset of the set of domain elements (recall that $\mathfrak{D}_n$ is a boolean algebra, closed under $+$), but E^* is the important part of $\mathfrak{D}_n$ in the following sense. For any non-domain element w, we have $\mathsf{d}(w),\mathsf{r}(w)\in E^*$. Hence, for any non-domain element w and any domain element d, we have $w\,;(d\,;\mathsf{r}(w))=w\,;d$ and $(d\,;\mathsf{d}(w))\,;w=d\,;w$, and both $d\,;\mathsf{r}(w)$ and $d\,;\mathsf{d}(w)$ belong to E^*. Hence we may replace any domain element among x,y,z by an element from E^* without altering either side of (16). So we will assume that the only domain elements occurring in (16) belong to E^*.

First suppose $\mathsf{d}(y)=e$. Since $\mathsf{r}(x)\,;\mathsf{d}(y)\neq 0$, we have $x=e$ by (9) and (11). Hence both sides of (16) equal $y;z$. Next suppose $\mathsf{d}(y)=f$. Then $y=z=f$, by (9) and (10), and both sides of (16) equal x. Hence we may assume that $\mathsf{d}(y)\in\{e_i^{01},e_i^{10},e_i^{11}:i<n\}$ and similarly $\mathsf{r}(y)$ also belongs to this set. In $\mathfrak{A}_n$, if the domain and range of an element belong to $\{e_i^{01},,e_i^{10},e_i^{11}:i<n\}$, then the element itself belongs to this set. Therefore, $y=e_i^{\lambda,\mu}$, for some i,λ,μ. Since $\mathsf{r}(x)\,;\mathsf{d}(y)\neq 0$, we have either $x=e_i^{\lambda',\mu'}$ or $x=a_i^{\lambda',\mu'}$, for some λ',μ'. Similarly, either $z=e_i^{\lambda^*,\mu^*}$ or $z=b_i^{\lambda^*,\mu^*}$, for some λ^*,μ^*. Let $l=\lambda\cdot\lambda'\cdot\lambda^*$ and $m=\mu\cdot\mu'\cdot\mu^*$. Then both sides of (16) evaluate to

$$\begin{cases} 0 & \text{if } l=m=0\\ e_i^{l,m} & \text{if } x=e_i^{\lambda',\mu'},\ y=e_i^{\lambda^*,\mu^*}\\ b_i^{l,m} & \text{if } x=e_i^{\lambda',\mu'},\ y=b_i^{\lambda^*,\mu^*}\\ a_i^{l,m} & \text{if } x=a_i^{\lambda',\mu'},\ y=e_i^{\lambda^*,\mu^*}\\ c_i & \text{if } x=a_i^{\lambda',\mu'},\ y=b_i^{\lambda^*,\mu^*},\ l=1\\ c_{i-1} & \text{if } x=a_i^{\lambda',\mu'},\ y=b_i^{\lambda^*,\mu^*},\ l=0,\ m=1 \end{cases}$$

This completes the proof that $\mathfrak{A}_n$ is associative. ∎

It might be instructive to check the definitions of $\sqsubseteq$ and $\preccurlyeq$ in $\mathfrak{A}_n$.

Lemma 3.8 *Let $2\leq n<\omega$.*

1. *For every $x\in A_n$, we have $0\sqsubseteq x\sqsubseteq x$.*
2. *For domain elements d,e, we have $d\sqsubseteq e\iff d\leq e$.*

3. *Whenever $\lambda \leq \nu$ and $\mu \leq \pi$, we have*

$$a_i^{\lambda,\mu} \sqsubseteq a_i^{\nu,\pi} \quad \text{and} \quad b_i^{\lambda,\mu} \sqsubseteq b_i^{\nu,\pi}$$

and no other pairs of elements (other than those listed here and above) are related by $\sqsubseteq$.

4. *The binary relation $\preccurlyeq$ in $\mathfrak{A}_n$ is*

$$\sqsubseteq \cup \{(c_i, c_{i+1}), (c_{n-1}, c_0) : i < n-1\}$$

Proof: The first part holds since $0 = 0 ; x ; 0$ and $x = \mathsf{d}(x) ; x ; \mathsf{r}(x)$. The second and third parts are easily verified. For the fourth part, we know that $x \sqsubseteq y$ implies $x \preccurlyeq y$ and

$$c_i = a_{i+1}^{01} ; b_{i+1}^{01} = (a_{i+1}^{11} ; e_{i+1}^{01}) ; (e_{i+1}^{01} ; b_{i+1}^{11}) = a_{i+1}^{11} ; e_{i+1}^{01} ; b_{i+1}^{11} \prec a_{i+1}^{11} ; b_{i+1}^{11} = c_{i+1}$$

(all $i < n$, addition modulo n), so $c_0 \prec c_1 \prec \ldots \prec c_{n-1} \prec c_0$.

Conversely, suppose $x \prec y$, i.e., there are non-domain elements $u_0, u_1, \ldots, u_{k-1}$ (some k) and domain elements $e_0, e_1, \ldots, e_k$ such that $y = u_0 ; u_1 ; \ldots ; u_{k-1}$ and $x = e_0 ; u_0 ; e_1 ; \ldots ; u_{k-1} ; e_k$ and $x \neq y$. If $k \leq 1$, then $x \sqsubset y$. If $k \geq 2$, then y is the product of at least two non-domain elements. By looseness, either $y = 0$ (so $x = y = 0$, but of course $0 \sqsubseteq 0$) or y is not a domain element. In $\mathfrak{A}_n$, this can only happen if $k = 2$ and the product is of the form $a_i^{\lambda,\mu} ; b_i^{\nu,\pi}$ (some $i < n$, some $\lambda, \mu, \nu, \pi < 2$). Therefore if $x \prec y$ and $x \not\sqsubseteq y$, then $x = a_i^{\lambda,\mu} ; e_i^{\rho,\sigma} ; b_i^{\nu,\pi}$ and $y = a_i^{\lambda,\mu} ; b_i^{\nu,\pi}$. Given that $x \neq 0$ and $x \neq y$, we must have $x = c_{i-1}$ and $y = c_i$, as required. ∎

Corollary 3.9 *Let $2 \leq n < \omega$. $\mathfrak{A}_n$ has an n-cycle but no k-cycle for $k < n$.*

Hence, by Lemma 3.7 and Lemma 3.4 and Corollary 3.9, we have the following.

Corollary 3.10 *Let τ be a signature containing $\{\mathsf{d}, ;\}$. For each $2 \leq n < \omega$, the τ-reduct of $\mathfrak{A}_n$ is not representable.*

Remark 3.11 A special case of the relation $\preccurlyeq$ defined in (8) is the relation $\preccurlyeq_2$ defined by

$$a \preccurlyeq_2 b \text{ iff } a = x ; \epsilon ; y \text{ and } b = x ; y$$

for some elements x, y and domain element ϵ. The non-representability of $\mathfrak{A}_n$ is witnessed by the following quasiequation q_n that is valid in representable algebras but fails in $\mathfrak{A}_n$:

$$(\bigwedge_{i<n} z_i \preccurlyeq_2 z_{i+1}) \rightarrow z_0 = z_1 \tag{17}$$

where the addition is modulo n. Expanding this, q_n may be expressed as

$$\bigwedge_{i<n} (\mathsf{d}(\epsilon_i) = \epsilon_i \wedge z_i = x_{i+1} ; \epsilon_{i+1} ; y_{i+1} \wedge z_{i+1} = x_{i+1} ; y_{i+1}) \rightarrow z_0 = z_1 \tag{18}$$

To see that q_n fails in $\mathfrak{A}_n$ consider the valuation that maps $\epsilon_i, x_i, y_i, z_i$ to $e_i^{01}, a_i^{11}, b_i^{11}, c_i$, respectively (the antecedent of q_n is true, but the consequent is false, since $c_0 \neq c_1$). On the other hand, since representable algebras cannot contain a cycle, q_n must be valid in representable algebras.

4 Representability

In this section, we show that any non-principal ultraproduct $\mathfrak{A}$ of $\mathfrak{A}_n$ over ω is representable.

Game for representability of domain algebras

First we work out the details for domain algebras without the antidomain operation, and show the necessary modifications for including antidomain later. The game we define gives a set of conditions that are sufficient (though not necessary) for representability of a domain algebra.

Let $\mathfrak{A} = (A, \mathsf{d}, \mathsf{r}, ;, 1')$ be a domain algebra with a set D of domain elements. Recall that the domain elements of $\mathfrak{A}$ together with composition form a lower semilattice. Let $\delta^{\uparrow}$ denote $\{d \in D : \delta \leq d\}$, for any $\delta \in D$. A *network* $N = (N^1, N^2)$ over $\mathfrak{A}$ consists of a finite set N^1 (of nodes) and a map $N^2 : (N^1 \times N^1) \to \wp(A)$ satisfying the following *coherence* conditions:

C1 there is $\delta_i \in D$ such that $N^2(i,i) = \delta_i^{\uparrow}$, and if $i \neq j$, then $N^2(i,j) \cap D = \emptyset$

C2 $\mathsf{d}(N^2(i,j)) \subseteq N^2(i,i)$ and $\mathsf{r}(N^2(i,j)) \subseteq N^2(j,j)$

C3 $N^2(i,j) ; N^2(j,k) \subseteq N^2(i,k)$

for all $i, j, k \in N^1$, and

C4 $\{(k,l) : k, l \in N^1,\ N^2(k,l) \neq \emptyset\}$ is reflexive, transitive and antisymmetric.

A network $N = (N^1, N^2)$ over $\mathfrak{A}$ is *saturated* if it satisfies

S1 if $\mathsf{d}(a) \in N^2(i,i)$, then $a \in N^2(i,l)$ for some l

S2 if $\mathsf{r}(a) \in N^2(j,j)$, then $a \in N^2(l,j)$ for some l

S3 if $a ; b \in N^2(i,j)$, then $a \in N^2(i,k)$ and $b \in N^2(k,j)$ for some k

for all $i, j \in N^1$ and $a, b \in A$. Note that in saturated networks the following condition holds:

S4 if $a \in N^2(i,j)$ and $a \preccurlyeq b$, then $b \in N^2(i,j)$.

Indeed, assuming $a \in N^2(i,j)$ and $a \preccurlyeq b$, we have $a = e_0 ; u_0 ; e_1 ; u_1 ; \ldots ; u_{n-1} ; e_n$ and $b = u_0 ; u_1 ; \ldots ; u_{n-1}$ for some natural number n, domain elements $e_0, e_1, \ldots, e_n$ and elements $u_0, \ldots, u_{n-1}$. Then by S3 and C1, we have $i = k_0, k_1, \ldots, k_{n-1}, k_n = j$ such that $u_i \in N^2(k_i, k_{i+1})$ and $e_i \in N^2(k_i, k_i)$, hence by C3, $b = u_0 ; u_1 ; \ldots ; u_{n-1} \in N^2(i,j)$.

Let $N = (N^1, N^2)$ and $M = (M^1, M^2)$ be networks. We write $N \subseteq M$ if $N^1 \subseteq M^1$ and for all $i, j \in N^1$, we have $N^2(i,j) \subseteq M^2(i,j)$. We sometimes drop the superscripts and write N for the network, the set of nodes and the map, distinguishing cases by context, though we may write $\mathsf{nodes}(N)$ to denote the set N^1 of nodes of N.

Let $t \leq \omega$. The two player game $G_t(\mathfrak{A})$ has t rounds numbered $0, 1, \ldots, i, \ldots$, for $i < t$. In the initial round, player $\forall$ (male) picks a set $\{\alpha, \beta\}$ of distinct elements of $\mathfrak{A}$. Player $\exists$ (female) responds with either α or β. In the former case, she has to prove that there is a pair of nodes witnessing α but not β, in the latter case the other way round — this suffices to prove that the representation she constructs is faithful. See Figure 2(a). Without loss of generality, suppose she picks α. In the initial round, $\exists$ has to define a network N_0 with nodes 0 and 1 such that $\alpha \in N_0(0,1)$ but $\beta \notin N_0(0,1)$. Of course, she identifies 0 with 1 precisely when α is a domain element. Suppose $0 < i < t$ and a network N_{i-1} has just been played. $\forall$ can choose from the following types of move.

Domain move He can demand to see a domain witness. He picks $j \in \mathsf{nodes}(N_{i-1})$ and an element $a \in A$ such that $\mathsf{d}(a) \in N_{i-1}(j,j)$. Such a move is denoted (j, a). Then $\exists$ has to play a network $N_i \supseteq N_{i-1}$, such that $a \in N_i(j,k)$ for some $k \in \mathsf{nodes}(N_i)$. See the top of Figure 2(b) where the newly added edges are indicated by dotted arrows — $\exists$ might have to add more edges to ensure that the new structure is indeed a network, see below.

Range move He can demand to see a range witness. This is completely symmetric to the domain move. See the bottom of Figure 2(b).

Composition move He can demand to see a composition witness. He picks $j, k \in \mathsf{nodes}(N_{i-1})$ and $a, b \in A$ such that $a ; b \in N_{i-1}(j,k)$. Such a move is denoted (j,k,a,b). $\exists$ has to play a network $N_i \supseteq N_{i-1}$ where there is $l \in \mathsf{nodes}(N_i)$ such that $a \in N_i(j,l)$ and $b \in N_i(l,k)$. See Figure 2(c).

If at any stage $\exists$ fails to define the required network or if a network is played such that $\beta \in N(0,1)$, then $\forall$ wins. Otherwise $\exists$ wins.

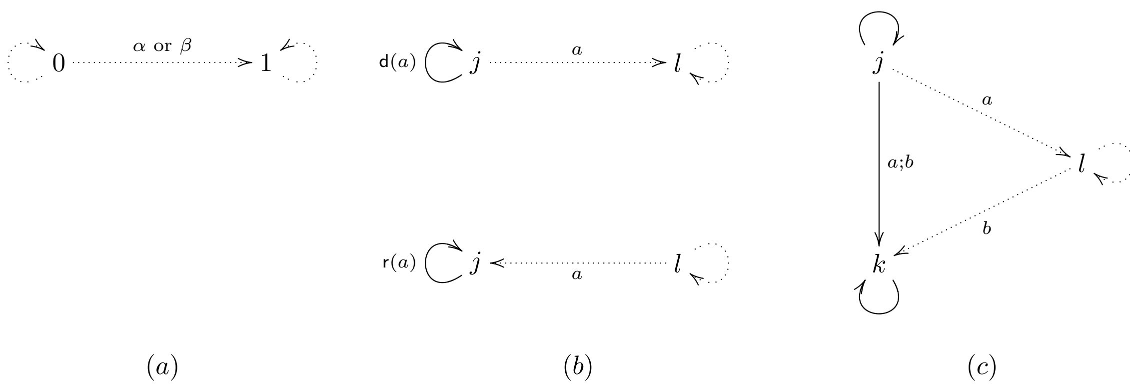

Figure 2: The initial move $\{\alpha, \beta\}$, a domain move (j,a), a range move (j,a) and a composition move (j,k,a,b).

Lemma 4.1 *Let $\mathfrak{A}$ be a loose domain algebra (see Definition 3.2) and t be a natural number. If for every $x \in A$ there is no cycle of length less than or equal to $2t$, then $\exists$ has a winning strategy in $G_t(\mathfrak{A})$.*

Proof: Suppose that no such cycle exists. We describe a winning strategy for $\exists$. Let $\forall$ play $\{\alpha, \beta\}$ in the initial round. If there is no chain from α to β of length less than or equal to t, then $\exists$ plays α. Otherwise there can be no chain from β to α of length less than or equal to t, and $\exists$ plays β in this case. Assume the former case holds, $\exists$ plays α. Then she defines N_0 by $N_0(0,0) = \mathsf{d}(\alpha)^{\uparrow}$, $N_0(1,1) = \mathsf{r}(\alpha)^{\uparrow}$, $N_0(0,1) = \{\alpha\}$, and $N_0(1,0) = \emptyset$ if α is not a domain element, or by identifying 0 with 1 and $N_0(0,0) = \alpha^{\uparrow}$ if $\alpha = \mathsf{d}(\alpha)$. In either case, N_0 is a network and we have $\alpha \in N_0(0,1)$ and $\beta \notin N_0(0,1)$.

We will prove, by induction over the round i, that

1. N_i is a network,
2. if $\gamma \in N_i(0,1)$, then there is a m-chain from α to γ for some $m \leq i$.

Suppose $0 < i < t$ and a network N_{i-1} has just been played.

Domain move Assume $\forall$ plays (j,a), where $\mathsf{d}(a) \in N_{i-1}(j,j)$. Recall by C1 that there is a domain element δ_j such that $N_{i-1}(j,j) = \delta_j^{\uparrow}$, so $\delta_j \leq \mathsf{d}(a)$. If there is a witness $l \in N_{i-1}$ such that $a \in N_{i-1}(j,l)$, then she lets $N_i = N_{i-1}$, so assume that there is no such witness in N_{i-1}. Then $\exists$ plays the network N_i defined as follows. She lets $\mathsf{nodes}(N_i) = \mathsf{nodes}(N_{i-1}) \cup \{l\}$, for some new node $l \notin \mathsf{nodes}(N_{i-1})$. For edges not incident with the new node l, the labelling in N_i is the same as in N_{i-1} (hence the minimal domain label δ_p is the same in N_i as in N_{i-1}, for $p \in N_{i-1}$). $\exists$ lets $\delta_l = \mathsf{r}(\delta_j ; a)$ and

$$N_i(l,l) = \delta_l^{\uparrow}$$
$$N_i(p,l) = N_{i-1}(p,j) ; \{a\} ; \delta_l^{\uparrow}$$
$$N_i(l,p) = \emptyset$$

for $p \in \mathsf{nodes}(N_{i-1})$. See Figure 3, where typical elements of labels are shown. Note that

$$a = \mathsf{d}(a) ; a ; \mathsf{r}(a) \in \delta_j^{\uparrow} ; \{a\} ; \delta_l^{\uparrow} = N_{i-1}(j,j) ; \{a\} ; \delta_l^{\uparrow} = N_i(j,l)$$

since $\delta_j \leq \mathsf{d}(a)$ (by assumption) and $\delta_l \leq \mathsf{r}(a)$ (by definition of δ_l).

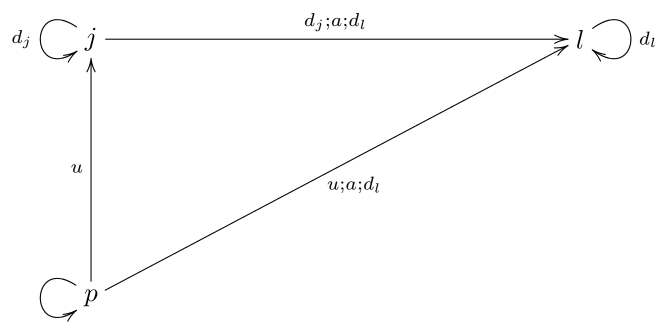

Figure 3: Domain move: $d_j \geq \delta_j, d_l \geq \delta_l$, $u \in N_{i-1}(p,j)$.

Range move Range moves are handled symmetrically.

Composition move Assume $\forall$ plays (j,k,a,b) where $a ; b \in N_{i-1}(j,k)$. If there is already a witness $l \in N_{i-1}$ such that $a \in N_{i-1}(j,l)$ and $b \in N_{i-1}(l,k)$, then $\exists$ lets $N_i = N_{i-1}$. Note that this includes the case when $j = k$, by looseness of $\mathfrak{A}$ and the coherence of N_{i-1}. So assume that $j \neq k$ and that there is no witness $l \in N_{i-1}$. Then $\exists$ plays the network N_i defined below. We have four cases according to whether a and b are domain elements.

1. $a, b \in D$. Then $a ; b \in D$, and hence $j = k$ by C1 for N_{i-1}, contrary to our assumption.
2. $a \in D$ and $b \notin D$. First note that, since $\mathsf{d}(a ; b) \leq \mathsf{d}(a) = a$ and $\mathsf{d}(a ; b) \in N_{i-1}(j,j)$ by C2, we have $a \in N_{i-1}(j,j)$ by C1. In this case, $\mathsf{nodes}(N_i) = \mathsf{nodes}(N_{i-1})$ and $\exists$ re-defines the labels as follows.

$$N_i(p,q) = N_{i-1}(p,q) \cup N_{i-1}(p,j) ; \{b\} ; N_{i-1}(k,q)$$

for all $p, q \in \mathsf{nodes}(N_{i-1})$. See Figure 4, where typical (mostly new) elements of labels are shown. Note that $N_i(p,q) = N_{i-1}(p,q)$ if $N_{i-1}(p,j) = \emptyset$ or $N_{i-1}(k,q) = \emptyset$. Considering the case $p = q$, we have $N_{i-1}(p,j) = \emptyset$ or $N_{i-1}(k,q) = \emptyset$, since $a;b \in N_{i-1}(j,k) \neq \emptyset$ and N_{i-1} satisfies C4. Hence we have $N_i(p,p) = N_{i-1}(p,p)$. Note that

$$b \in \delta_j^{\uparrow} ; \{b\} ; \delta_k^{\uparrow} = N_{i-1}(j,j) ; \{b\} ; N_{i-1}(k,k) \subseteq N_i(j,k)$$

since $\delta_j \leq \mathsf{d}(a ; b) \leq \mathsf{d}(b)$ and $\delta_k \leq \mathsf{r}(a ; b) \leq \mathsf{r}(b)$ by Proposition 3.1.
3. $a \notin D$ and $b \in D$. This case is completely symmetric to the previous one.
4. $a, b \notin D$. In this case, $\mathsf{nodes}(N_i) = \mathsf{nodes}(N_{i-1}) \cup \{l\}$ (some $l \notin \mathsf{nodes}(N_{i-1})$), with minimal domain element $\delta_l = \mathsf{r}(\delta_j ; a) ; \mathsf{d}(b ; \delta_k)$ and the labelling is defined by

$$\begin{aligned} N_i(l,l) &= \delta_l^{\uparrow} \\ N_i(p,l) &= N_{i-1}(p,j) ; \{a\} ; \delta_l^{\uparrow} \\ N_i(l,q) &= \delta_l^{\uparrow} ; \{b\} ; N_{i-1}(k,q) \\ N_i(p,q) &= N_{i-1}(p,q) \cup N_{i-1}(p,j) ; \{a\} ; \delta_l^{\uparrow} ; \{b\} ; N_{i-1}(k,q) \end{aligned}$$

for all $p, q \in \mathsf{nodes}(N_{i-1})$. See Figure 5, where typical (mostly new) elements of labels are shown. Observe that the new part $N_{i-1}(p,j) ; \{a\} ; \delta_l^{\uparrow} ; \{b\} ; N_{i-1}(k,q)$ of $N_i(p,q)$ will be empty if $N_{i-1}(p,j) = \emptyset$ or $N_{i-1}(k,q) = \emptyset$. So

$$N_i(p,q) = N_{i-1}(p,q) \text{ if either } N_{i-1}(p,j) = \emptyset \text{ or } N_{i-1}(k,q) = \emptyset \tag{19}$$

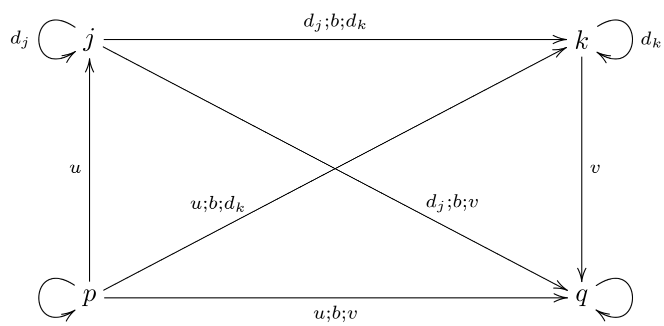

Figure 4: Composition move — case 2: $d_j \geq \delta_j, d_k \geq \delta_k$, $u \in N_{i-1}(p,j), v \in N_{i-1}(k,q)$.

If $N_{i-1}(p,j) \neq \emptyset \neq N_{i-1}(k,q)$, then $p \neq q$ by C4 for N_{i-1}, whence $N_i(p,p) = N_{i-1}(p,p)$. Note that

$$a \in N_i(j,l) = N_{i-1}(j,j) ; \{a\} ; N_i(l,l)$$

since $N_{i-1}(j,j) \ni \delta_j \leq \mathsf{d}(a ; b) \leq \mathsf{d}(a)$, $N_i(l,l) \ni \delta_l \leq \mathsf{r}(\delta_j ; a) \leq \mathsf{r}(a)$. Similarly, $b \in N_i(l,k)$. Also note that $N_{i-1}(j,k) \ni \delta_j ; a ; \delta_l ; b ; \delta_k$, since

$$\delta_j ; a ; b ; \delta_k = \delta_j ; a ; \mathsf{r}(\delta_j ; a) ; \mathsf{d}(b ; \delta_k) ; b ; \delta_k = \delta_j ; a ; \delta_l ; b ; \delta_k \tag{20}$$

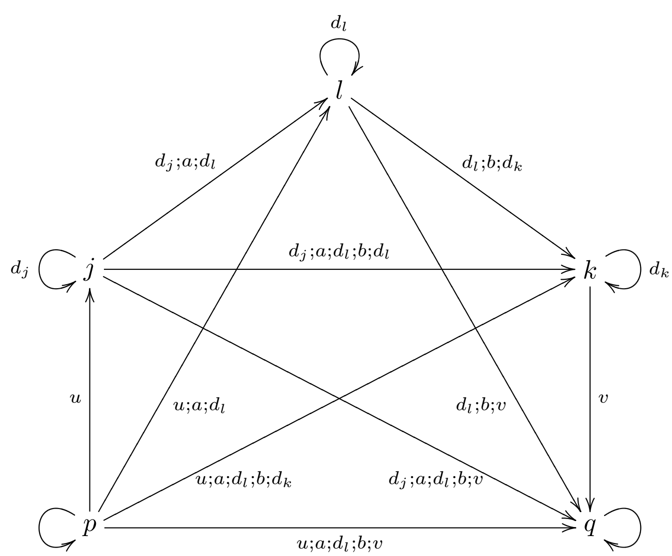

Figure 5: Composition move — case 4: $d_j \geq \delta_j, d_k \geq \delta_k, d_l \geq \delta_l$, $u \in N_{i-1}(p,j), v \in N_{i-1}(k,q)$.

Next we show that the structure N_i defined above satisfies the induction hypothesis.

First we check that $\gamma \in N_i(0,1)$ implies the existence of a chain of length at most i from α to γ. We show that in each round i of the game, if a new element z is included in the label $N_i(0,1)$, then $x \preccurlyeq z$ for some $x \in N_{i-1}(0,1)$. In fact, we show for every $p,q \in \mathsf{nodes}(N_{i-1})$,

$$N_i(p,q) \subseteq \{z : x \preccurlyeq z \text{ for some } x \in N_{i-1}(p,q)\} \tag{21}$$

Since $x \preccurlyeq x$, we know that $N_{i-1}(p,q) \subseteq \{z : x \preccurlyeq z \text{ for some } x \in N_{i-1}(p,q)\}$, so have to show that $N_i(p,q) \setminus N_{i-1}(p,q) \subseteq \{z : x \preccurlyeq z \text{ for some } x \in N_{i-1}(p,q)\}$. Observe that the only moves of $\exists$ that add additional elements to the label of (p,q) are in response to composition moves. So assume that, in round $i > 0$, $\forall$ plays (j, k, a, b) where $a ; b \in N_{i-1}(j, k)$. First suppose that this is covered by case 2, i.e., $a \in D$. Note that $a ; b \preccurlyeq b$ in this case and thus $u ; a ; b ; v \preccurlyeq u ; b ; v$ for any u, v, and in particular, for every $u \in N_{i-1}(p, j)$ and $v \in N_{i-1}(k, q)$. Since N_{i-1} satisfies C3, we have $u ; a ; b ; v \in N_{i-1}(p, q)$. Hence any new element z included in $N_i(p, q) \smallsetminus N_{i-1}(p, q)$ satisfies $x \preccurlyeq z$ for some $x \in N_{i-1}(p, q)$. Since case 3 is completely symmetric, next we assume that $a, b \notin D$, as in case 4 for composition moves. Consider a new element z in the label of (p, q). From the definition of N_i in case 4 we have $z = u ; a ; d_l ; b ; v$ for some $u \in N_{i-1}(p, j)$, $d_l \geq \delta_l$ and $v \in N_{i-1}(k, q)$. Then $z = u ; a ; d_l ; b ; v \succcurlyeq u ; \delta_j ; a ; \delta_l ; b ; \delta_k ; v \in N_{i-1}(p, q)$, by (20) and C3 for N_{i-1}. This proves (21) and hence the second induction hypothesis. Since there is no t-chain from α to β it follows that $\beta \notin N_t(0, 1)$.

It remains to show that N_i is a network. C1 holds by looseness of $\mathfrak{A}$, C4 can be checked by inspection. We check C2. First consider an old edge (p, q), where $p, q \in N_{i-1}$, and $z \in N_i(p, q)$. By (21), $x \preccurlyeq z$ for some $x \in N_{i-1}(p, q)$. Hence, by Proposition 3.5(3), $\mathsf{d}(x) \leq \mathsf{d}(z)$ and $\mathsf{r}(x) \leq \mathsf{r}(z)$. By C2 for N_{i-1} we have $\mathsf{d}(x) \in N_{i-1}(p, p)$ and $\mathsf{r}(x) \in N_{i-1}(q, q)$. Since $N_{i-1} \subseteq N_i$, it follows that $\mathsf{d}(z) \in N_i(p, p)$ and $\mathsf{r}(z) \in N_i(q, q)$.

It remains to check C2 for new edges of N_i — edges incident with the new node l played in response to a domain move, a range move, or case 4 of a composition move. First suppose N_i was played in response to a domain move (j, a). Let l be the new node, $p \in N_{i-1}$, and consider $u ; a ; d_l \in N_i(p, l)$ where $u \in N_{i-1}(p, j)$ and $d_l \geq \delta_l$. Recall that $\delta_l = \mathsf{r}(\delta_j ; a)$, so $\delta_j ; a ; \delta_l = \delta_j ; a$. Then

$$\mathsf{d}(u ; a ; d_l) \geq \mathsf{d}(u ; \delta_j ; a ; \delta_l) = \mathsf{d}(u ; \mathsf{d}(\delta_j ; a ; \delta_l)) = \mathsf{d}(u ; \mathsf{d}(\delta_j ; a)) = \mathsf{d}(u ; \delta_j) \in N_{i-1}(p, p)$$

using the last part of Proposition 3.1, $\mathsf{d}(a) \geq \delta_j$ and C2 and C4 for N_{i-1}. Also,

$$\mathsf{r}(u ; a ; d_l) \geq \mathsf{r}(u ; \delta_j ; a ; \delta_l) = \mathsf{r}(u ; \delta_j ; a) = \mathsf{r}(\mathsf{r}(u ; \delta_j) ; a) = \mathsf{r}(\delta_j ; a) = \delta_l \in N_i(l, l)$$

since $\mathsf{r}(u) \geq \delta_j$. So C2 holds on the edge (p, l), for $p \in N_{i-1}$, and the edge (l, l) is trivial to check. Range moves are similar.

Now we check C2 for edges incident with the new node l in response to case 4 of composition moves. First consider $u ; a ; d_l \in N_i(p, l)$ for some $u \in N_i(p, j)$ and $d_l \in N_i(l, l)$. We need $\mathsf{d}(u ; a ; d_l) \in N_i(p, p)$ and $\mathsf{r}(u ; a ; d_l) \in N_i(l, l)$. Well,

$$\mathsf{d}(u ; a ; d_l) \geq \mathsf{d}(u ; \delta_j ; a ; \delta_l) = \mathsf{d}(u ; \delta_j ; a ; (\mathsf{r}(\delta_j ; a) ; \mathsf{d}(b ; \delta_k))) = \mathsf{d}(u ; \delta_j ; a ; b ; \delta_k) \in N_{i-1}(p, p)$$

by Proposition 3.1, C2 and C4 for N_{i-1}. Furthermore,

$$\mathsf{r}(u ; a ; d_l) \geq \mathsf{r}(u ; \delta_j ; a ; \delta_l) \geq \delta_l \in N_i(l, l)$$

since $\mathsf{r}(u ; \delta_j ; a) = \mathsf{r}(\mathsf{r}(u ; \delta_j) ; a) = \mathsf{r}(\delta_j ; a) \geq \delta_l$. This shows that C2 holds for edges (p, l) in case 4 composition moves. Similarly C2 holds for edges (l, q). This establishes C2 for N_i.

Next we check that N_i satisfies C3 as well. We must show that $u ; v \in N_i(p, r)$, whenever $u \in N_i(p, q)$ and $v \in N_i(q, r)$, for any $p, q, r \in N_i$. We consider a domain move (j, a) first. Let l be the new node included in N_i. Labels of edges not involving l are the same in N_i as they are in N_{i-1}, so we may assume that $l \in \{p, q, r\}$. Also $N_i(l, k) = \emptyset$, for $k \in N_{i-1}$, so we may assume $r = l$. If $q = l$, then

$$N_i(p, l) ; N_i(l, l) = (N_{i-1}(p, j) ; \{a\} ; \delta_l^{\uparrow}) ; \delta_l^{\uparrow} \subseteq N_i(p, l)$$

If $r = l$ but $q \neq l$, then we can assume $p \neq l$ (as $N_i(l, q) = \emptyset$). Then

$$N_i(p, q) ; N_i(q, l) = N_{i-1}(p, q) ; (N_{i-1}(q, j) ; \{a\} ; \delta^{\uparrow}) \subseteq N_{i-1}(p, j) ; \{a\} ; \delta^{\uparrow} = N_i(p, l)$$

by C3 for N_{i-1}. So C3 holds, when N_i is played in response to a domain or (similarly) a range move.

Next we consider a composition move (j, k, a, b). Again, let the new node be l. Consider three nodes $p, q, r \in N_i$. We have to show, for all $u \in N_i(p, q)$ and $v \in N_i(q, r)$, that $u ; v \in N_i(p, r)$.

Suppose first that $p, q, r \in N_{i-1}$. If $u \in N_{i-1}(p,q)$ and $v \in N_{i-1}(q,r)$, then $u ; v \in N_{i-1}(p,r) \subseteq N_i(p,r)$, inductively. Next suppose that $u \in N_i(p,q) \smallsetminus N_{i-1}(p,q)$. Whether the composition move is case 2, 3 or 4, we have $N_{i-1}(p,j) \neq \emptyset \neq N_{i-1}(k,q)$ in this case, by (19). Recall that $a ; b \in N_{i-1}(j,k) \neq \emptyset$. Since N_{i-1} satisfies C4, we must have $N_{i-1}(q,j) = \emptyset$ and therefore $N_i(q,r) = N_{i-1}(q,r)$, again by (19). Thus $v \in N_{i-1}(q,r)$. If a is a domain element but b is not (case 2), then, since $u \in N_i(p,q) \smallsetminus N_{i-1}(p,q)$, we must have $u = w ; b ; y$, for some $w \in N_{i-1}(p,j)$ and $y \in N_{i-1}(k,q)$. Hence $y;v \in N_{i-1}(k,r)$ by C3 for N_{i-1}. Then $u;v = (w;b;y);v = w;b;(y;v) \in N_i(p,r)$. Case 3 of composition moves is similar. For case 4, neither a nor b is a domain element and we have $u = w ; a ; d_l ; b ; y$, for some $w \in N_{i-1}(p,j)$, $d_l \geq \delta_l$ and $y \in N_{i-1}(k,q)$. By C3 for N_{i-1}, $y ; v \in N_{i-1}(k,r)$. By definition of N_i, $u ; v = w ; a ; d_l ; b ; (y ; v) \in N_i(p,r)$, as required. The case where $v \in N_i(q,r) \smallsetminus N_{i-1}(q,r)$ (and thus $N_i(p,q) = N_{i-1}(p,q)$) is similar.

Finally we must consider cases where $l \in \{p,q,r\}$, for case 4 of composition moves. We consider $p = l$ first. We have to show that $N_i(l,q) ; N_i(q,r) \subseteq N_i(l,r)$. If $N_i(l,q) = \emptyset$, then the inclusion is trivial, so suppose not. Then $N_{i-1}(k,q) \neq \emptyset$. Since N_{i-1} satisfies C4 and $a ; b \in N_{i-1}(j,k) \neq \emptyset$, we must have $N_{i-1}(q,j) = \emptyset$, whence $N_i(q,r) = N_{i-1}(q,r)$. So

$$N_i(l,q) ; N_i(q,r) = (\delta_l^{\uparrow} ; \{b\} ; N_{i-1}(k,q)) ; N_{i-1}(q,r) \subseteq \delta_l^{\uparrow} ; \{b\} ; N_{i-1}(k,r) = N_i(l,r)$$

The case where $l = r$ is similar. Finally, if $l = q$, then $N_i(p,l) ; N_i(l,r) \subseteq N_i(p,r)$, by the final line in the definition of N_i in case 4 of composition moves. This completes the proof that C3 holds for N_i. Thus N_i is indeed a network, hence $\exists$ can win $G_t(\mathfrak{A})$, finishing the proof of Lemma 4.1. ∎

Game with antidomain

In this section, we describe the necessary modifications of networks and games for dealing with antidomain as well.

Let $\mathfrak{A} = (A, \mathsf{d}, \mathsf{r}, \mathsf{a}, ;, 1')$ be a domain algebra with a set D of domain elements forming the boolean algebra $\mathfrak{D}$. An additional requirement in the definition of a network is that

$$N(i,i) \text{ is an ultrafilter of } \mathfrak{D} \tag{22}$$

Note that this extra condition for antidomain networks is necessary and sufficient for the antidomain operation to be properly represented in a saturated antidomain network. Indeed, if N is a saturated antidomain network and $i \in \mathsf{nodes}(N)$, we have

$$\mathsf{a}(x) \in N(i,i) \iff \mathsf{d}(x) \notin N(i,i) \iff \neg(\exists j \in \mathsf{nodes}(N)) x \in N(i,j)$$

by (22), C2 and S1. It is included in the definition of a domain network that $N(i,i) = \delta^{\uparrow}$, for some $\delta \in D$. But $\delta^{\uparrow}$ is an ultrafilter iff δ is an atom of the boolean algebra $\mathfrak{D}$. Thus, for antidomain algebras, condition (22) is equivalent to

$$N(i,i) = e^{\uparrow} \text{ for some atom } e \text{ of } \mathfrak{D} \tag{23}$$

For the remainder of this section, all networks are antidomain networks (i.e., they satisfy (22)/(23)).

The game $G_t^a(\mathfrak{A})$ is almost identical to the previously defined game $G_t(\mathfrak{A})$, the only difference is that the networks played have to be antidomain networks, i.e., there has to be an atom in the label of a reflexive edge. In all other respects, the definition of the game $G_t^a(\mathfrak{A})$ is the same as the definition of $G_t(\mathfrak{A})$. Instead of Lemma 4.1 we have the following.

Lemma 4.2 *Let t be a natural number and let $\mathfrak{A}$ be a loose antidomain algebra with no cycles of length $2t$ or less. Suppose for all non-domain elements $x, y \in A$, there is an atom e of $\mathfrak{D}$ such that $x ; y = x ; e ; y$. Then player $\exists$ has a winning strategy in $G_t^a(\mathfrak{A})$.*

Proof: The winning strategy for $\exists$ is very similar to the one we gave before. This time, in response to a domain, range or composition move in round i, if $\exists$ has to include a new node l in N_i, then she has to let δ_l be an atom of $\mathfrak{D}$. In response to a domain move (j, a), she lets δ_l be any atom below $\mathsf{r}(\delta_j \,;\, a)$, range moves are similar. In response to a composition move (j, k, a, b), where neither a nor b are domain elements, she lets δ_l be any atom below $\mathsf{r}(\delta_j \,;\, a) \,;\, \mathsf{d}(b \,;\, \delta_k)$ such that $\delta_j \,;\, a \,;\, b \,;\, \delta_k = \delta_j \,;\, a \,;\, \delta_l \,;\, b \,;\, \delta_k$. Such an atom δ_l exists, by the assumption in the lemma. In other respects, the definition of N_i is the same as before. The proofs that N_i is a network and that every element of $N_i(0,1)$ can be reached from an element of $N_0(0,1)$ by a chain of length at most i are the same as before. ∎

Corollary 4.3 *Let t be a natural number. $\exists$ has a winning strategy in the antidomain network game $G^a_t(\mathfrak{A}_{2t})$.*

Proof: By Lemma 3.7 and Corollary 3.9, $\mathfrak{A}_{2t}$ is a loose algebra with no cycles of length less than $2t$, and for all non-domain elements $x, y \in A_{2t}$, there is a domain atom e such that $x \,;\, e \,;\, y = x \,;\, y$. By Lemma 4.2, $\exists$ has a winning strategy in $G^a_t(\mathfrak{A}_{2t})$. ∎

We are ready to formulate the result connecting representability and $\exists$'s winning strategy.

Lemma 4.4 *Let $\mathfrak{A}$ be a finite or countable domain algebra. If $\exists$ has a winning strategy in $G_\omega(\mathfrak{A})$, then $\mathfrak{A}$ is representable. If $\mathfrak{A}$ is a finite or countable antidomain algebra and $\exists$ has a winning strategy in $G^a_\omega(\mathfrak{A})$, then $\mathfrak{A}$ is representable.*

Proof: We prove the lemma for domain algebras but the same proof works *mutatis mutandis* for antidomain algebras. Let us assume that $\forall$ picked $\{\alpha, \beta\}$ in the initial round of the game and $\exists$ responded with α. Since $\exists$ has a winning strategy, there is a saturated network N such that $\alpha \in N(0,1)$ but $\beta \notin N(0,1)$. It is easy to see that saturated networks define representable algebras. Hence there is a representable, homomorphic image $\mathfrak{M}$ of $\mathfrak{A}$ such that $\alpha^{\mathfrak{M}} \neq \beta^{\mathfrak{M}}$. The homomorphism h is given by

$$h(a) = \{(j,k) \in N \times N : a \in N(j,k)\}$$

for every $a \in A$.

Assume that the players repeat the game for every pair $\{\alpha, \beta\}$ of distinct elements of A and $\exists$ applies her winning strategy in each of these games. It follows that $\mathfrak{A}$ can be isomorphically embedded into the product of representable algebras, which is again a representable algebra. ∎

Representing the ultraproduct

Lemma 4.5 *Let U be a non-principal ultrafilter over ω and for each $n \in \omega$ let $\mathfrak{B}_n$ be an antidomain algebra such that $\exists$ has a winning strategy in $G^a_n(\mathfrak{B}_n)$. Then $\exists$ has a winning strategy in $G^a_\omega(\prod_U \mathfrak{B}_n)$, where $\prod_U \mathfrak{B}_n$ is the non-principal ultraproduct of the $\mathfrak{B}_n$s based on U.*

Proof: The ultraproduct $\prod_U \mathfrak{B}_n$ is defined as follows. Let $\sim$ be the equivalence relation over the cartesian product $\prod_{n\in\omega} \mathfrak{B}_n$ defined by $(b_0, b_1, \ldots) \sim (b'_0, b'_1, \ldots) \iff \{n : b_n = b'_n\} \in U$. The elements of the ultraproduct $\prod_U \mathfrak{B}_n$ are the equivalence classes $\prod_{n\in\omega} \mathfrak{B}_n / \sim$. We write $[(a_0, a_1, \ldots)]$ for the equivalence class of $(a_0, a_1, \ldots)$. Domain, range, antidomain and composition operations may be defined as follows: $\mathsf{d}([a_0, a_1, \ldots)] = [(\mathsf{d}(a_0), \mathsf{d}(a_1), \ldots)]$, range and antidomain are similar and $[(a_0, a_1, \ldots)] ; [(b_0, b_1, \ldots)] = [(a_0 ; b_0, a_1 ; b_1, \ldots)]$. It can easily be checked that these operators are well defined (not dependent on choice of representatives of equivalence classes). The ultraproduct $\prod_U \mathfrak{B}_n$ is defined as $\prod_{n\in\omega} \mathfrak{B}_n / \sim$ with these operations.

Consider a play of the game $G^a_\omega(\mathfrak{B}_n)$. In the initial round, $\forall$ picks elements $[(a_0, a_1, \ldots)] \neq [(b_0, b_1, \ldots)]$. Let $S_0 = \{n \in \omega : a_n \neq b_n\} \in U$. For each $n \in S_0$, $\exists$ starts a game of $G^a_n(\mathfrak{B}_n)$ and supposes that $\forall$ plays $\{a_n, b_n\}$ in the initial round. In each of these games, she responds using her winning strategy for the initial round in $G^a_n(\mathfrak{B}_n)$ by choosing either a_n or b_n. Let $S_a = \{n \in S_0 : \exists \text{ chooses } a_n\}$, $S_b = \{n \in S : \exists \text{ chooses } b_n\}$. Since $S_0 = S_a \cup S_b$ either $S_a \in U$ or $S_b \in U$, by ultrafilter properties, so there is $S_1 \in \{S_a, S_b\}$ with $S_1 \in U$. Without loss we will assume $S_1 = S_a$ in the following. For each $n \in S_1$, $\exists$ plays the initial network N^n_0 which has either one or two nodes. Without loss we may suppose the nodes of N^n_0 are contained in $\{0,1\}$ and there is a subset $S_2 \subseteq S_1$ with $S_2 \in U$ and a set of nodes $X \subseteq \{0,1\}$ such that $n \in S_2 \Rightarrow \mathsf{nodes}(N^n_0) = X$. For all $n \in \omega \setminus S_2$ let N^n_0 be an arbitrary $\mathfrak{B}_n$-network with nodes X, so that all networks N^n_0 $(n \in \omega)$ have nodes X. We may now define the *ultraproduct* N_0 of the N^n_0s as the network with nodes X and labelling defined by $N_0(x,y) = [(N^n_0(x,y) : n \in \omega)]$, for $x, y \in X$. In the initial round of the ultraproduct game, $\exists$ plays N_0. It can be checked that N_0 is a $\prod_U \mathfrak{B}_n$-network, since each N^n_0 is a $\mathfrak{B}_n$-network.

In round $t > 0$ suppose inductively that (i) the $\prod_U \mathfrak{A}_n$-network N_{t-1} has just been played with finite set of nodes, Y say, (ii) for each $n \in \omega$ there is a $\mathfrak{B}_n$-network N^n_{t-1} with nodes Y such that N_{t-1} is the ultraproduct of the N^n_{t-1}s, (iii) there is $W_0 \in U$ such that for all $n \in W_0$, we have $n \geq t$ and N^n_{t-1} is the tth network in a play of $G^a_n(\mathfrak{B}_n)$ in which $\exists$ has been using her winning strategy. We established these properties for the initial round. $\forall$ has three types of moves he could make, here we consider only composition moves (domain and range moves are simpler). Suppose $\forall$ plays the composition move $(j, k, [c], [d])$. The set W_1 of indices $n \in W_0$ such that $n \geq t+1$ and (j, k, c_n, d_n) is a valid composition move in $G^a_n(\mathfrak{A}_n)$ must belong to U. For each $n \in W_1$, $\exists$ responds using her winning strategy in $G^a_t(\mathfrak{B}_n)$ with the network N^n_t. We may assume that she adds at most a single new node to the previous network and the choice of name for this node is the same in all games. Observe that at most one of $\exists$'s strategies will have 'expired' in this round, namely the strategy in $G^a_t(\mathfrak{B}_t)$. There will be a subset W_2 of W_1 with $W_2 \in U$ and each of N^n_t, $(n \in W_2)$ has the same finite set of nodes, say Z. For $n \notin W_2$ we can let N^n_t be an arbitrary $\mathfrak{B}_n$-network with nodes Z. N_t is then defined as the ultraproduct of the N^n_ts. Again, N_t, so defined, will in fact be a network, since each of N^n_t is a $\mathfrak{B}_n$-network. Also, $\exists$ has not lost in the tth round of $G^a_n(\mathfrak{B}_n)$ for $n \in W_2$, since she is using a winning strategy. Hence $b_n \notin N^n_t(0,1)$ for $n \in W_2$. It follows that $[(b_0, b_1, \ldots)] \notin N_t(0,1)$, so $\exists$ does not lose the play in round t of $G^a_\omega(\prod_U \mathfrak{B}_n)$. ∎

Lemma 4.6 *Suppose $\exists$ has a winning strategy in $G^a_\omega(\mathfrak{B})$. Then there is a countable elementary subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ such that $\exists$ still has a winning strategy in $G^a_\omega(\mathfrak{C})$.*

Proof: We may suppose that $\exists$'s winning strategy in $G^a_\omega(\mathfrak{B})$ is deterministic and depends only on the current network and $\forall$'s move in any situation, i.e., it does not depend on the previous history of the game. Use the downward Löwenhein–Skolem theorem to find a countable elementary subalgebra $\mathfrak{C}_0 \prec \mathfrak{B}$. We will define an elementary chain $\mathfrak{C}_0 \prec \mathfrak{C}_1 \prec \ldots \prec \mathfrak{B}$ as follows. Suppose a countable algebra $\mathfrak{C}_n$ has already been defined. Consider a play of $G_\omega(\mathfrak{B})$ in which the elements chosen by $\forall$ for any of his moves are restricted to $\mathfrak{C}_n$. Let S_{n+1} be the set of all elements used by $\exists$ using her winning strategy in plays of $G_\omega(\mathfrak{B})$ in which $\forall$'s moves are restricted to elements in $\mathfrak{C}_n$. Then S_{n+1} is a countable set and by the downward Löwenheim–Skolem theorem again, there is a countable elementary algebra $\mathfrak{C}_{n+1}$ containing $\mathfrak{C}_n \cup S_{n+1}$. This defines $\mathfrak{C}_{n+1}$.

Now let $\mathfrak{C} = \bigcup_{n \in \omega} \mathfrak{C}_n$. By the elementary chain theorem (see [CK90, 3.1.9]) this is a countable elementary subalgebra of $\mathfrak{B}$, and $\exists$ has a winning strategy in $G^a_\omega(\mathfrak{C})$. ∎

Theorem 4.7 *If $\exists$ has a winning strategy in $G^a_n(\mathfrak{B}_n)$ (all $n \in \omega$) and U is a non-principal ultrafilter, then $\prod_U \mathfrak{B}_n \in \mathsf{R}(\{\mathsf{d}, \mathsf{r}, \mathsf{a}, ;, 1'\})$.*

Proof: Consider the countable algebra $\mathfrak{C}$ of Lemma 4.6. Since $\mathfrak{C}$ is countable and $\exists$ has a winning strategy in $G^a_\omega(\mathfrak{C})$, Lemma 4.4 shows that $\mathfrak{C}$ is representable. Now $\mathsf{R}(\tau)$ is elementary and $\mathfrak{C} \equiv \mathfrak{B}$, hence $\mathfrak{B}$ is also representable. ∎

Now letting $\mathfrak{B}_n = \mathfrak{A}_{2n}$ for $n \in \omega$ (cf. Definition 3.6) we have the following.

Corollary 4.8 *Non-principal ultraproducts $\mathfrak{B} = \prod_U \mathfrak{B}_n$ of the $\mathfrak{B}_n$s are representable.*

This finishes the proof of Theorem 2.3, since for any $\{\mathsf{d}, ;\} \subseteq \tau \subseteq \{\mathsf{d}, \mathsf{r}, \mathsf{a}, ;, 1', 0\}$ we know that $\mathfrak{B}_n \notin \mathsf{R}(\tau)$ by Corollary 3.10 and $\mathfrak{B} \in \mathsf{R}(\tau)$ by Corollary 4.8.

5 Extending the similarity type

A natural question from both the theoretical and application point of view is whether the same non-finite axiomatizability holds for larger similarity types. Obvious choices for the extra operations include join $+$, meet $\cdot$, converse $\smile$, top 1 and bottom 0 constants and the Kleene star * (reflexive–transitive closure).

Recall from Definition 3.6 that for each $x \in A_n$ we have either $x ; x = x$ (for domain element x) or $x ; x = 0$. Hence it easy to extend the signature of these algebras to include the transitive (but not reflexive) closure operation by letting the transitive closure of each element be itself. Thus our non-finite axiomatizability result holds for signatures including the transitive closure operation. But it is not easy to see how to extend the signature of the algebras $\mathfrak{A}_n$ to include the reflexive and transitive closure operation *, since for non-domain elements x the natural definition of x^* would be $1' + x$, but our algebras do not include such elements.

It might be possible to modify the definition of $\mathfrak{A}_n$ to include a definition of $+$ or $\cdot$, but the resulting algebra is likely to have $c_i \leq c_{i+1} \pmod n$ for all $i < n$, and then $c_0 = c_1 = \ldots = c_{n-1}$, whence $\mathfrak{A}_n$ would be representable. Instead, for signatures including $+$, we recall the following from [An88], see [AM10] for a full proof.

Theorem 5.1 *Let $\{+, ;\} \subseteq \tau \subseteq \{0, 1, +, 1', \smile, ;, ^*\}$. Then the class $\mathsf{R}(\tau)$ of representable τ-algebras is not finitely axiomatizable.*

Our contribution here is to make the fairly trivial observation that there is an obvious way to define domain and range operations for the algebras used in [An88]. Since [An88] is not widely available, we recall the key steps of the proof. For every natural number m, Andréka constructs an algebra $\mathfrak{A}_m = (A_m, 0, 1, +, 1', \smile, ;, ^*)$ such that

1. the $\{+, ;\}$-reduct of $\mathfrak{A}_m$ is not representable
2. any non-trivial ultraproduct $\mathfrak{A}$ of $\mathfrak{A}_m$ (for $m \in \omega$) is representable.

Define

$$G = \{a, a'_1, a''_1, \ldots, a'_m, a''_m, b, b'_1, b''_1, \ldots, b'_m, b''_m, o, 1', 0\}$$

Let $(A_m, +)$ be the free upper semilattice generated freely by G under the defining relations:

$$a \leq a'_i + a''_i \quad b \leq b'_i + b''_i \quad 0 + x = x$$

for $1 \leq i \leq m$ and $x \in G$. Let S denote the following set of two-element subsets of A_m:

$$S = \{\{a, b'_1\}\} \cup \{\{a'_i, b''_1\} : 1 \leq i \leq m\} \cup \{\{a''_i, b'_{i+1}\} : 1 \leq i < m\} \cup \{\{a''_m, b\}\}$$

The other operations on A_m are defined as follows.

$$0 = \emptyset \qquad 1 = \sum G \qquad x^{\smile} = x$$
$$0^* = 0 \qquad 1'^* = 1' \qquad x^* = 1 \ \text{ if } x \notin \{0, 1'\}$$
$$0 ; x = 0 = x ; 0 \qquad 1' ; x = x = x ; 1'$$
$$\text{for } x, y \notin \{0, 1'\} \quad x ; y = \begin{cases} o & \text{if } \{x, y\} \in S \\ 1 & \text{otherwise} \end{cases}$$

1. The quasiequation q_m is defined as

$$\bigwedge_{i=1}^{m} (x \leq x'_i + x''_i \wedge y \leq y'_i + y''_i) \rightarrow$$
$$x ; y \leq x ; y'_1 + \sum_{i=1}^{m-1} (x'_i ; y''_i + x''_i ; y'_{i+1}) + x'_m ; y''_m + x''_m ; y$$

By an induction on m one can show that q_m is valid in representable algebras. On the other hand, the evaluation ϵ given by

$$\epsilon(x) = a \quad \epsilon(x'_i) = a'_i \quad \epsilon(x''_i) = a''_i \quad \epsilon(y) = b \quad \epsilon(y'_i) = b'_i \quad \epsilon(y''_i) = b''_i$$

falsifies q_m in $\mathfrak{A}_m$. Since q_m uses only the operations ; and $+$, it follows that already the $\{+, ;\}$-reduct of $\mathfrak{A}_m$ is not representable.

2. By a step-by-step argument one can build a representation of the ultraproduct $\mathfrak{A}$.

Since $\mathfrak{A}_m$ has a monoid reduct and $1'$ is a minimal non-zero element, one can define the (anti)domain and range operations by letting

$$\begin{aligned} \mathsf{d}(0) = \mathsf{r}(0) = 0 \quad &\text{and} \quad \mathsf{d}(x) = \mathsf{r}(x) = 1' \text{ for } x \neq 0 \\ \mathsf{a}(0) = 1' \quad &\text{and} \quad \mathsf{a}(x) = 0 \text{ for } x \neq 0 \end{aligned}$$

Obviously, $\mathfrak{A}_m$ expanded with (anti)domain and/or range remains non-representable, while the representation of the ultraproduct respects both the domain, range and antirange operations. Indeed, if $x \notin \{0, 1'\}$, then $x^* = 1$, whence the representation of x is a relation such that both its domain and range contain all elements of the base, thus it is sound to represent $\mathsf{d}(x)$ and $\mathsf{r}(x)$ as the identity relation and $\mathsf{a}(x)$ as the empty relation. Hence we have the following.

Corollary 5.2 *Let* $\{+, ;\} \subseteq \tau \subseteq \{0, 1, +, \mathsf{d}, \mathsf{r}, \mathsf{a}, 1', ;, {}^*, {}^\smile\}$. *Then the class* $\mathsf{R}(\tau)$ *of representable* τ*-algebras is not finitely axiomatizable.*

Including meet $\cdot$ into the similarity type does not seem promising either. Let τ be a similarity type such that the elements of τ are definable in representable relation algebras (i.e., using the booleans, composition, converse and identity). Andréka [An91] shows non-finite axiomatizability for representable algebras of similarity type $\tau \supseteq \{\cdot, +, ;\}$. In [HM07], we defined non-representable algebras of the similarity type $\{\cdot, 1', ;\}$ whose ultraproduct is representable. Since $1'$ is a minimal non-zero element in these algebras, defining domain, range and antidomain operations should not be a problem. Hence we conjecture that representable algebras of the similarity type $\tau \supseteq \{\cdot, \mathsf{d}, ;\}$ form a non-finitely axiomatizable quasivariety.

6 Conclusion

As we have seen the quasivarieties of representable domain algebras in general are not finitely axiomatizable. It would be interesting to see simple characterizations of representable domain algebras, cf. [Ko06] where additional separation properties provide representability of KATs as relational KATs. Note that representing monoids or domain–range monoids is easier than representing antidomain algebras, since the labels on reflexive arrows do not have to be ultrafilters.

Problem 6.1 Let $\mathfrak{A}$ be a domain monoid or domain–range monoid (but do not assume that $\mathfrak{A}$ is loose). If $\mathfrak{A}$ has no cycles, must it be representable?

If so, we can find a simple, infinite, recursive axiomatisation of the representation class.

As [Ko00] notes, inference rules for partial correctness assertions can be translated to quasiequations of KATs of the following form:

$$\Big(\bigwedge_{1 \le i \le n} b_i \,;\, p_i = b_i \,;\, p_i \,;\, c_i\Big) \to b_0 \,;\, p_0 = b_0 \,;\, p_0 \,;\, c_0 \tag{24}$$

with tests b_i, c_i and programs p_i for $0 \le i \le n$. [Ko00] also shows that KAT is deductively complete for quasiequations of the form (24) over relational models, i.e., a quasiequation of the above form is valid iff it is a theorem of KAT. We can write the above quasiequation as

$$\Big(\bigwedge_{0 \le i \le n} (\mathsf{d}(b_i) = b_i \wedge \mathsf{d}(c_i) = c_i) \wedge \bigwedge_{1 \le i \le n} b_i \,;\, p_i = b_i \,;\, p_i \,;\, c_i\Big) \to b_0 \,;\, p_0 = b_0 \,;\, p_0 \,;\, c_0 \tag{25}$$

in domain algebras. Note that our quasiequations (18) showing the non-finite axiomatizability of representable domain algebras are not in this form. Hence we can ask the following for various classes of domain algebras.

Problem 6.2 Is there a finite set of quasiequations that is deductively complete for quasiequations of the form (25) over representable domain algebras.

Next we mention the problem of finitely axiomatizing the equational theories of representable domain algebras. [Ho97] shows finite axiomatizability of the variety generated by representable antidomain algebras. We conjecture that the same can be achieved for domain and domain–range semigroups/monoids and their expansions with lattice operations join and meet.

A challenging problem is to *finitely quasiaxiomatize* those varieties V that are not finitely based, i.e., find a finitely axiomatizable quasivariety Q such that the variety generated by Q and V coincide. This is the case for the variety generated by representable (or relational) Kleene algebras where several such quasivarieties have been found, see [ÉB95] for short descriptions of these quasivarieties and for references. In particular, we may ask whether these finite quasiaxiomatizations could be used for representable domain algebras with Kleene star.

Problem 6.3 Let $\{+, ;, 0, 1', ^*\} \subset \tau \subseteq \{+, ;, 0, 1', ^*, \mathsf{d}, \mathsf{r}, \mathsf{a}\}$. Is the variety generated by $\mathsf{R}(\tau)$ finitely axiomatizable over the variety generated by $\mathsf{R}(+, ;, 0, 1', ^*)$?

Acknowledgements We are grateful to the anonymous referees for their comments and to Peter Jipsen and Georg Struth for drawing our attention to the problem of finite axiomatizability of domain algebras and for stimulating discussions.

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