Games in Algebraic Logic

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In 1860, Augustus de Morgan published [DeM60], thereby launching an investigation into the algebra of relations. This developed into the subject now called *Algebraic Logic*, though in the 19th century it was simply thought of as mathematical logic. This work, along with Frege's quantifier logic, became the foundation of modern logic and model theory.

In De Morgan's writing there is no sharp separation of philosophy and mathematics and the central problem for him was to unveil the laws of rational thought. Of course the word 'rational' is critical and problematic here: he did not wish to consider how an insane person might think, nor the effect of one's mood on the thought process, nor any subjective features of thinking. Rational thought is considered here as a purely objective process, quite independent from the real cognitive process — a strange concept admittedly, but a pervasive idea in the philosophy of mathematics. De Morgan was particularly interested in discovering the principles of everyday thinking and of mathematical argument.

Two authors influenced him greatly: Aristotle and Boole. Aristotle's syllogism had held sway for 2000 years. Indeed Kant [Kan72] had argued that

> Since Aristotle's time Logic has not gained much in extent, as indeed nature forbids it should. ... Aristotle has omitted no essential point of the understanding; we have only become more accurate, methodical, and orderly.

But De Morgan was among a number of philosophers who found the Aristotelian syllogism inadequate to model the laws of thought. Consider the following quote from De Morgan

> Accordingly, all logical relation is affirmed to be reducible to *identity* A is A, to *non-contradiction*, Nothing both A and not-A, and to *excluded middle*, Everything either A or not-A. These three principles, it is affirmed, dictate all the forms of inference, and evolve all the canons of syllogism. I am not prepared to deny the truth of either of these propositions, at least when A is not self-contradictory, but I cannot see how, alone, they are competent to the functions assigned. I see that they distinguish truth from falsehood: but I do not see that they, again alone, either distinguish or evolve one truth from another. Every transgression of these laws is an invalid inference: every valid inference is not a transgression of these laws. But I cannot admit that every thing which is not a transgression of these laws is a valid inference. And I cannot make out how just the only propositions which are true of all things conceivable can *be* or *lead to* any distinction between one thing and another. I believe these three principles to be of the soil, and not of the seed, though the seed may possess some materials of the soil; of the foundation, not of the building, though the bricks may partake of the nature of the foundation; of the rails, not of the locomotive, though both may have iron in their structure.

> The canons of ordinary syllogism cannot be established without help from our knowledge of the *convertible* and *transitive* character of identification: that is, we must know and use the properties 'A is B gives B is A' and 'A is B and B is C, compounded, give A is C'. Can these principles be established by concession of 'A is A, nothing is both A and not-A, and every thing is one or the other'?

De Morgan thought not. Of course De Morgan's writings came long before we had the Tarski semantics and a notion of completeness, but De Morgan was dissatisfied with Aristotle's syllogism because it could not prove enough and because considerations about the laws of equality led him to believe that the syllogism was inadequate for dealing with relations of rank greater than one.

The other decisive influence was George Boole, who had put forward a highly successful algebra of propositions [Boo51]. De Morgan wrote

> When the ideas thrown out by Mr Boole shall have borne their full fruit, algebra, though only founded on ideas of number in the first instance, will appear like a sectional model of the whole form of thought. Its forms, considered apart from their matter, will be seen to contain all the forms of thought in general. The anti-mathematical logician says that it makes thought a branch of algebra, instead of algebra a branch of thought. It *makes* nothing; it *finds*: and it finds the laws of thought symbolized in the forms of algebra.

De Morgan's project then, was to use an algebraic formalism, like boolean algebra, to reason about higher order relations and particularly about binary relations.

Now binary relations, which are considered by mathematicians as simply sets of pairs, have all the structure of boolean algebras: there is a smallest (empty) binary relation, a biggest binary relation (denoted 1), you can take the union and intersection of two binary relations, and you can find the complement of a binary relation r (meaning the set of all pairs which belong to 1 but not to the relation r). So all the axioms for boolean algebra still hold over a field of binary relations. But there is more: it is natural to define the *identity* relation $1' = \{(x,x) : x \in X\}$ over a domain X, the *converse* of a relation r, $\breve{r} = \{(x,y) : (y,x) \in r\}$ and the *composition* of two relations $r;s = \{(x,y) : \exists z \in X, (x,z) \in r \wedge (z,y) \in s\}$. Observe that composition gives us relativized quantification over the variable z.

In order to consider these binary relations and the new operators from the point of view of algebra, we consider a *field of binary relations*. This is just a set of binary relations containing the identity, 0 and 1, and closed under the boolean operations, converse and composition. There are many properties that any field of binary relations must satisfy: for example composition must be associative and the identity law says that $1';r = r;1' = r$ for any r. De Morgan, Peirce, Schröder and others investigated many, many axioms like this as well as additional operators that can be defined over binary relations.

Tarski re-launched the subject in the mid-twentieth century with his formalisations of relation algebra and, for n-ary relations, n-dimensional cylindric algebra [JT48, JT51, JT52, HMT]. In this formalisation, a relation algebra was defined to be any algebraic structure with the appropriate operators that obeyed the axioms mentioned above plus a few others.

Definition 1 *A* relation algebra *is a structure* $\mathcal{A} = (A, 0, 1, ., -, 1', ^{\smile}, ;)$, *where* A *is a non-empty set (the domain or universe of* $\mathcal{A}$*),* . *and* ; *are binary functions,* $^{\smile}$ *and* $-$ *are unary functions, and* $0, 1$, *and* $1'$ *are constants. We require that* $(A, 0, 1, ., -)$ *is a boolean algebra (so we can use* $+$ *and* $\leq$ *as abbreviations), and* *that the following hold, for all $r, s, t \in A$:*

1. $r;1' = 1';r = r$	2. $(r;s);t = r;(s;t)$
3. $r^{\smile\smile} = r$	4. $(r+s)^{\smile} = \breve{r} + \breve{s}$
5. $(-r)^{\smile} = -\breve{r}$	6. $(r;s)^{\smile} = \breve{s};\breve{r}$
7. $(r;s).\breve{t} = 0 \ \rightarrow \ (s;t).\breve{r} = 0.$	

However, although it is fairly immediate that any field of binary relations must obey these axioms and is therefore a relation algebra, it turned out that not all relation algebras are isomorphic to genuine fields of binary relations [Lyn50] (or to put it another way, not all relation algebras are *representable* as fields of binary relations). Worse, there could be no finite axiomatisation that exactly defined the representable relation algebras [Mon64]. On the other hand, the class of representable relation algebras (or cylindric algebras) was shown, by methods of universal algebra, to form a *variety* with a recursive (equational) axiomatisation [JT48]. This led to the problem of finding a nice, recursive axiomatisation of the representable algebras. Lyndon and Monk did produce recursive axiomatisations, but these were rather complex.

A game-theoretic approach to the representability problem was described in [HH97b, HH97a]. We start by considering finite directed complete graphs N where each edge (x, y) is labelled by an element $N(x, y)$ of a relation algebra $\mathcal{A}$. One can think of these graphs as candidates to be finite fragments of some representation. These graphs must obey certain consistency conditions — for any node $x \in N$, $N(x, x) \leq 1'$, and for any three nodes x, y, z of N we have $N(x, y) \cdot (N(x, z); N(z, y)) \neq 0$. If a labelled graph failed one of these conditions, it certainly could not approximate a representation. We call labelled graphs that satisfy these consistency conditions *networks*. We want to know whether a given network actually defines a representation, or at least if it approximates some finite fragment of some representation of $\mathcal{A}$.

The consistency conditions on their own do not ensure that N defines a representation, as it may contain a *defect* — perhaps there is an edge (x, y) with $N(x, y) \leq a; b$ for some elements $a, b \in \mathcal{A}$, but there is no node z of the network with $N(x, z) \leq a$ and $N(z, y) \leq b$. Such a defect means that the map $a \mapsto \{(x, y) \in N : N(x, y) \leq a\}$ cannot be an isomorphism from $\mathcal{A}$ to a field of binary relations because it does not preserve composition.

Notation: If M, N are networks we write $M \subseteq N$ and say that M is a *subnetwork* of N, or equivalently that N is an *extension* of M, if the nodes of M form a subset of those of N and for any nodes $m, m' \in M$ we have $M(m, m') \geq N(m, m')$. Think of this as meaning that N carries more information about the constraints on the nodes of M.

We define a game $G_\omega(\mathcal{A})$, in which two players build a countably infinite sequence of graphs (or networks) labelled by elements of the algebra $\mathcal{A}$, each one a subnetwork of the next. The first player ($\forall$) is trying to prove that the relation algebra has no representation (or at least that the current network does not approximate a fragment of any representation) by picking potential defects in the current network, while the second player ($\exists$) is trying to make the networks approximate a representation better and better either by repairing the defect $\forall$ has picked or by demonstrating that it is not really a defect after all. If, at any stage, $\exists$ cannot repair the defect, or if she produces a graph that fails one of the consistency conditions, then player $\forall$ wins immediately. On the other hand, if she survives each round of the game without losing, we say that player $\exists$ has won.

More precisely, in round 0, $\forall$ picks any element $a \neq 0$ in $\mathcal{A}$, and $\exists$ must respond with a network N_0 with nodes m, n such that $N_0(m, n) = a$. In round $t > 0$, if the play so far has been $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{t-1}$ (where each N_i is a network for $i < t$), then $\forall$ may pick any edge (x, y) of N_{t-1} and any pair of elements $r, s \in \mathcal{A}$. $\exists$ must respond in one of two ways. She may *reject* $\forall$'s move, by letting N_t be the same as N_{t-1} except that $N_t(x, y) = N_{t-1}(x, y) \cdot -(r; s)$. Effectively she is saying that there is no defect because the edge (x, y) is not in the binary relation $r; s$. Alternatively, she may *accept* $\forall$'s move, by choosing a network N_t with the nodes of N_{t-1} plus a single extra node z with labelling

- $N_t(x, z) = r, \;\; N_t(z, z) = 1', \;\;\; N_t(z, y) = s, \;\;\; N_t(x, y) = N_{t-1}(x, y) \cdot (r; s)$
- if $u, v \in N_{t-1}$ and $(u, v) \neq (x, y)$ then $N_t(u, v) = N_{t-1}(u, v)$
- all other labels of N_t not yet mentioned are 1.

Here, $\exists$ is repairing the defect presented to her by $\forall$. This is an example of the 'cut-and-choose' games mentioned in Wilfrid Hodges' abstract — $\forall$ *cuts* by picking nodes x, y and elements r, s and $\exists$ *chooses* by either accepting or rejecting.

A winning strategy for player $\exists$ in this game is equivalent to the representability of the algebra. We can see this, at least for countable relation algebras, by considering a play of the game in which $\forall$ picks all possible defects at some stage in the play. In this case, if $\exists$ uses her winning strategy, we can define a map h from elements of $\mathcal{A}$ to binary relations over the nodes of the networks occurring in the play, defined by $h(a) = \{(x, y) : N_i(x, y) \leq a \text{ for some } i < \omega\}$, for any $a \in \mathcal{A}$. Because all network defects were eliminated during the game, h yields a representation of $\mathcal{A}$. (There are some minor technical complications here, and we refer the interested reader to [HH97b, HH97a].) In fact, by a compactness argument, a winning strategy for $\exists$ implies representability for arbitrary (even uncountable) relation algebras.

We can define a sequence of approximations to representability by asking if $\exists$ has a winning strategy in the game, $G_n(\mathcal{A})$, which is like $G_\omega(\mathcal{A})$ but curtailed after n rounds $(n < \omega)$. By a König tree lemma argument, a winning strategy for her in each of these curtailed games gives a winning strategy for her in the full, infinite length game. To see this, suppose $\exists$ has a winning strategy in each of the games $G_n(\mathcal{A})$, for $n < \omega$. In a play of $G_\omega(\mathcal{A})$ let her adopt the following strategy. In round t let $\forall$ pick the edge (x, y) from N_{t-1} and elements $r, s \in \mathcal{A}$. Inductively, suppose that infinitely many of her strategies for the finite length games have been used in her choice of moves so far and they are still running. By assumption this is true initially. Each of these strategies will tell her either to reject or accept $\forall$'s move.[1] If infinitely many of the strategies tell her to reject, then she rejects in the infinite length game. Otherwise, infinitely many finite game strategies tell her to accept, which she does in the infinite length game. As we move to the next round, her strategy for the finite length game $G_t(\mathcal{A})$ will 'run out', but all her other strategies that are running are still running in the next round. Thus we re-establish the induction hypothesis: she is in a situation where infinitely many strategies have been followed and are still running. Thus she can continue in this way forever and win the play.

To obtain a recursive axiomatisation, it remains to find a formula saying that $\exists$ has a winning strategy in the game of length n. This is not hard to do. *Very roughly,* ϕ^0_n is a formula that says "the current graph is a network with no more than n nodes" and ϕ^{m+1}_n says "for any possible $\forall$-move, either ϕ^m_{n+1} holds on the graph obtained by rejecting or ϕ^m_{n+1} holds on the graph obtained by accepting his move". Thus, ϕ^m_2 holds of a network N of size ≤ 2 iff $\exists$ has a winning strategy in $G_n(\mathcal{A})$ where play starts with N. It is now easy to turn the ϕ^m_2 $(m < \omega)$ into sentences axiomatising the representable relation algebras.

Games can also be used to reprove Monk's non-finite axiomatisability result, by constructing a sequence of algebras $\mathcal{A}_n$ such that $\exists$ has a winning strategy in

[1] It is crucial for this argument that $\exists$ has only a finite number of choices, in fact just two choices, for her move. There are other games that we may consider where this is not so, and then it may be possible to find relation algebras where $\exists$ has a winning strategy for all finite length games but not for the ω length game. We will return to this point later.

$G_n(\mathcal{A}_n)$ (for each $n < \omega$), but no winning strategy in $G_\omega(\mathcal{A}_n)$. Game-theoretic arguments then show that $\exists$ has a winning strategy in $G_\omega(\mathcal{B})$ where $\mathcal{B}$ is any non-principal ultraproduct of the $\mathcal{A}_n$. Non-finite axiomatisability follows from this, using Łós theorem. This theorem states that if any first-order formula σ is true in $\mathcal{B}$ then is is true in 'many' of the $\mathcal{A}_n$. Now if the representable relation algebras could be axiomatised by a finite set of formulas, it could be axiomatised by a single formula σ (just take the conjunction of the finite set of formulas). But then, since $\mathcal{B}$ is representable, σ must be true in $\mathcal{B}$. By Łós theorem, σ must be true in many of the $\mathcal{A}_n$. But *none* of the $\mathcal{A}_n$ are representable, so σ is true in none of them, which is a contradiction.

Game-theoretic techniques can also be used to show that the class of 'completely representable' relation algebras is not elementary (it cannot be defined by *any* set of first-order formulas). A complete representation of a relation algebra $\mathcal{A}$ is a representation in which arbitrary disjunctions (and conjunctions) are preserved, wherever they are defined. So for any set $S \subseteq \mathcal{A}$ with a least upper bound ΣS in $\mathcal{A}$, a complete representation h satisfies $h(\Sigma S) = \bigcup\{h(s) : s \in S\}$. A completely representable relation algebra must be atomic.

To deal with complete representations, we define another game, $H_\omega(\mathcal{A})$. This game is similar to $G_\omega(\mathcal{A})$, except that we use graphs in which the edges are labelled by atoms of $\mathcal{A}$. One complication for $\exists$ is that she cannot 'accept' in the way she did before by leaving many of the new edges labelled by the top element 1; she must actually choose atoms to label all the edges. Thus, she may have infinitely many choices in each round. It can be shown, using similar arguments to those above, that for a countable relation algebra $\mathcal{A}$, a winning strategy for $\exists$ in $H_\omega(\mathcal{A})$ is equivalent to the existence of a complete representation of $\mathcal{A}$. To show that the class of completely representable relation algebras is not elementary, we construct a countable relation algebra $\mathcal{A}$ such that $\exists$ has a winning strategy in the finite length game $H_n(\mathcal{A})$ for each $n < \omega$, but not in $H_\omega(\mathcal{A})$. Though $\mathcal{A}$ is not completely representable, it turns out that $\exists$ has a winning strategy in $H_\omega(\mathcal{B})$, where $\mathcal{B}$ is a non-principal ultrapower of $\mathcal{A}$. Although $\mathcal{B}$ may be uncountable, we can find a countable elementary subalgebra $\mathcal{C}$ of $\mathcal{B}$ in which $\exists$ still has a winning strategy. So $\mathcal{C}$ is completely representable, but elementarily equivalent to $\mathcal{A}$ which is not completely representable. Thus, the class of completely representable relation algebras is not closed under elementary equivalence, and cannot form an elementary class.

Other games have been devised to show that the atom structure of an atomic relation algebra does not determine its representability; that it is undecidable whether a finite relation algebra is representable; and to give non-finite axiomatisability results for such classes as the relational reducts of n-dimensional cylindric algebras, etc. Not all these results are negative, though: we have already seen how games can be used to derive a recursive axiomatisation of the representable relation algebras.

The work in algebraic logic ties in with work in other fields. Here in Amsterdam, van Benthem, Venema, Marx, de Rijke and others developed arrow logic in part as a link from relation algebra to modal logic. The Budapest algebraic logic group has explored many aspects including the link between algebra and logic, decidability and complexity results for various algebraic logics, and they have considered the whole 'finitization problem' in depth. Maddux, in the US, has developed the field significantly and has explored relativized representations among many other things; his and Lyndon's work is in many ways a foundation for the game-theoretic results sketched here. There are numerous applications in planning and temporal reasoning. Many other groups, in the US, South Africa, New Zealand and so on, are working in this field, but we cannot summarise the field here.

The game-theoretic approach appears to have a number of advantages. One of them is that it is very natural to define a sequence of approximations to a property characterised by an infinite length game. These approximations are obtained by considering finite length games that are curtailed after n rounds. Often these approximations characterise classes of interest in their own right, and they can be useful for proving (e.g.) non-finite axiomatisability results.

Another advantage is that it makes the proofs more intuitive. For example, there are other axiomatisations of the representable relation algebras, but they really are very difficult to comprehend. With games, the axioms say "$\exists$ has a winning strategy in the game of length n", so the meaning is clear. In many of the games we construct, the second player $\exists$ is effectively trying to build a representation or prove the theorem, so her strategy corresponds to the proof.

Future developments based on this approach might involve work on finite representations — where we want a representation with a finite domain. Another line of enquiry is to ask for which values of n does a winning strategy in the game of length n ensure a winning strategy in the ω-length game. This might be used to obtain tighter complexity results. Designing algebraic logics to model various systems, such as interactive systems, is a field that is currently being investigated. Indeed, games have already been used to model interactive systems, and we believe that they could provide a useful tool for more sophisticated modelling.

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