ORDERED DOMAIN ALGEBRAS

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Abstract. We give a finite axiomatisation to representable ordered domain algebras and show that finite algebras are representable on finite bases.

1. Introduction

Domain algebras provide an elegant, one-sorted formalism for automated reasoning about program and system verification, see [DS11, DS08] and [HM11] for details and further motivation. The algebraic behaviour of domain algebras have been investigated, e.g. in [DJS09a, DJS09b]. Their primary models are algebras of relations, viz. representable domain algebras. P. Jipsen and G. Struth raised the question whether the class $\mathsf{R}(;, \mathsf{dom})$ of representable domain algebras of the minimal signature $(;, \mathsf{dom})$ is finitely axiomatisable. To formulate the question precisely, let us recall the definition of representable domain algebras $\mathsf{R}(;, \mathsf{dom})$.

Definition 1.1. *The class* $\mathsf{R}(;, \mathsf{dom})$ *is defined as the isomorphs of* $\mathcal{A} = (A, ;, \mathsf{dom})$ *where* $A \subseteq \wp(U \times U)$ *for some base set* U *and*

$$x \,;\, y = \{(u, v) \in U \times U : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w \in U\}$$
$$\mathsf{dom}(x) = \{(u, u) \in U \times U : (u, v) \in x \text{ for some } v \in U\}$$

for every $x, y \in A$.

The signature $(;, \mathsf{dom})$ can be expanded to larger signatures τ by including other operations. For instance, we can define

$$\mathsf{ran}(x) = \{(v, v) \in U \times U : (u, v) \in x \text{ for some } u \in U\}$$
$$x^{\smile} = \{(v, u) \in U \times U : (u, v) \in x\}$$
$$1' = \{(u, v) \in U \times U : u = v\}$$

and also include the bottom element 0 (interpreted as the empty set $\emptyset$) and the ordering $\leq$ (interpreted as the subset relation $\subseteq$) to yield representable algebraic structures. The corresponding representation classes $\mathsf{R}(\tau)$ for larger signatures τ are defined analogously to the definition of $\mathsf{R}(;, \mathsf{dom})$.

It turned out that the answer to the above problem is negative.

Theorem 1.2. *[*[HM11]*] Let* τ *be a similarity type such that* $(;, \mathsf{dom}) \subseteq \tau \subseteq (;, \mathsf{dom}, \mathsf{ran}, 0, 1')$. *The class* $\mathsf{R}(\tau)$ *of representable* τ*-algebras is not finitely axiomatisable in first-order logic.*

Note that the above theorem does not apply to signatures where the ordering $\leq$ is present. In fact, D.A. Bredikhin proved [Bre77] that the class $\mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, {}^{\smile}, \leq)$ of representable algebraic structures is finitely axiomatisable. Bredikhin proves finite axiomatisability by reducing the problem of representing an ordered domain algebra (see the definition below for this axiomatically given class of algebraic structures) to that of an ordered involuted semigroup. Representable (ordered) involuted semigroups were characterised independently by Bredikhin [Bre75] and B.M. Schein [Sch74]. These characterisations use infinitely many quasi-equations (in fact, the class of representable involuted semigroups is a non-finitely axiomatisable quasi-variety [Bre77]) and the representations are rather involved (in the case of Schein's proof, an infinitary construction called "graph-theoretic scaffolding" is used).

In this paper we take a more direct approach in proving finite axiomatisation of representable ordered domain algebras. The advantage of our proof is that it uses a Cayley-type representation of abstract algebraic structures that also shows *finite representability*, i.e. that finite elements of $\mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \leq)$ can be represented on finite bases. In passing we note that if composition is not definable in τ, then $\mathsf{R}(\tau)$ has the finite representation property, but it can be shown that every signature containing $(\cdot, ;, 1')$ or $(\cdot, ;, ^\smile)$ (where $\cdot$ is interpreted as intersection) fails to have the finite representation property.

2. Main result

Let Ax denote the following formulas. These axioms are essentially the same as in [Bre77], we just made slight adjustments to include the constants $1'$ and 0 as well.

Partial order: The ordering $\leq$ is reflexive, transitive and antisymmetric, with lower bound 0.

Monotonicity and normality: The operations $;, \mathsf{dom}, \mathsf{ran}, ^\smile$ are monotonic, e.g. $a \leq b$ implies $a ; c \leq b ; c$ etc. and normal $0^\smile = 0 ; a = a ; 0 = \mathsf{dom}(0) = \mathsf{ran}(0) = 0$.

Involuted monoid: The operation ; is associative, the constant $1'$ is a left and right identity for ;, $^\smile$ is an involution $(a^\smile)^\smile = a$ and $(a;b)^\smile = b^\smile;a^\smile$, and $1'^\smile = 1'$.

Domain/range axioms:

$$\mathsf{dom}(a) = (\mathsf{dom}(a))^\smile \leq 1' = \mathsf{dom}(1') \tag{1}$$

$$\mathsf{dom}(a) \leq a ; a^\smile \tag{2}$$

$$\mathsf{dom}(a^\smile) = \mathsf{ran}(a) \tag{3}$$

$$\mathsf{dom}(\mathsf{dom}(a)) = \mathsf{dom}(a) = \mathsf{ran}(\mathsf{dom}(a)) \tag{4}$$

$$\mathsf{dom}(a) ; a = a \tag{5}$$

$$\mathsf{dom}(a ; b) = \mathsf{dom}(a ; \mathsf{dom}(b)) \tag{6}$$

$$\mathsf{dom}(\mathsf{dom}(a) ; \mathsf{dom}(b)) = \mathsf{dom}(a) ; \mathsf{dom}(b) = \mathsf{dom}(b) ; \mathsf{dom}(a) \tag{7}$$

A model $\mathcal{A} = (A, ;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \leq)$ of these axioms is called an *ordered domain algebra*, and the class of ordered domain algebras is denoted by ODA.

Each of the axioms (1)–(7) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions. We denote the dual axiom by a ∂ superscript, thus for example, $(6)^\partial$ is $\mathsf{ran}(b ; a) = \mathsf{ran}(\mathsf{ran}(b) ; a)$. The dual axioms can be obtained from the axioms above, using the involution axioms and (3).

The operations can be easily extended to subsets of elements as follows. Let $\mathcal{A} \in \mathsf{ODA}$ and $X, Y \subseteq A$. We define

$$X^{\smile} = \{x^{\smile} : x \in X\}$$
$$X \,;\, Y = \{x \,;\, y : x \in X,\ y \in Y\}$$
$$\mathsf{dom}(X) = \{\mathsf{dom}(x) : x \in X\}$$
$$\mathsf{ran}(X) = \{\mathsf{ran}(x) : x \in X\}$$

Note that we do not claim that the algebra of subsets satisfy the axioms.

We will need the following notation

$$X^{\uparrow} = \{a \in A : a \geq x \text{ for some } x \in\ X\}$$

that is, we will consider subsets which are closed upward. If $X = \{x\}$ is a singleton set, then we simply write $x^{\uparrow}$ for $\{x\}^{\uparrow}$ and similarly $x \,;\, Y$ for $\{x\} \,;\, Y$ and $x^{\uparrow} \,;\, Y$ for $\{x\}^{\uparrow} \,;\, Y$, etc.

Next we define closed sets that we will use as a base of the representation of abstract algebras.

Definition 2.1. *We say that $X \subseteq A$ is* closed *if $0 \notin X$ and*

$$X = \{\mathsf{dom}(x) \,;\, y \,;\, \mathsf{ran}(z) : x, y, z \in X\}^{\uparrow}$$

and let $\Gamma[\mathcal{A}]$ denote the set of closed subsets of $\mathcal{A}$.

Our main result is the following.

Theorem 2.2. *The class $\mathsf{R}(;, \mathsf{dom}, \mathsf{ran}, {}^{\smile}, 0, 1', \leq)$ is finitely axiomatisable:*

$$\mathcal{A} \in \mathsf{R}(;, \mathsf{dom}, \mathsf{ran}, {}^{\smile}, 0, 1', \leq) \text{ iff } \mathcal{A} \in \mathsf{ODA}$$

and has the finite representation property, i.e. every finite ODA is isomorphic to some $\mathsf{R}(;, \mathsf{dom}, \mathsf{ran}, {}^{\smile}, 0, 1', \leq)$ for some finite base U.

Proof. Let $\mathcal{A} \in \mathsf{ODA}$. We define the map h from $\mathcal{A}$ to a structure with base $\Gamma[\mathcal{A}]$ by setting

$$(X, Y) \in h(a) \iff X \,;\, a \subseteq Y \text{ and } Y \,;\, a^{\smile} \subseteq X \tag{8}$$

for every $a \in A$.

We will show that h is injective (Lemma 4.1) and that the ordering and the operations are correctly represented, see Lemma 4.2, 4.3 and 4.5, whence h is indeed an isomorphism.

Clearly, when $\mathcal{A}$ is finite, the base $\Gamma[\mathcal{A}]$ of this representation is also finite. □

The rest of the paper is devoted to make the above proof complete.

3. Closed sets

We mention some easy consequences of the axioms. A consequence of axioms (4) and (5) is

$$\mathsf{dom}(a) \,;\, \mathsf{dom}(a) = \mathsf{dom}(a) \tag{9}$$

Let $\Delta[\mathcal{A}]$ denote the set of *domain elements* of $\mathcal{A} \in \mathsf{ODA}$ — those elements $d \in A$ for which $\mathsf{dom}(d) = d$. Then it is easy to check that $(\Delta[\mathcal{A}], ;)$ forms a lower semilattice (ordered by $\leq$).

Another consequence of the axioms is the following lemma, which we shall use later.

Lemma 3.1. *Let* $\mathcal{A} \in \mathsf{ODA}$ *and* $b, c \in A$. *Then*

$$\mathsf{dom}(b\,;c)\,;b \geq b\,;\mathsf{dom}(c) \text{ and } b\,;\mathsf{ran}(c\,;b) \geq \mathsf{ran}(c);b$$

Proof.

$$\begin{aligned}
\mathsf{dom}(b\,;c)\,;b &= \mathsf{dom}(b\,;\mathsf{dom}(c))\,;b && \text{by (6)}\\
&\geq \mathsf{dom}(b\,;\mathsf{dom}(c))\,;b\,;\mathsf{dom}(c) && \text{by (1)}\\
&= b\,;\mathsf{dom}(c) && \text{by (5)}
\end{aligned}$$

The other part is similar using the dual axioms. □

We will need some properties of closed sets. Note that it is enough to show that $\mathsf{dom}(X)\,;X \subseteq X$ and $X\,;\mathsf{ran}(X) \subseteq X$ for establishing that X is closed.

Lemma 3.2. *Let* $\mathcal{A} \in \mathsf{ODA}$.

(1) *For any* $a \in A$, $a^{\uparrow}$ *is closed.*
(2) *If* X *is closed, then so are* $\mathsf{dom}(X)$ *and* $\mathsf{ran}(X)$.
(3) *If* X *is closed and* $a \in A$, *then* $(X\,;a^{\uparrow})\,;\mathsf{ran}(X\,;a^{\uparrow}) \subseteq (X\,;a^{\uparrow})^{\uparrow}$ *and* $(\mathsf{ran}(X\,;a^{\uparrow}))^{\uparrow} = (\mathsf{ran}(X\,;a))^{\uparrow}$ *is closed.*
(4) *If* X *is closed,* $a \in A$ *and* $\mathsf{dom}(a) \in \mathsf{ran}(X)$, *then* $(X\,;a^{\uparrow})^{\uparrow}$ *is closed.*
(5) *If* X, Y *are closed, and* $\mathsf{dom}(X) = \mathsf{dom}(Y)$ *and* $\mathsf{ran}(X) = \mathsf{ran}(Y)$, *then* $X \cup Y$ *is closed.*

Proof. (1): By monotonicity and (5).

(2): To prove that $\mathsf{dom}(X)$ is closed we must check that

$$\mathsf{dom}(\mathsf{dom}(X))\,;\mathsf{dom}(X) \subseteq \mathsf{dom}(X) \text{ and } \mathsf{dom}(X)\,;\mathsf{ran}(\mathsf{dom}(X)) \subseteq \mathsf{dom}(X)$$

First note that $\mathsf{dom}(\mathsf{dom}(X)) = \mathsf{dom}(X)$ by (4). Thus we need that $\mathsf{dom}(x)\,;\mathsf{dom}(x') \in \mathsf{dom}(X)$ for every $x, x' \in X$. By (7) and (6), $\mathsf{dom}(x)\,;\mathsf{dom}(x') = \mathsf{dom}(\mathsf{dom}(x)\,;x')$. Since X is closed, $\mathsf{dom}(x)\,;x' \in X$, whence $\mathsf{dom}(\mathsf{dom}(x)\,;x') \in \mathsf{dom}(X)$ as desired. The other requirement follows similarly by observing that $\mathsf{ran}(\mathsf{dom}(x)) = \mathsf{dom}(x)$, by (4).

Showing that $\mathsf{ran}(X)$ is closed is completely analogous.

(3): For every $x, x' \in X$,

$$\begin{aligned}
(x\,;a)\,;\mathsf{ran}(x'\,;a) &= (x\,;a)\,;\mathsf{ran}(\mathsf{ran}(x')\,;a) && \text{by } (6)^{\partial}\\
&\geq x\,;\mathsf{ran}(x')\,;a\,;\mathsf{ran}(\mathsf{ran}(x')\,;a) && \text{by } (1)^{\partial}\\
&= x\,;\mathsf{ran}(x')\,;a && \text{by } (5)^{\partial}\\
&\in X\,;a && \text{as } X \in \Gamma[\mathcal{A}]
\end{aligned}$$

whence $(X\,;a^{\uparrow})\,;\mathsf{ran}(X\,;a^{\uparrow}) \subseteq (X\,;a^{\uparrow})^{\uparrow}$ follows by monotonicity.

For the second part, let $x_i \in X$ (for $i = 1, 2, 3$). We show that

$$\mathsf{dom}(\mathsf{ran}(x_1\,;a))\,;\mathsf{ran}(x_2\,;a)\,;\mathsf{ran}(\mathsf{ran}(x_3\,;a)) \in (\mathsf{ran}(X\,;a))^{\uparrow}.$$

Well,

$$\begin{aligned}
&\mathsf{dom}(\mathsf{ran}(x_1 \,;\, a)) \,;\, \mathsf{ran}(x_2 \,;\, a) \,;\, \mathsf{ran}(\mathsf{ran}(x_3 \,;\, a)) \\
&\quad = \mathsf{ran}(x_1 \,;\, a) \,;\, \mathsf{ran}(x_2 \,;\, a) \,;\, \mathsf{ran}(x_3 \,;\, a) && \text{by } (4)^{\partial} \\
&\quad \geq \mathsf{ran}(\underbrace{x_1 \,;\, \mathsf{ran}(x_2) \,;\, \mathsf{ran}(x_3)}_{\in X} \,;\, a) && \text{by } (1)^{\partial},\ (6)^{\partial},\ X \in \Gamma[\mathcal{A}] \\
&\quad \in \mathsf{ran}(X \,;\, a)
\end{aligned}$$

as desired. Thus the claim follows by monotonicity.

(4): We have seen in the previous item that $(X \,;\, a) \,;\, \mathsf{ran}(X \,;\, a) \subseteq (X \,;\, a)^{\uparrow}$. Thus it remains to show that $\mathsf{dom}(X \,;\, a) \,;\, (X \,;\, a) \subseteq X \,;\, a$.

$$\begin{aligned}
\mathsf{dom}(X \,;\, a) &= \mathsf{dom}(X \,;\, \mathsf{dom}(a)) && \text{by } (6) \\
&= \mathsf{dom}(X \,;\, \mathsf{ran}(X) \,;\, \mathsf{dom}(a)) && \text{as } X \in \Gamma[\mathcal{A}] \\
&\subseteq \mathsf{dom}(X \,;\, \mathsf{ran}(X)) && \text{as } \mathsf{ran}(X) \in \Gamma[\mathcal{A}],\ \mathsf{dom}(a) \in \mathsf{ran}(X) \\
&= \mathsf{dom}(X) && \text{as } X \in \Gamma[\mathcal{A}]
\end{aligned}$$

whence $\mathsf{dom}(X \,;\, a) \,;\, (X \,;\, a) \subseteq \mathsf{dom}(X) \,;\, X \,;\, a = X \,;\, a$ as X is closed.

(5): Immediate from the definition of closed and the fact that $Z \,;\, (X \cup Y) = (Z \,;\, X) \cup (Z \,;\, Y)$. ☐

4. Representing ordered domain algebras

We recall that h was defined in (8) by

$$(X, Y) \in h(a) \iff X \,;\, a \subseteq Y \text{ and } Y \,;\, a^{\smile} \subseteq X$$

for every $X, Y \in \Gamma[\mathcal{A}]$.

Lemma 4.1. *The map h is injective.*

Proof. Let $a \not\leq b \in \mathcal{A}$. By (5) $\mathsf{dom}(a) \,;\, a = a$ and by (2) $a \,;\, a^{\smile} \geq \mathsf{dom}(a)$, so $((\mathsf{dom}(a))^{\uparrow}, a^{\uparrow}) \in h(a)$ by monotonicity. Also, we cannot have $a \leq \mathsf{dom}(a) \,;\, b$, else $a = \mathsf{dom}(a) \,;\, a \leq \mathsf{dom}(a) \,;\, b \leq b$ by (1) and (5) contrary to our assumption that $a \not\leq b$. Thus $((\mathsf{dom}(a))^{\uparrow}, a^{\uparrow}) \notin h(b)$, and we are done. ☐

Lemma 4.2. *The operation $^{\smile}$, the constants 0 and $1'$ and the ordering $\leq$ are correctly represented.*

Proof. For any closed set X, we have $0 \in X \,;\, 0$, hence there is no closed set containing $X \,;\, 0$, so $h(0) = \emptyset$. If $(X, Y) \in h(a)$ and $a \leq b$, then $X \,;\, b \subseteq X \,;\, a \subseteq Y$, similarly $Y \,;\, b^{\smile} \subseteq X$, so $(X, Y) \in h(b)$. If $(X, Y) \in h(1')$, then $X = X \,;\, 1' \subseteq Y$ and $Y = Y \,;\, 1'^{\smile} \subseteq X$, so $X = Y$. Conversely for any closed set X, we have $(X, X) \in h(1')$, hence $h(1') = \{(X, X) : X \in \Gamma[\mathcal{A}]\}$. Finally, $(X, Y) \in h(a)$ iff $X ; a \subseteq Y$ and $Y ; a^{\smile} \subseteq X$ iff $Y ; a^{\smile} \subseteq X$ and $X ; (a^{\smile})^{\smile} \subseteq Y$ iff $(Y, X) \in h(a^{\smile})$. ☐

Lemma 4.3. *The operation $;$ is correctly represented.*

Proof. If $(X, Y) \in h(a)$ and $(Y, Z) \in h(b)$, then $X \,;\, a \subseteq Y$, $Y \,;\, a^{\smile} \subseteq X$, $Y \,;\, b \subseteq Z$ and $Z \,;\, b^{\smile} \subseteq Y$. Hence $X \,;\, (a \,;\, b) \subseteq Z$ and $Z \,;\, ((a \,;\, b)^{\smile}) = Z \,;\, (b^{\smile} \,;\, a^{\smile}) \subseteq X$ by associativity and the involution axioms. So $(X, Z) \in h(a \,;\, b)$.

Conversely, assume that $(X,Z)\in h(a;b)$, i.e. $X;(a;b)\subseteq Z$ and $Z;(b^\smile;a^\smile)\subseteq X$ for some $Z\in\Gamma[\mathcal{A}]$. Let

$$\alpha=(X\,;a\,;\mathsf{ran}(Z\,;b^\smile))^\uparrow \text{ and } \beta=(Z\,;b^\smile\,;\mathsf{ran}(X\,;a))^\uparrow$$

and $Y=\alpha\cup\beta$. We need the following claim.

Claim 4.4. *The subsets α,β and $\alpha\cup\beta$ defined above are closed.*

Proof. Consider $\alpha=(X\,;a\,;\mathsf{ran}(Z\,;b^\smile))^\uparrow$ first. If $z\in Z$, then

$$\begin{aligned}
\mathsf{dom}(a\,;\mathsf{ran}(z\,;b^\smile)) &= \mathsf{dom}(a\,;\mathsf{dom}(b\,;\mathsf{ran}(z)) && \text{by } (3),(6),(4)^\partial\\
&= \mathsf{dom}(a\,;b\,;\mathsf{ran}(z)) && \text{by } (6)\\
&= \mathsf{ran}(\mathsf{ran}(z);b^\smile;a^\smile) && \text{by } (3)\\
&= \mathsf{ran}(z\,;b^\smile\,;a^\smile) && \text{by } (6)^\partial\\
&\in \mathsf{ran}(X) && \text{as } Z;b^\smile;a^\smile\subseteq X
\end{aligned}$$

whence $\mathsf{dom}(a\,;\mathsf{ran}(Z\,;b^\smile))\subseteq\mathsf{ran}(X)$.

Write $D=\mathsf{ran}(Z\,;b^\smile)$. Observe that $D^\uparrow$ is closed by Lemma 3.2(3). Let $x_i\in X$ and $d_i\in D$ (for $i=1,2,3$). We are required to prove that

$$\mathsf{dom}(x_1\,;a\,;d_1)\,;(x_2\,;a\,;d_2)\,;\mathsf{ran}(x_3\,;a\,;d_3)\in(X\,;a\,;D)^\uparrow.$$

For this,

$$\begin{aligned}
&\mathsf{dom}(x_1\,;a\,;d_1)\,;(x_2\,;a\,;d_2)\,;\mathsf{ran}(x_3\,;a\,;d_3)\\
&\quad= \mathsf{dom}(x_1\,;\mathsf{dom}(a\,;d_1))\,;x_2\,;a\,;d_2\,;\mathsf{ran}(\mathsf{ran}(x_3\,;a)\,;d_3) && \text{by } (6),\ (6)^\partial\\
&\quad= \mathsf{dom}(x_1\,;\mathsf{dom}(a\,;d_1))\,;x_2\,;a\,;\mathsf{ran}(x_3\,;a)\,;d_2\,;d_3 && \text{by } (7)^\partial\\
&\quad= \underbrace{\mathsf{dom}(x_1\,;\mathsf{dom}(a\,;d_1))\,;x_2}_{=x_4\in X}\,;a\,;\mathsf{ran}(x_3\,;a)\,;d_2\,;d_3 && \text{as } X\in\Gamma[\mathcal{A}],\ \mathsf{ran}(X)\supseteq\mathsf{dom}(a\,;D)\\
&\quad\geq x_4\,;\mathsf{ran}(x_3)\,;a\,;d_2\,;d_3 && \text{by Lemma 3.1}\\
&\quad\in X\,;a\,;D^\uparrow && \text{as } X\in\Gamma[\mathcal{A}],\ D^\uparrow\in\Delta[\mathcal{A}]\\
&\quad\subseteq (X\,;a\,;D)^\uparrow && \text{by monotonicity}
\end{aligned}$$

Thus α is closed. Similarly β is closed.

For any closed sets V,W, we have $\mathsf{dom}(V)\subseteq\mathsf{dom}(V\,;W)^\uparrow$ by (1) and (6), so $\mathsf{dom}(X)\subseteq(\mathsf{dom}(X\,;a\,;\mathsf{ran}(Z\,;b^\smile)))^\uparrow=\mathsf{dom}(\alpha)=\mathsf{dom}((X\,;a\,;b\,;Z^\smile)^\uparrow)\subseteq\mathsf{dom}((Z\,;Z^\smile)^\uparrow)\subseteq\mathsf{dom}((Z\,;\mathsf{ran}(Z))^\uparrow)=\mathsf{dom}(Z)$. Similarly $\mathsf{dom}(Z)\subseteq\mathsf{dom}(\beta)\subseteq\mathsf{dom}(X)$. Hence $\mathsf{dom}(\alpha)=\mathsf{dom}(\beta)$. In the same way, $\mathsf{ran}(\alpha)=\mathsf{ran}(\beta)$. Then $\alpha\cup\beta$ is also closed, by Lemma 3.2(5). □

By Claim 4.4, Y is closed. We claim that $(X,Y)\in h(a)$ (similarly $(Y,Z)\in h(b)$). To prove the claim we must show that $X\,;a\subseteq Y$ and $Y\,;a^\smile\subseteq X$. For the first inclusion, we have $X\,;a\subseteq\alpha\subseteq Y$. For the other inclusion, let $y\in Y$. We have to prove that $y;a^\smile\in X$. Since $y\in Y=(X;a;\mathsf{ran}(Z;b^\smile))^\uparrow\cup(Z;b^\smile;\mathsf{ran}(X;a))^\uparrow$, there are $x\in X$ and $z\in Z$ such that either $y\geq x\,;a\,;\mathsf{ran}(z;b^\smile)$ or $y\geq z\,;b^\smile\,;\mathsf{ran}(x\,;a)$.

In the former case,

$$\begin{aligned}
y \,;\, a^{\smile} &\geq x \,;\, a \,;\, \mathsf{ran}(z \,;\, b^{\smile}) \,;\, a^{\smile} \\
&\geq x \,;\, a \,;\, \mathsf{ran}(z \,;\, b^{\smile}) \,;\, \mathsf{ran}(z \,;\, b^{\smile}) \,;\, a^{\smile} \\
&\geq x \,;\, a \,;\, \mathsf{ran}(z \,;\, b^{\smile}) \,;\, (a \,;\, \mathsf{ran}(z \,;\, b^{\smile}))^{\smile} \\
&\geq x \,;\, \mathsf{ran}(\mathsf{ran}(z \,;\, b^{\smile}) \,;\, a^{\smile}) && \text{by } (2)^{\partial}, (3)^{\partial} \\
&\geq x \,;\, \mathsf{ran}(z \,;\, b^{\smile} \,;\, a^{\smile}) && \text{by } (6)^{\partial} \\
&\in X && \text{as } Z \,;\, b^{\smile} \,;\, a^{\smile} \subseteq X, X \in \Gamma[\mathcal{A}]
\end{aligned}$$

while in the latter case

$$\begin{aligned}
y \,;\, a^{\smile} &\geq z \,;\, b^{\smile} \,;\, \mathsf{ran}(x \,;\, a) \,;\, a^{\smile} \\
&\geq z \,;\, b^{\smile} \,;\, \mathsf{dom}(a^{\smile} \,;\, \mathsf{ran}(x)) \,;\, a^{\smile} && \text{by } (3), (6) \\
&\geq z \,;\, b^{\smile} \,;\, a^{\smile} \,;\, \mathsf{ran}(x) && \text{by Lemma 3.1, } (6) \\
&\in X \,;\, \mathsf{ran}(X) && \text{as } X \in \Gamma[\mathcal{A}] \\
&= X
\end{aligned}$$

as desired. □

Lemma 4.5. *The operations* dom *and* ran *are correctly represented.*

Proof. Suppose $(X, Y) \in h(\mathsf{dom}(a))$. We must prove that $X = Y$ and there is a closed set Z with $(X, Z) \in h(a)$. Since $(X, Y) \in h(\mathsf{dom}(a))$, we have $X ; \mathsf{dom}(a) \subseteq Y$ and $Y ; \mathsf{dom}(a)^{\smile} \subseteq X$. Now $\mathsf{dom}(a) \leq 1'$ by (1), so we have that for every $x \in X$, there is $y \in Y$ such that $x \geq x ; \mathsf{dom}(a) \geq y$. Since Y is (upward) closed, we get $X \subseteq Y$. Similarly, we get $Y \subseteq X$ by $Y \subseteq Y ; \mathsf{dom}(a)^{\smile} = Y ; \mathsf{dom}(a) \subseteq X$, thus $X = Y$. Note also that $\mathsf{dom}(a) \in \mathsf{ran}(X)$, since for every $y \in Y$, $\mathsf{dom}(a) = \mathsf{dom}(a)^{\smile} \geq \mathsf{ran}(y ; \mathsf{dom}(a)^{\smile}) \in \mathsf{ran}(X)$, by (1) and $(6)^{\partial}$. Now define $Z = (X ; a^{\uparrow})^{\uparrow}$. By Lemma 3.2(4), Z is closed. Then $(X, Z) \in h(a)$, since $X ; a \subseteq Z$ by definition, and $X ; a ; (a^{\smile}) \subseteq X ; \mathsf{dom}(a) \subseteq X$ by (2), (5), since $\mathsf{dom}(a) \in \mathsf{ran}(X)$.

Conversely, suppose $(X, Z) \in h(a)$ for some $Z \in \Gamma[\mathcal{A}]$. Then $X ; a \subseteq Z$ and $Z ; a^{\smile} \subseteq X$. Since $Z ; a^{\smile} \subseteq X$, we have $\mathsf{dom}(a) = \mathsf{ran}(a^{\smile}) \in (\mathsf{ran}(Z ; a^{\smile}))^{\uparrow} \subseteq (\mathsf{ran}(X))^{\uparrow}$, by (3) and $(6)^{\partial}$. Hence $X ; \mathsf{dom}(a) \subseteq X$, i.e. $(X, X) \in h(\mathsf{dom}(a))$. So dom is correctly represented.

Showing that ran is properly represented is similar. □

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