PROVABILITY WITH FINITELY MANY VARIABLES

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Abstract. For every finite $n \geq 4$ there is a logically valid sentence φ_n with the following properties: φ_n contains only 3 variables (each of which occurs many times); φ_n contains exactly one nonlogical binary relation symbol (no function symbols, no constants, and no equality symbol); φ_n has a proof in first-order logic with equality that contains exactly n variables, but no proof containing only $n-1$ variables. This result was first proved using the machinery of algebraic logic developed in several research monographs and papers. Here we replicate the result and its proof entirely within the realm of (elementary) first-order binary predicate logic with equality. We need the usual syntax, axioms, and rules of inference to show that φ_n has a proof with only n variables. To show that φ_n has no proof with only $n-1$ variables we use alternative semantics in place of the usual, standard, set-theoretical semantics of first-order logic.

§1. Introduction. Questions about what can be expressed and proved with only finitely many variables began with Peirce's [24] calculus of binary relations, as developed by Schröder [25]. They let letters a, b, c, ... denote binary relations on some fixed underlying set U (called the "domain", "universe of discourse", or "Denkbereich"). New relations are made from these according to operations chosen by Peirce: the union $a+b$, the intersection $a \cdot b$, the complement $\overline{a}$, the relative product $a;b$, the relative sum $a \dagger b$, and the converse $\breve{a}$. There are four distinguished relations: 1 is the universal relation on U (namely $U \times U$), 0 is the empty relation, 1' is the identity relation on U, and 0' is the diversity relation on U, consisting of all pairs of distinct elements of U. Peirce and Schröder write, for example, a_{ij} to denote the statement that element i is related to element j by relation a. The union, intersection, complement, relative product, sum, converse, and the four distinguished relations can then be defined as follows:

†Research partially supported by UK EPSRC grants GR/K54946, GR/L82441, GR/L85961.

*Research partially supported by UK EPSRC grants GR/K54946, GR/L85978.

$$\begin{aligned}
(a+b)_{ij} &\text{ iff } a_{ij} \text{ or } b_{ij},\\
(a\cdot b)_{ij} &\text{ iff } a_{ij} \text{ and } b_{ij},\\
\overline{a}_{ij} &\text{ iff } \text{ not } a_{ij},\\
(a;b)_{ij} &\text{ iff for some } k\in U, a_{ik} \text{ and } b_{kj},\\
(a\dagger b)_{ij} &\text{ iff for every } k\in U, a_{ik} \text{ or } b_{kj},\\
(\breve{a})_{ij} &\text{ iff } a_{ji},\\
\text{it is never the case that } 0_{ij}&,\\
\text{it is always the case that } 1_{ij}&,\\
1'_{ij} &\text{ iff } i=j,\\
0'_{ij} &\text{ iff } i\neq j.
\end{aligned}$$

Elementary statements about relations and the domain can be made by equations between relations. Here are some examples. The domain has at least two elements iff $1 = 1;0';1$. The domain has at least three elements iff $1 = 1;(0' \cdot (0';0'));1$. If the domain is not empty, then the relation a is nonempty iff $1 = 1;a;1$. The relation a is transitive and contains the identity relation on the domain iff $a = \breve{\overline{a}} \dagger a$. Every equation in the calculus of relations can be translated into an elementary statement about the domain and the binary relations used in the equation. For example, the associative law for relative multiplication, namely

$$a;(b;c) = (a;b);c, \tag{1}$$

is equivalent to the following sentence, which uses only three variables denoting elements of the domain (namely, i, j, and k):

$$\forall_i\forall_j(\exists_k(a_{ik}\wedge\exists_i(b_{ki}\wedge c_{ij}))\Leftrightarrow\exists_k(\exists_j(a_{ij}\wedge b_{jk})\wedge c_{kj})). \tag{2}$$

It is straightforward to prove that every equation in the calculus of relations is equivalent to a 3-sentence, that is, a first-order sentence that uses only three variables. The converse also holds: every 3-sentence is equivalent to an equation in the calculus of binary relations [28, §3.9], [4, p 204]. Even if a sentence uses more than three variables, it may still be expressible as an equation if it can be equivalently rewritten in a way that uses only three variables. An example for which this is true but far from obvious is the "pairing axiom", $\forall_i\forall_j\exists_k\forall_h(a_{hk}\Leftrightarrow 1'_{hi}\vee 1'_{hj})$. Although it contains four variables in the usual formulation, it is equivalent to a 3-sentence, and so to an equation in the calculus of relations [28, p 137].

Schröder seems to have been the first to consider the problem of whether every sentence can be expressed by an equation [25, p 551]. This problem was solved by Alwin Korselt. In a letter to Löwenheim [12], Korselt proved that no equation is equivalent to the following sentence, which asserts that the domain has at least four elements:

$$\exists_h \exists_i \exists_j \exists_k (0'_{hi} \wedge 0'_{hj} \wedge 0'_{hk} \wedge 0'_{ij} \wedge 0'_{ik} \wedge 0'_{jk}). \tag{3}$$

Since every 3-sentence can be translated into an equation, it follows that (3) is a 4-sentence that is not equivalent to any 3-sentence. Korselt's theorem was considerably generalized by Tarski [26, p 89], [28, p 61].

Tarski proposed several equations as axioms for the calculus of relations, including the six equations listed below. These equations, together with the associative law (1) and any finite set of equational axioms for Boolean algebras, axiomatize the class of *relation algebras.*

$$(a+b);c = a;c+b;c, \tag{4}$$

$$a;1' = a, \tag{5}$$

$$\breve{\breve{a}} = a, \tag{6}$$

$$(a+b)\breve{} = \breve{a}+\breve{b}, \tag{7}$$

$$(a;b)\breve{} = \breve{b};\breve{a}, \tag{8}$$

$$\breve{a};\overline{a;b}+\overline{b} = \overline{b}. \tag{9}$$

A 3-sentence equivalent to the last equation is

$$\forall_i \forall_j (\exists_k (a_{ki} \wedge \forall_i (\overline{a}_{ki} \vee \overline{b}_{ij})) \Rightarrow \overline{b}_{ij}). \tag{10}$$

This sentence is logically valid, and has a proof in any complete system of first-order logic. A proof of (10), in a natural deductive style, goes as follows. Let i and j be arbitrary elements of the domain. Assume there is some element k such that a_{ki} and for every i, either $\overline{a}_{ki}$ or $\overline{b}_{ij}$. The latter assumption, applied to the arbitrary i we are considering, implies that either $\overline{a}_{ki}$ or $\overline{b}_{ij}$. But a_{ki}, so $\overline{b}_{ij}$. We have so far proved that $\exists_k (a_{ki} \wedge \forall_i (\overline{a}_{ki} \vee \overline{b}_{ij})) \Rightarrow \overline{b}_{ij}$. Since we proved this for arbitrary i and j, we may generalize and conclude that (10) holds. Note that we only needed the three variables that actually occur in (10). We suspect that, for practically any textbook axiomatization of first order logic (with or without equality), (10) will still be provable if the axiom set is restricted to those formulæ that do not contain any variables outside a fixed group of three variables. Principles involving equality aren't needed to prove (10), but they are used in the proof of the following 3-sentence, which is equivalent to (5):

$$\forall_i \forall_j (\exists_k (a_{ik} \wedge 1'_{kj}) \Leftrightarrow a_{ij})). \tag{11}$$

To prove this, let i and j be arbitrary elements of the domain. For one direction, assume there is some k such that a_{ik} and $1'_{kj}$. By an axiom asserting that equality is a congruence relation (that is, an equivalence relation on the domain with the property that a_{ik} and $1'_{kj}$ imply a_{ij}, and a_{kj} and $1'_{ki}$ imply a_{ij}), this gives us a_{ij}. For the converse, assume a_{ij}. Assuming that everything is equal to something, we conclude there is some

k such that 1'_{jk}. Since equality is a congruence relation, it follows that a_{ik} and 1'_{kj}, hence $\exists_k(a_{ik} \wedge 1\text{'}_{kj})$. So far we have proved $\exists_k(a_{ik} \wedge 1\text{'}_{kj}) \Leftrightarrow a_{ij}$. Generalizing, we obtain (11).

Similar 3-variable proofs can be given for the 3-sentences obtained from (4), (6), (7), and (8). On the other hand, while it is easy to prove (2) using four variables, the difficulties encountered when trying to do it with only three led Tarski to explicitly include (2) in the axiomatization of the 3-variable logic $\mathcal{L}_3^+$ defined in [28], where the problem of axiomatizing n-variable logic for $n \geq 3$ is discussed at length. Let $\mathcal{L}s_3^+$ be obtained from $\mathcal{L}_3^+$ by deleting (2) [28, p 89]. Tarski established that (2) is not provable in $\mathcal{L}s_3^+$ by considering an algebra, constructed by J. C. C. McKinsey, that satisfies (4)–(9): *i.e.* all of the equational axioms for relation algebras except the associative law for relative multiplication (1). See [28, p 68]; Tarski's proof was never published, but another proof was given by Henkin [9].

Four variables are sufficient to prove not only the 3-sentences obtained from the equational axioms (1) & (4)–(9) for relation algebras, but also the 3-variable translations of all equations derivable from these axioms. It follows that if an equation is true in every relation algebra then its 3-variable translation has a 4-variable proof. It turns out that the converse is also true: if the 3-variable translation of a given equation has a 4-variable proof, then the equation holds in every relation algebra [14], [16], [17], [28, pp 92–93], [4, pp 206–207]. Therefore, if an equation E fails in some relation algebra and its equivalent 3-sentence T is logically valid, then T is provable but not with fewer than five variables. Here is an example of an equation whose equivalent 3-sentence is logically valid and provable with five variables but not four [28, pp 54, 93], [4, p 198]:

$$1;\big((a \dagger a) + [(a;a) + 1\text{'} + (\overline{a + \breve{a}};\overline{a + \breve{a}}) \cdot \overline{\breve{a}}] \cdot \overline{a} + \overline{a;\breve{a}}\big);1 = 1. \tag{12}$$

Both this equation and its equivalent 3-sentence use only a single binary relation symbol, because they arise from the analysis of a 16-element non-representable relation algebra that is generated by a single element [18].

Suppose a 3-sentence T is logically valid, hence provable using some number of variables in some textbook axiomatization of first order logic with equality. Let n be the least number of variables that occur in any proof of T. Then T requires n variables to prove, in the sense that no smaller number of variables will suffice; in particular, T has no proof that uses only $n-1$ variables. It has been known for some time that there are 3-sentences that require arbitrarily large numbers of variables to prove, that is, for every integer n, there is a 3-sentence that requires more than n variables to prove. Monk [21, Th 1.11] proved an algebraic formulation of this result in the course of showing that the class of n-dimensional cylindric algebras is not finitely axiomatizable whenever $n \geq 3$. His proof involves the construction of cylindric algebras that were modified into nonrepresentable polyadic algebras by Johnson [11]. Monk [22] outlined the connections between the algebra and logic, and applied one of Johnson's results to show that no finite set of axiom schemata for first-order logic with equality and only finitely many variables is both sound and complete. Monk [21, §4] made a conjecture about cylindric algebras. In logical terms, Monk's conjecture was that for every $n \geq 3$ there is a 3-sentence that requires n variables to prove. Both the algebraic and logical versions of the conjecture are confirmed in [10]. It is perhaps an interesting historical fact that the algebras constructed already by Monk [21] can be used to do so. These algebras cannot, however, be used to obtain similar results for languages with a fixed, finite number of binary relation symbols. The reason is that the number of generators of one of Monk's algebras depends on the size of the algebra; larger algebras require more generators. The same feature afflicts both the nonrepresentable relation algebras that were constructed from projective geometries by Lyndon [13] and used by Monk [19] to show that the class of representable relation algebras is not finitely axiomatizable, and also the corresponding 3-dimensional cylindric algebras that were used by Monk [20] to show that the class of 3-dimensional cylindric algebras is not finitely axiomatizable.

The first-order language $\mathcal{L}$ used in [28] has an equality symbol and exactly one binary relation symbol, the appropriate choice for the development of set theory, in which the sole nonlogical predicate is the membership relation. To prove for this language that there are 3-sentences that require arbitrarily large numbers of variables to prove, it was necessary to modify Monk's algebras into ones that are generated by a single element [17, §4]. This introduces complexity into the construction; the automorphism group goes from large to trivial. Similarly modified versions of Monk's algebras are used in [10] to show that for every $n \geq 3$ there is a 3-sentence in $\mathcal{L}$ that requires n variables to prove, thus solving a problem posed by Tarski and Givant [28, p 93], [4, p 208], [23, p 735].

Here we replicate this result and its proof entirely within the realm of first-order logic. The structure of the modified algebras shows up in the 3-sentences. Our proof proceeds as follows. In §2, we recall a finite Ramsey theorem whose formalization in first-order logic will provide an upper bound on the number of variables required to prove our 3-sentences. The n-variable fragments of first-order logic, and their proof theory, are defined formally in §3 and §4. Some proof-theoretic lemmas are given in §5. The main work begins in §6, where the 'Ramsey 3-sentence' (for each $n \geq 4$) is specified and shown (in §7) to be provable with n variables. In §8, we show that it is not provable with fewer variables by showing that it is not valid in a certain unorthodox semantics for $(n-1)$-variable logic that validates theorems provable with $n-1$ variables. Finally, in §9, we obtain a similar sentence written with only one binary relation symbol.

We believe most of this paper can be read as if it were a chapter in a (fairly terse and sophisticated) text on first-order logic with equality.

§2. A Ramsey theorem. Define $\rho : \omega \to \omega$ by $\rho(0) = 1$ and $\rho(k+1) = 1 + (k+1) \cdot \rho(k)$ for every $k < \omega$. This function gives an upper bound for a certain Ramsey number [7], [6, p 6], [8, Cor 3], [2, p 440–443]. Its first five values are, respectively, 1, 2, 5, 16, and 65. For any set X, let $[X]^2$ be the set of 2-element subsets of X:

$$[X]^2 = \{\{x, y\} : x, y \in X,\ x \neq y\}.$$

For $1 \leq k < \omega$, a *k-coloring of X* is a partition of $[X]^2$ into k pieces (called 'colors') so that no 'monochromatic triangle' appears, that is, there is no 3-element subset of X with all three of its 2-element subsets in the same piece of the partition.

PROPOSITION 2.1. *Let $k \geq 1$. If a set X has more than $\rho(k)$ elements then X has no k-coloring.*

PROOF. Here is the standard proof. First case: $k = 1$. No 1-coloring can exist if X has more than $\rho(1) = 2$ elements, because every 3-element subset is a monochromatic triangle for every 1-coloring. Next, we assume the result holds for a particular $k \geq 1$ and prove it for $k+1$. Assume X has a $(k+1)$-coloring. Choose such a coloring. Fix one element $x \in X$, and define an equivalence relation $\sim$ on $X \setminus \{x\}$ by letting $y \sim y'$ if and only if $\{x, y\}$ and $\{x, y'\}$ have the same color in the chosen coloring. For distinct $y \sim y'$ in $X \setminus \{x\}$, if $\{x, y\}$ and $\{x, y'\}$ belong to color c, say, then $\{y, y'\}$ cannot also belong to c, else $\{x, y, y'\}$ is monochromatic. Hence we can partition $X \setminus \{x\}$ into at most $k+1$ parts, given by the $\sim$-classes, and within each part only k colors are used. Thus, each of the parts has a k-coloring and inductively has size not exceeding $\rho(k)$. So $|X| \leq 1 + (k+1) \cdot \rho(k) = \rho(k+1)$. ⊣

Partitions of $[X]^2$ can be treated in first-order logic by means of symmetric binary relations on X. In a first-order language with binary relation symbols and equality, it is possible with 2-sentences to assert that the relations (denoted by the relation symbols) are symmetric, pairwise disjoint, and their union is the diversity relation (that holds between any two distinct elements). That monochromatic triangles do not occur can be expressed by 3-sentences, such as $\forall_i \forall_j \forall_k \neg\,(a_{ij} \wedge a_{ik} \wedge a_{kj})$. Thus, for any $k \geq 1$, there is a single 3-sentence that is satisfiable in X just in case some set of k binary relations on X forms a partition of the diversity relation on X into disjoint symmetric binary relations with no monochromatic triangles—in short, that there is a k-coloring of X. By Proposition 2.1, such a sentence has no model of cardinality greater than $\rho(k)$. By considering more complicated types of partitions, we can further assert with only 3 variables that there are more than $\rho(k)$ elements. A 3-sentence obtained this way, asserting the existence of a k-coloring on a set with more than $\rho(k)$ elements, has no model, so its negation is logically valid and provable. It is easy to see that it can be proved with $1+\rho(k)$ variables (Remark 6.2 below gives a corresponding semantic argument). The aim of the next few sections is to get this number down, by formalizing the inductive proof of Proposition 2.1.

§3. Relational languages. Let $\mathcal{L}$ be a first order language with equality symbol $=$, any set of binary relation symbols, no function symbols, no constants, and variables v_i for $i < \omega$. For every $n \leq \omega$ let

$$\mathsf{Var}_n = \{v_i : i < n\}.$$

Let Rel be the set of relation symbols of $\mathcal{L}$. (The equality symbol $=$ is not in Rel.) Let atomic_n be the set of atomic formulæ of $\mathcal{L}$ written with variables from Var_n. Thus

$$\mathsf{atomic}_n = \{v_i{=}v_j,\, v_i R v_j : R \in \mathsf{Rel},\ i, j < n\}.$$

The only propositional connectives in $\mathcal{L}$ are $\neg$ and $\Rightarrow$, and the only quantifier is $\forall$. The other connectives and the existential quantifier are introduced as abbreviations in the usual way. We will need to use conjunctions and disjunctions of formulæ indexed by various finite index sets. The order in which the formulæ appear is assumed to be determined by some standard ordering of the index set. Parentheses are restored by association to the left. For example,

$$\bigvee_{i \leq k} \varphi_i = (\cdots((\varphi_0 \vee \varphi_1) \vee \varphi_2) \vee \cdots \vee \varphi_{k-1}) \vee \varphi_k.$$

At one point (Lemma 7.4) we encounter a disjunction over a possibly empty index set, in which case we let $\bigvee_{i\in\emptyset} \varphi_i = \bot$, where $\bot = \neg\forall_{v_0}(v_0{=}v_0)$. Similarly, $\bigwedge_{i\in\emptyset} \varphi_i = \top$, where $\top = \forall_{v_0}(v_0{=}v_0)$.

For an $\mathcal{L}$-formula φ, let $\mathsf{free}(\varphi)$ be the set of variables that occur free in φ, defined for any $R \in \mathsf{Rel}$, $i, j < n$, and formulæ φ, ψ, by:

$$\begin{aligned}
\mathsf{free}(v_i{=}v_j) &= \{v_i, v_j\},\\
\mathsf{free}(v_i R v_j) &= \{v_i, v_j\},\\
\mathsf{free}(\varphi \Rightarrow \psi) &= \mathsf{free}(\varphi) \cup \mathsf{free}(\psi),\\
\mathsf{free}(\neg\varphi) &= \mathsf{free}(\varphi),\\
\mathsf{free}(\forall_{v_i}\varphi) &= \mathsf{free}(\varphi) \setminus \{v_i\}.
\end{aligned}$$

A formula φ is a **sentence** if $\mathsf{free}(\varphi)=\emptyset$. For every $n\leq\omega$ let Fm_n be the set of formulæ of $\mathcal{L}$ that contain only variables in Var_n, *i.e.*,

$$\begin{aligned}\mathsf{Fm}_n = \bigcap\{\, X : \mathsf{atomic}_n \subseteq X,\ \ &\text{if } \varphi\in X,\ \psi\in X, \text{ and } x\in\mathsf{Var}_n,\\ &\text{then } \neg\varphi\in X,\ \varphi\Rightarrow\psi\in X, \text{ and } \forall_x\varphi\in X\,\}.\end{aligned}$$

We say that φ is an n**-formula** if $\varphi\in\mathsf{Fm}_n$, and an n**-sentence** if $\varphi\in\mathsf{Fm}_n$ and $\mathsf{free}(\varphi)=\emptyset$.

§4. **Axiomatization.** We now develop the proof theory of $\mathcal{L}$. For every $\varphi\in\mathsf{Fm}_\omega$ and all $i,j<\omega$ let $\mathsf{S}_{ij}\varphi$ be the result of interchanging v_i and v_j, that is, simultaneously replacing every occurrence of v_i in φ with v_j and every occurrence of v_j in φ with v_i. For example, if $R\in\mathsf{Rel}$ then $\mathsf{S}_{01}(\forall_{v_0}\exists_{v_1}v_0Rv_1) = \forall_{v_1}\exists_{v_0}v_1Rv_0$ and $\mathsf{S}_{02}(\forall_{v_0}\exists_{v_1}v_0Rv_1) = \forall_{v_2}\exists_{v_1}v_2Rv_1$. Note that if $i,j<n$ and $\varphi\in\mathsf{Fm}_n$, then $\mathsf{S}_{ij}\varphi\in\mathsf{Fm}_n$ as well. A **tautology** is a formula which is mapped to 1 (truth) by the canonical extension to all formulæ of any map from atomic and universally quantified formulæ to the two-element set $\{1,0\}$. For every $n\leq\omega$, let Ax_n be the set of all formulæ of the following types.

Name	Axiom	Restrictions
(A1)	φ	$\varphi\in\mathsf{Fm}_n$ and φ is a tautology
(A2)	$\forall_{v_i}(\varphi\Rightarrow\psi)\Rightarrow(\varphi\Rightarrow\forall_{v_i}\psi)$	$\varphi,\psi\in\mathsf{Fm}_n$, $i<n$, and $v_i\notin\mathsf{free}(\varphi)$
(A3)	$\forall_{v_i}\varphi\Rightarrow\varphi$	$\varphi\in\mathsf{Fm}_n$ and $i<n$
(A4)	$\exists_{v_i}(v_i{=}v_j)$	$i,j<n$ and $i\neq j$
(A5)	$(v_i{=}v_j)\Rightarrow(\varphi\Rightarrow\psi)$	$\varphi,\psi\in\mathsf{atomic}_n$, $i,j<n$, and ψ is obtained from φ by replacing some occurrence of v_i in φ by v_j
(A6)	$(v_i{=}v_j)\Rightarrow(\varphi\Rightarrow\mathsf{S}_{ij}\varphi)$	$\varphi\in\mathsf{Fm}_n$ and $i,j<n$

The rules of inference are *modus ponens* (infer ψ from φ and $\varphi\Rightarrow\psi$) and n-generalization (infer $\forall_x\varphi$ from φ, for any $x\in\mathsf{Var}_n$). These rules can be displayed in a more graphical style as follows.

$$\frac{\varphi\quad\varphi\Rightarrow\psi}{\psi}\qquad\qquad\frac{\varphi}{\forall_x\varphi}$$

A formula $\varphi\in\mathsf{Fm}_\omega$ is n**-provable**, in symbols $\vdash_n\varphi$, if φ belongs to every set of formulæ that contains Ax_n and is closed under *modus ponens* and n-generalization. As Fm_n is such a set, if $\vdash_n\varphi$ then $\varphi\in\mathsf{Fm}_n$. By an n**-proof of** φ we mean a finite sequence of formulæ in Fm_n whose last element is φ, having the property that every formula in the sequence is either a member of Ax_n or follows from one or two previous formulæ by n-generalization or *modus ponens*. Evidently, φ is n-provable iff there exists an n-proof of φ. Two formulæ $\varphi, \psi \in \mathsf{Fm}_n$ are **n-equivalent**, in symbols $\varphi \equiv_n \psi$, iff $\vdash_n \varphi \Leftrightarrow \psi$.

The axiom set (A1)–(A6) is based on the axiom set Σ_2 of Tarski [27]. Σ_2 is defined by eight axiom schemata, (B1) through (B8). Tarski's axioms (B1)–(B3) are tautologies from which every other tautology can be obtained by *modus ponens*. Axiom (A1) is our replacement for (B1)–(B3). Axiom (A2) substitutes for the following two axioms (see Lemmas 5.5 and 5.4 below):

(B4) $\forall_x(\varphi \Rightarrow \psi) \Rightarrow (\forall_x\varphi \Rightarrow \forall_x\psi)$,

(B6) $\varphi \Rightarrow \forall_x\varphi$ for $x \notin \mathsf{free}(\varphi)$.

Tarski's (B5), (B7), and (B8) coincide with our (A3), (A4), and (A5), respectively. Axiom (A6) can be derived from (A1)–(A5) when $n = \omega$. However, proofs of (A6) (and related schemata) usually involve the introduction of variables that do not occur in φ. This may not be possible when n is finite since every variable in the language can then occur in a single formula. Therefore, (A6) has been explicitly included in Ax_n.

Axioms (A1)–(A6) are sound with respect to the usual semantics: all formulæ in Ax_n are logically valid (true in all models under all assignments to their free variables). The rules of inference preserve logical validity. It follows that if $\vdash_n \varphi$ then φ is logically valid. To prove the converse when $n = \omega$, it suffices to observe that Σ_2 is complete [27, Th 5], and that every formula in Σ_2 is ω-provable. On the other hand, as was proved by Monk [22], completeness fails when $n < \omega$.

We say that a formula **requires n variables to prove** if it *can* be proved with n variables, but *cannot* be proved with only $n-1$ variables, *i.e.*, $\vdash_n \varphi$ and $\nvdash_{n-1} \varphi$. Evidently, any tautology in $\mathsf{Fm}_n \setminus \mathsf{Fm}_{n-1}$ requires n variables to prove. A more interesting challenge is to find 3-formulæ in Fm_3 that require n variables to prove.

§5. Provability.

The results on provability in this section will be used later. Here are three lemmata that hold for every $n \leq \omega$. Their proofs, which we omit, use only axioms (A1)–(A3) and the two inference rules. None of the equality axioms are involved.

LEMMA 5.1. *If $\varphi \in \mathsf{Ax}_n$ then $\vdash_n \varphi$.*

LEMMA 5.2. *If $\vdash_n \varphi$ and $i < n$ then $\vdash_n \forall_{v_i}\varphi$.*

LEMMA 5.3. 1. *If $\vdash_n \varphi$ and $\vdash_n \varphi \Rightarrow \psi$, then $\vdash_n \psi$. In particular, if $\vdash_n \varphi$ and $\varphi \Rightarrow \psi \in \mathsf{Ax}_n$, then $\vdash_n \psi$.*

2. *If $\vdash_n \varphi_0, \ldots, \vdash_n \varphi_k$ and $(\bigwedge_{i\leq k} \varphi_i) \Rightarrow \psi$ is a tautology in Fm_n, then $\vdash_n \psi$.*

Lemma 5.3 implies that $\equiv_n$ is an equivalence relation on Fm_n, and that $\varphi \equiv_n \psi$ iff $\vdash_n \varphi \Rightarrow \psi$ and $\vdash_n \psi \Rightarrow \varphi$.

LEMMA 5.4. *If $\varphi \in \mathsf{Fm}_n$, $i < n$, and $v_i \notin \mathsf{free}(\varphi)$, then $\vdash_n \varphi \Rightarrow \forall_{v_i}\varphi$ and $\vdash_n \exists_{v_i}\varphi \Rightarrow \varphi$.*

PROOF. By (A1) and Lemma 5.1, $\vdash_n \varphi \Rightarrow \varphi$. By Lemma 5.2, $\vdash_n \forall_{v_i}(\varphi \Rightarrow \varphi)$. By (A2) and Lemma 5.1, $\vdash_n \forall_{v_i}(\varphi \Rightarrow \varphi) \Rightarrow (\varphi \Rightarrow \forall_{v_i}\varphi)$. So by Lemma 5.3, $\vdash_n \varphi \Rightarrow \forall_{v_i}\varphi$. As $v_i \notin \mathsf{free}(\neg\varphi)$, this also gives $\vdash_n \neg\varphi \Rightarrow \forall_{v_i}\neg\varphi$. By Lemma 5.3, we obtain $\vdash_n \neg\forall_{v_i}\neg\varphi \Rightarrow \varphi$, *i.e.* $\vdash_n \exists_{v_i}\varphi \Rightarrow \varphi$. $\dashv$

LEMMA 5.5. *If $\varphi, \psi \in \mathsf{Fm}_n$ and $i < n$ then*

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\forall_{v_i}\varphi \Rightarrow \forall_{v_i}\psi).$$

PROOF. By (A3) and Lemma 5.1,

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi) \text{ and } \vdash_n \forall_{v_i}\varphi \Rightarrow \varphi.$$

So by Lemma 5.3,

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\forall_{v_i}\varphi \Rightarrow \psi).$$

By Lemma 5.2,

$$\vdash_n \forall_{v_i}(\forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\forall_{v_i}\varphi \Rightarrow \psi)).$$

Since v_i is not free in $\forall_{v_i}(\varphi \Rightarrow \psi)$, we can use (A2) and Lemma 5.3 to get

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow \forall_{v_i}(\forall_{v_i}\varphi \Rightarrow \psi).$$

By (A2) and Lemma 5.1, $\vdash_n \forall_{v_i}(\forall_{v_i}\varphi \Rightarrow \psi) \Rightarrow (\forall_{v_i}\varphi \Rightarrow \forall_{v_i}\psi)$. By Lemma 5.3, we obtain

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\forall_{v_i}\varphi \Rightarrow \forall_{v_i}\psi),$$

proving the lemma. $\dashv$

LEMMA 5.6. *Assume that $\varphi, \varphi', \psi \in \mathsf{Fm}_n$ and that ψ' is the result of replacing an occurrence of φ in ψ by φ'. If $\varphi \equiv_n \varphi'$, then $\psi \equiv_n \psi'$. Hence, if $\vdash_n \psi$ then $\vdash_n \psi'$.*

PROOF. By induction on ψ. If ψ is φ, the result is clear. Inductively assume the result for ψ, so that $\psi \equiv_n \psi'$, *i.e.* $\vdash_n \psi \Leftrightarrow \psi'$. For $\chi \in \mathsf{Fm}_n$, consider $\psi \Rightarrow \chi$. By (A1) and Lemma 5.1, $\vdash_n (\psi \Leftrightarrow \psi') \Rightarrow ((\psi \Rightarrow \chi) \Leftrightarrow (\psi' \Rightarrow \chi))$. By Lemma 5.3, $\vdash_n (\psi \Rightarrow \chi) \Leftrightarrow (\psi' \Rightarrow \chi)$, so that $\psi \Rightarrow \chi \equiv_n \psi' \Rightarrow \chi$. The cases $\chi \Rightarrow \psi$ and $\neg\psi$ are similar. Finally, consider $\forall_{v_i}\psi$. We have $\vdash_n \psi \Rightarrow \psi'$. Lemma 5.2 gives $\vdash_n \forall_{v_i}(\psi \Rightarrow \psi')$, and Lemmas 5.5 and 5.3 give $\vdash_n \forall_{v_i}\psi \Rightarrow \forall_{v_i}\psi'$. Similarly, $\vdash_n \forall_{v_i}\psi' \Rightarrow \forall_{v_i}\psi$, giving $\forall_{v_i}\psi \equiv_n \forall_{v_i}\psi'$ via Lemma 5.3, as required. $\dashv$

LEMMA 5.7. *If $\varphi, \psi \in \mathsf{Fm}_n$ and $i < n$ then*

$$\vdash_n \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow (\exists_{v_i}\varphi \Rightarrow \exists_{v_i}\psi).$$

PROOF. Observe that Lemmas 5.3, 5.5, and 5.6 give

$$\begin{aligned}
&\vdash_n\ \forall_{v_i}(\varphi \Rightarrow \psi) \Rightarrow \forall_{v_i}(\neg\psi \Rightarrow \neg\varphi),\\
&\vdash_n\ \forall_{v_i}(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\forall_{v_i}\neg\psi \Rightarrow \forall_{v_i}\neg\varphi),\\
&\vdash_n\ (\forall_{v_i}\neg\psi \Rightarrow \forall_{v_i}\neg\varphi) \Rightarrow (\neg\forall_{v_i}\neg\varphi \Rightarrow \neg\forall_{v_i}\neg\psi).
\end{aligned}$$

The lemma now follows by Lemma 5.3. ⊣

We will find the following 'monotonicity' lemma handy.

LEMMA 5.8. *Assume* $\varphi_i, \psi_i, \chi \in \mathsf{Fm}_n$ *and* $\vdash_n \varphi_i \Rightarrow \psi_i$ *for each* $i \in I$. *Then* $\vdash_n \bigwedge_{i\in I}\varphi_i \Rightarrow \bigwedge_{i\in I}\psi_i$ *and* $\vdash_n \bigvee_{i\in I}\varphi_i \Rightarrow \bigvee_{i\in I}\psi_i$. *Furthermore, for each* $i \in I, j < n$, $\vdash_n \exists_{v_j}\varphi_i \Rightarrow \exists_{v_j}\psi_i$, *and if* $\vdash_n \chi \Rightarrow \varphi_i$ *then* $\vdash_n \chi \Rightarrow \psi_i$.

PROOF. All parts but the third follow immediately from Lemma 5.3. For the third part, if $\vdash_n \varphi_i \Rightarrow \psi_i$ then Lemma 5.2 gives $\vdash_n \forall_{v_j}(\varphi_i \Rightarrow \psi_i)$, and Lemmas 5.7 and 5.3 give $\vdash_n \exists_{v_j}\varphi_i \Rightarrow \exists_{v_j}\psi_i$. ⊣

LEMMA 5.9. *Let* $i, j < n$ *and* $\varphi, \psi \in \mathsf{Fm}_n$. *Then*

$$\begin{aligned}
\exists_{v_i}(\varphi \vee \psi) &\equiv_n \exists_{v_i}\varphi \vee \exists_{v_i}\psi,\\
\exists_{v_j}(\varphi \wedge \psi) &\equiv_n \varphi \wedge \exists_{v_j}\psi \qquad \textit{if } v_j \notin \mathsf{free}(\varphi),\\
\forall_{v_i}(\varphi \wedge \psi) &\equiv_n \forall_{v_i}\varphi \wedge \forall_{v_i}\psi,\\
\forall_{v_j}(\varphi \vee \psi) &\equiv_n \varphi \vee \forall_{v_j}\psi \qquad \textit{if } v_j \notin \mathsf{free}(\varphi).
\end{aligned}$$

PROOF. By Lemmas 5.8 and 5.3, $\vdash_n (\exists_{v_i}\varphi \vee \exists_{v_i}\psi) \Rightarrow \exists_{v_i}(\varphi \vee \psi)$. For the converse, (A1) and Lemma 5.1 give $\vdash_n \neg\varphi \Rightarrow (\neg\psi \Rightarrow \neg(\neg\varphi \Rightarrow \psi))$. Lemma 5.2 yields $\vdash_n \forall_{v_i}(\neg\varphi \Rightarrow (\neg\psi \Rightarrow \neg(\neg\varphi \Rightarrow \psi)))$, and two applications of Lemmas 5.5 and 5.3 give us

$$\vdash_n \forall_{v_i}\neg\varphi \Rightarrow (\forall_{v_i}\neg\psi \Rightarrow \forall_{v_i}\neg(\neg\varphi \Rightarrow \psi)).$$

As $(p \Rightarrow (q \Rightarrow r)) \Rightarrow (\neg r \Rightarrow (\neg\neg p \Rightarrow \neg q))$ is a tautology, Lemma 5.3 now yields $\vdash_n \neg\forall_{v_i}\neg(\neg\varphi \Rightarrow \psi) \Rightarrow (\neg\neg\forall_{v_i}\neg\varphi \Rightarrow \neg\forall_{v_i}\neg\psi)$, which, abbreviated, is

$$\vdash_n \exists_{v_i}(\varphi \vee \psi) \Rightarrow (\exists_{v_i}\varphi \vee \exists_{v_i}\psi).$$

For the second part, (A3) and Lemma 5.1 give $\vdash_n \forall_{v_j}\neg\psi \Rightarrow \neg\psi$. An application of Lemma 5.3 yields $\vdash_n (\varphi \Rightarrow \forall_{v_j}\neg\psi) \Rightarrow (\varphi \Rightarrow \neg\psi)$, and by Lemma 5.2,

$$\vdash_n \forall_{v_j}((\varphi \Rightarrow \forall_{v_j}\neg\psi) \Rightarrow (\varphi \Rightarrow \neg\psi)).$$

As v_j is not free in $\varphi \Rightarrow \forall_{v_j}\neg\psi$, (A2) and Lemmas 5.1 and 5.3 give us

$$\vdash_n (\varphi \Rightarrow \forall_{v_j}\neg\psi) \Rightarrow \forall_{v_j}(\varphi \Rightarrow \neg\psi).$$

(A2) and Lemma 5.1 immediately give the converse of this, so

$$(\varphi \Rightarrow \forall_{v_j}\neg\psi) \equiv_n \forall_{v_j}(\varphi \Rightarrow \neg\psi).$$

Lemmas 5.3 and 5.6 now yield

$$\neg(\varphi \Rightarrow \neg\neg\forall_{v_j}\neg\psi) \equiv_n \neg\forall_{v_j}\neg\neg(\varphi \Rightarrow \neg\psi),$$

that is, $\varphi \wedge \exists_{v_j}\psi \equiv_n \exists_{v_j}(\varphi \wedge \psi)$, as required. The last two statements follow from the first two by Lemmas 5.3 and 5.6. ⊣

Now we make use of the equality axioms. Note that we never use (A5).

LEMMA 5.10. *If $i, j < n$, $\varphi \in \mathsf{Fm}_n$, and $v_i, v_j \notin \mathsf{free}(\varphi)$, then $\varphi \equiv_n \mathsf{S}_{ij}\varphi$.*

PROOF. If $i = j$ then the result is obvious, so assume $i \neq j$. By (A6) and Lemma 5.1,

$$\vdash_n (v_i{=}v_j) \Rightarrow (\varphi \Rightarrow \mathsf{S}_{ij}\varphi).$$

By Lemma 5.2,

$$\vdash_n \forall_{v_i}\left((v_i{=}v_j) \Rightarrow (\varphi \Rightarrow \mathsf{S}_{ij}\varphi)\right).$$

By Lemmas 5.7 and 5.3,

$$\vdash_n \exists_{v_i}(v_i{=}v_j) \Rightarrow \exists_{v_i}(\varphi \Rightarrow \mathsf{S}_{ij}\varphi).$$

By (A4) and Lemmas 5.1 and 5.3,

$$\vdash_n \exists_{v_i}(\varphi \Rightarrow \mathsf{S}_{ij}\varphi).$$

Clearly, $v_i \notin \mathsf{free}(\varphi \Rightarrow \mathsf{S}_{ij}\varphi)$, so by Lemmas 5.4 and 5.3,

$$\vdash_n \varphi \Rightarrow \mathsf{S}_{ij}\varphi.$$

Similarly, $\vdash_n \neg\varphi \Rightarrow \neg\mathsf{S}_{ij}\varphi$, since $\neg\mathsf{S}_{ij}\varphi = \mathsf{S}_{ij}(\neg\varphi)$ and $v_i, v_j \notin \mathsf{free}(\neg\varphi)$. So, by Lemma 5.3, $\vdash_n \mathsf{S}_{ij}\varphi \Rightarrow \varphi$, whence $\varphi \equiv_n \mathsf{S}_{ij}\varphi$. ⊣

§6. A special language. Now we put restrictions on n and $\mathcal{L}$. For the remainder of the paper, fix $n \geq 4$ and $\beta > 2\rho(n-3)$ (here, ρ is as in §2), and let $\mathcal{L}$ be the first-order language with the following set of binary relation symbols:

$$\mathsf{Rel} = \{\mathsf{Id}, \mathsf{E}, \mathsf{P}_1, \ldots, \mathsf{P}_{n-3}, \mathsf{Q}_0, \ldots, \mathsf{Q}_{\beta-1}\}.$$

The smallest choices for n and β are $n = 4$ and $\beta = 1 + 2\rho(1) = 5$, and $\mathsf{Rel} = \{\mathsf{Id}, \mathsf{E}, \mathsf{P}_1, \mathsf{Q}_0, \ldots, \mathsf{Q}_4\}$. It can be checked that $\beta - 1 \geq n - 3$ for all $n \geq 4$. Partition $\mathsf{Rel}^3 = \mathsf{Rel} \times \mathsf{Rel} \times \mathsf{Rel}$ into two parts, as follows. A triple $\langle R, S, T\rangle \in \mathsf{Rel}^3$ is said to be **forbidden** if one of the following conditions holds:

(F1) $R = \mathsf{Id}$ and $S \neq T$, or $S = \mathsf{Id}$ and $R \neq T$, or $T = \mathsf{Id}$ and $R \neq S$,

(F2) $\{R, S, T\} \subseteq \{\mathsf{E}, \mathsf{Q}_0\}$,

(F3) $\{R, S, T\} \subseteq \{\mathsf{Q}_i : i < \beta\}$,

(F4) $\{R, S, T\} \subseteq \{\mathsf{P}_i, \mathsf{Q}_i\}$ for some i with $1 \leq i \leq n-3$,

(F5) $\{R, S, T\} = \{\mathsf{E}, \mathsf{Q}_i, \mathsf{Q}_j\}$ for some $i, j < \beta$ with $|i - j| > 1$.

Notice that any permutation of a forbidden triple is also forbidden. Any triple that is not forbidden is said to be **mandatory**. Let Mand be the set of mandatory triples, and let Forb be the set of forbidden triples. Then

$\mathsf{Rel}^3 = \mathsf{Mand} \cup \mathsf{Forb}$ and $\emptyset = \mathsf{Mand} \cap \mathsf{Forb}$. Now we define some special 3-sentences that make assertions about the binary relations denoted by the relation symbols of $\mathcal{L}$. They say, among other things, that the relations are symmetric and form a partition of the universal relation, and that Id is a congruence relation (an identity for relative multiplication from either side).

Any two objects are related by some relation:

$$\alpha_1 = \forall_{v_0}\forall_{v_1} \bigvee_{R \in \mathsf{Rel}} v_0 R v_1.$$

Forbidden triples do not occur:

$$\alpha_2 = \bigwedge_{\langle R,S,T\rangle \in \mathsf{Forb}} \forall_{v_0}\forall_{v_1}\forall_{v_2} \neg\Big(v_0 R v_1 \wedge v_0 S v_2 \wedge v_2 T v_1\Big).$$

Mandatory triples must occur wherever possible:

$$\alpha_3 = \bigwedge_{\langle R,S,T\rangle \in \mathsf{Mand}} \forall_{v_0}\forall_{v_1} \Big(v_0 R v_1 \Rightarrow \exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1)\Big).$$

Now let

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

Of course, α depends implicitly on n. α_2 can be expressed in a form similar to α_3. By Lemmas 5.1, 5.3, 5.6, and 5.9,

$$\alpha_2 \equiv_3 \bigwedge_{\langle R,S,T\rangle \in \mathsf{Forb}} \forall_{v_0}\forall_{v_1} \Big(v_0 R v_1 \Rightarrow \neg\exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1)\Big).$$

Using only Lemmas 5.1, 5.3, 5.6, and 5.9, we can derive useful but longer alternative formulations of these sentences.

$$\alpha_1 \equiv_3 \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_0}\forall_{v_1} \Big(\Big(\bigvee_{\langle R,S,T\rangle \in \mathsf{Forb}} v_0 R v_1\Big) \vee \Big(\bigvee_{\langle R,S,T\rangle \in \mathsf{Mand}} v_0 R v_1\Big)\Big)$$

$$\alpha_2 \equiv_3 \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_0}\forall_{v_1} \Big(\Big(\bigvee_{\langle R,S,T\rangle \in \mathsf{Forb}} v_0 R v_1\Big) \Rightarrow \neg\exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1)\Big)$$

$$\alpha_3 \equiv_3 \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_0}\forall_{v_1} \Big(\Big(\bigvee_{\langle R,S,T\rangle \in \mathsf{Mand}} v_0 R v_1\Big) \Rightarrow \exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1)\Big)$$

$$\alpha \equiv_3 \alpha_1 \wedge \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_0}\forall_{v_1} \Big(\neg\exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1) \Leftrightarrow \bigvee_{\langle R,S,T\rangle \in \mathsf{Forb}} v_0 R v_1\Big)$$

(13)

$$\alpha \equiv_3 \alpha_1 \wedge \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_0}\forall_{v_1} \Big(\exists_{v_2}\,(v_0 S v_2 \wedge v_2 T v_1) \Leftrightarrow \bigvee_{\langle R,S,T\rangle \in \mathsf{Mand}} v_0 R v_1\Big).$$

These conclusions follow only from the assumption that Mand and Forb partition Rel^3; their actual structure has not mattered, but it does in the next lemma. Let

$$\alpha_4 = \forall_{v_0}\exists_{v_1} v_0 \mathsf{Id} v_1.$$

LEMMA 6.1. $\vdash_3 \alpha_1 \wedge \alpha_3 \Rightarrow \alpha_4$.

PROOF. Let $R \in \mathsf{Rel}$. Since $\langle R, \mathsf{Id}, R\rangle \in \mathsf{Mand}$, we apply Lemma 5.3 to get $\vdash_3 \alpha_3 \Rightarrow \forall_{v_0}\forall_{v_1}[v_0 R v_1 \Rightarrow \exists_{v_2}(v_0 \mathsf{Id} v_2 \wedge v_2 R v_1)]$. By axiom (A3) and Lemma 5.3, $\vdash_3 \alpha_3 \Rightarrow [v_0 R v_1 \Rightarrow \exists_{v_2}(v_0 \mathsf{Id} v_2 \wedge v_2 R v_1)]$. As $\vdash_3 v_0 \mathsf{Id} v_2 \wedge v_2 R v_1 \Rightarrow v_0 \mathsf{Id} v_2$, Lemmas 5.8 and 5.3 give $\vdash_3 \alpha_3 \Rightarrow [v_0 R v_1 \Rightarrow \exists_{v_2} v_0 \mathsf{Id} v_2]$. By Lemma 5.3, we obtain $\vdash_3 \alpha_3 \Rightarrow \bigwedge_{R\in\mathsf{Rel}}[v_0 R v_1 \Rightarrow \exists_{v_2} v_0 \mathsf{Id} v_2]$.

Now by (A3) and Lemma 5.3 again, $\vdash_3 \alpha_1 \Rightarrow \bigvee_{R\in\mathsf{Rel}} v_0 R v_1$. Hence by Lemma 5.3, $\vdash_3 (\alpha_1 \wedge \alpha_3) \Rightarrow \exists_{v_2} v_0 \mathsf{Id} v_2$. Lemma 5.10 shows that $\exists_{v_2} v_0 \mathsf{Id} v_2 \equiv_3 \exists_{v_1} v_0 \mathsf{Id} v_1$. So by Lemma 5.6, $\vdash_3 (\alpha_1 \wedge \alpha_3) \Rightarrow \exists_{v_1} v_0 \mathsf{Id} v_1$. By Lemma 5.2, $\vdash_3 \forall_{v_0}((\alpha_1 \wedge \alpha_3) \Rightarrow \exists_{v_1} v_0 \mathsf{Id} v_1)$, and by (A2) and Lemma 5.3, $\vdash_3 (\alpha_1 \wedge \alpha_3) \Rightarrow \forall_{v_0}\exists_{v_1} v_0 \mathsf{Id} v_1$, as required. $\dashv$

REMARK 6.2. It is not hard to show that the 3-sentence α is unsatisfiable. Suppose, to the contrary, that α had a model $\mathcal{M}$, and pick an element a of $\mathcal{M}$. By Lemma 6.1, $\mathcal{M} \models a\ \mathsf{Id}\ b$ for some b. Let $\Sigma = \{i < \beta : i \text{ is even}\}$. As $\mathcal{M} \models \alpha_3$, the fact that $\langle \mathsf{Id}, \mathsf{Q}_i, \mathsf{Q}_i\rangle$ is a mandatory triple for all $i \in \Sigma$ forces $\mathcal{M}$ to contain points c_i ($i \in \Sigma$) such that $\mathcal{M} \models a\ \mathsf{Q}_i\ c_i \wedge c_i\ \mathsf{Q}_i\ b$. Let $i < j$ in Σ. By α_1, $\mathcal{M} \models c_i\ R\ c_j$ for some $R \in \mathsf{Rel}$. Choose such an R. So $\mathcal{M} \models a\ \mathsf{Q}_j\ c_j \wedge a\ \mathsf{Q}_i\ c_i \wedge c_i\ R\ c_j$. From α_2 and the definition of Forb, we see that $R \neq \mathsf{Id}$ since $\mathsf{Q}_i \neq \mathsf{Q}_j$, $R \neq \mathsf{E}$ since $|i-j| > 1$, and $R \neq \mathsf{Q}_k$ for every $k < \beta$ since $\langle \mathsf{Q}_i, \mathsf{Q}_j, \mathsf{Q}_k\rangle$ is forbidden. Hence, $\mathcal{M} \models c_i\ \mathsf{P}_l\ c_j$ for some l with $1 \leq l \leq n-3$. But, by the choice of β, $|\Sigma| > \rho(n-3)$, so by Proposition 2.1, there is no $(n-3)$-coloring of Σ. There must therefore be $i < j < k$ in Σ such that $\mathcal{M} \models c_i\ \mathsf{P}_l\ c_k \wedge c_i\ \mathsf{P}_l\ c_j \wedge c_j\ \mathsf{P}_l\ c_k$ for some l. However, this contradicts α_2, since $\langle \mathsf{P}_l, \mathsf{P}_l, \mathsf{P}_l\rangle$ is forbidden. Hence α has no model.

This argument can be easily formalized to give a proof of $\neg\alpha$ using $2 + |\Sigma|$ variables.

REMARK 6.3. We can translate $\neg\alpha$ into an equivalent equation in the calculus of relations in the way described in the introduction. (Of course, the preceding remark shows that $0 = 0$ is also equivalent to $\neg\alpha$.) α_1 states that $\sum_{R\in\mathsf{Rel}} R = 1$, or equivalently that $\prod_{R\in\mathsf{Rel}} \overline{R} = 0$. α_2 states that $R \cdot (S;T) = 0$ whenever $\langle R, S, T\rangle \in \mathsf{Forb}$. α_3 states that $R \cdot \overline{S;T} = 0$ whenever $\langle R, S, T\rangle \in \mathsf{Mand}$. For all $R, S, T \in \mathsf{Rel}$, let

$$\tau(R,S,T) \ = \ \begin{cases} R \cdot S;T & \text{if } \langle R,S,T\rangle \in \mathsf{Forb}, \\ R \cdot \overline{S;T} & \text{if } \langle R,S,T\rangle \in \mathsf{Mand}. \end{cases}$$

So α states that $(\prod_{R\in\mathsf{Rel}}\overline{R}) + (\sum_{R,S,T\in\mathsf{Rel}}\tau(R,S,T)) = 0$. The desired equation equivalent to $\neg\alpha$ is therefore

$$1;\Big(\Big(\prod_{R\in\mathsf{Rel}}\overline{R}\Big)+\Big(\sum_{R,S,T\in\mathsf{Rel}}\tau(R,S,T)\Big)\Big);1=1. \tag{14}$$

From (F1)–(F5) it can be shown that for every $R\in\mathsf{Rel}$ there is a term $t_R(x)$ in the calculus of relations such that if ψ is the elementary 3-variable translation of $t_R(\mathsf{E}) = R$ (cf. the introduction) then $\vdash_3 \alpha \Rightarrow \psi$. For example, $t_{\mathsf{Q}_0} = \overline{\overline{x};\overline{x}}\cdot\overline{x}$. Terms like this are used in §9 to eliminate all relation symbols other than E from α.

Remark 6.4. If we were only interested in 3-sentences that require n variables to prove, without regard for the number of binary relation symbols used in such 3-sentences, then we could simplify the definition of Forb and Mand, although these changes have almost no effect on the length of $\neg\alpha$ or the corresponding equation (14). First, delete E from the language. Conditions (F2) and (F5) in the definition of forbidden triples fall away. If α' is the resulting 3-sentence, then $\neg\alpha'$ still requires n variables to prove. Next, drop Q_i from (F4) and let α'' be the resulting 3-sentence. Again $\neg\alpha''$ requires n variables to prove. At this point the definition of forbidden and mandatory triples matches relation algebras that arise from Monk's cylindric algebras. These algebras can also be obtained by the process of splitting one atom [1, Ex 6], starting with the algebras called $\mathfrak{E}_k(\{2,3\})$ in [15, p 510]. These algebras are symmetric integral relation algebras whose forbidden triples are exactly the ones of the form $\langle a,a,a\rangle$, for every nonidentity atom a. Sentences similar to $\neg\alpha''$ and the corresponding equations (14) are given by Gordeev [5], where it is shown *inter alia* that for every $n\geq 4$ there is a 3-sentence that can be proved in a proof theory with equality and with n variables, but not with $n-2$. If we wish to construct the α that arises from the algebras $\mathfrak{E}_k(\{2,3\})$, we may achieve the same effect by deleting the relation symbols Q_i and condition (F3), leaving only Id, the symbols P_j, (F1), and the simplified (F4). It is not known, however, except for small n, whether $\neg\alpha$ has no model in this case [3], [23, Prob. 5].

We will show in the next section, by formalising the proof of Proposition 2.1, that the 3-sentence $\neg\alpha$ defined according to (F1)–(F5) is n-provable (Theorem 7.5). Then, in §8, we show that α is valid in certain $(n-1)$-dimensional, non-standard models; it follows (Theorem 8.9) that $\neg\alpha$ is not $(n-1)$-provable. In §9 we alter $\neg\alpha$ by replacing all atomic formulæ other than those containing E with formulæ containing only E. The resulting 3-sentences (for $n=4,5,\ldots$) still require n variables to prove, and the corresponding equations (14) can be used to continue the sequence begun with (1) and (12).

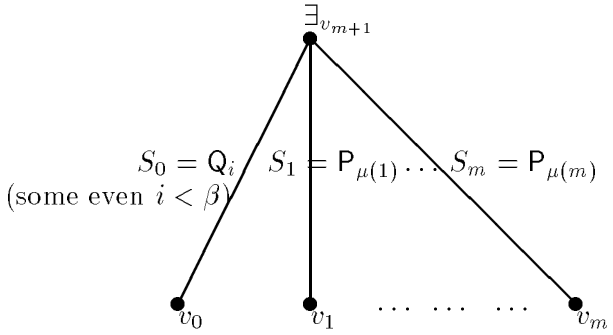

FIGURE 1. The conjunct of α_Σ corresponding to $S \in \Sigma$.

§7. $\neg\alpha$ is n-provable. To help the induction in the proof that $\vdash_n \neg\alpha$, we introduce a structure called a *spectrum*. A **0-spectrum** is a subset of $\{\mathsf{Q}_i : i < \beta,\ i \text{ even}\}$. If $1 \le m \le n-3$, we say that Σ is an m-**spectrum** with **color sequence** μ if μ is a one-to-one function $\mu : \{1, \dots, m\} \to \{1, \dots, n-3\}$, and

$$\begin{aligned}\Sigma &\subseteq \{\mathsf{Q}_i : i < \beta, i \text{ even}\} \times \{\mathsf{P}_{\mu(1)}\} \times \cdots \times \{\mathsf{P}_{\mu(m)}\} \\ &= \left\{\left\langle \mathsf{Q}_i, \mathsf{P}_{\mu(1)}, \dots, \mathsf{P}_{\mu(m)}\right\rangle : i < \beta, i \text{ even}\right\}.\end{aligned}$$

Distinct sequences S, S' in a spectrum have distinct first elements ($S_0 \neq S'_0$), but they agree on all other entries, if any ($S_i = S'_i$ for $1 \le i \le m$). For each m-spectrum Σ we also define a formula

$$\alpha_\Sigma = \bigwedge_{S \in \Sigma} \exists_{v_{m+1}} \bigwedge_{i < m+1} v_i S_i v_{m+1},$$

saying that for every sequence $S \in \Sigma$ there is v_{m+1} such that $v_0, \dots, v_m$ and v_{m+1} 'realize' S, as shown in Figure 1.

The following two propositions show that $\neg\alpha$ is n-provable.

PROPOSITION 7.1. *Let Σ be any 0-spectrum. Then $\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \alpha_\Sigma$.*

PROOF. Let $\mathsf{Q}_i \in \Sigma$. By (A1) and Lemma 5.1,

$$\vdash_n \alpha_3 \Rightarrow \forall_{v_0}\forall_{v_1}(v_0 \mathsf{Id} v_1 \Rightarrow \exists_{v_2}(v_0 \mathsf{Q}_i v_2 \wedge v_2 \mathsf{Q}_i v_1)).$$

By (A3) and Lemmas 5.1 and 5.3,

$$\begin{aligned}&\vdash_n \alpha_4 \Rightarrow \exists_{v_1}(v_0 \mathsf{Id} v_1),\\ &\vdash_n \alpha_3 \Rightarrow \forall_{v_1}(v_0 \mathsf{Id} v_1 \Rightarrow \exists_{v_2}(v_0 \mathsf{Q}_i v_2 \wedge v_2 \mathsf{Q}_i v_1)).\end{aligned}$$

By Lemmas 5.7 and 5.3,

$$\begin{aligned}&\vdash_n \alpha_3 \Rightarrow (\exists_{v_1}(v_0 \mathsf{Id} v_1) \Rightarrow \exists_{v_1}\exists_{v_2}(v_0 \mathsf{Q}_i v_2 \wedge v_2 \mathsf{Q}_i v_1)),\\ &\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \exists_{v_1}\exists_{v_2}(v_0 \mathsf{Q}_i v_2 \wedge v_2 \mathsf{Q}_i v_1).\end{aligned}$$

As $\vdash_n v_0 \mathsf{Q}_i v_2 \wedge v_2 \mathsf{Q}_i v_1 \Rightarrow v_0 \mathsf{Q}_i v_2$, by Lemmas 5.3 and 5.8 we get

$$\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \exists_{v_1}\exists_{v_2} v_0 \mathsf{Q}_i v_2.$$

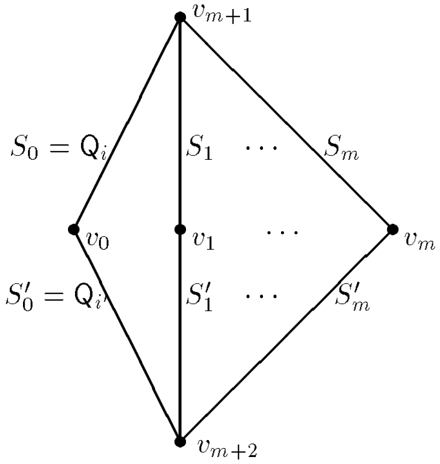

FIGURE 2. The meaning of $\psi(S, S')$.

By Lemma 5.4, $\vdash_n \exists_{v_1}\exists_{v_2} v_0\mathsf{Q}_i v_2 \Rightarrow \exists_{v_2} v_0\mathsf{Q}_i v_2$, so by Lemma 5.3,

$$\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \exists_{v_2} v_0\mathsf{Q}_i v_2.$$

But v_1, v_2 are not free in $\exists_{v_2} v_0\mathsf{Q}_i v_2$, so Lemma 5.10 gives $\vdash_n \exists_{v_2} v_0\mathsf{Q}_i v_2 \Rightarrow \exists_{v_1} v_0\mathsf{Q}_i v_1$. By Lemma 5.3,

$$\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \exists_{v_1} v_0\mathsf{Q}_i v_1.$$

This holds for each $\mathsf{Q}_i \in \Sigma$, so by Lemma 5.3 again we obtain $\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \alpha_\Sigma$, as required. $\dashv$

PROPOSITION 7.2. *Suppose that* $0 \leq m \leq n-3$ *and* Σ *is an* m*-spectrum with* $|\Sigma| > \rho(n-3-m)$. *Then* $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma$.

PROOF. The proof will proceed by downward induction on m, starting with $m = n-3$ and ending at $m = 0$. Therefore, we will need to construct an $(m+1)$-spectrum from an m-spectrum. The two preliminary lemmas 7.3 and 7.4 allow us to do this. For m, Σ as in the proposition, and $S, S' \in \Sigma$, let $\psi(S, S')$ be the following Fm_n-formula, illustrated in Figure 2:

$$\psi(S, S') = \bigwedge_{i<m+1} \left(v_i S_i v_{m+1} \wedge v_i S_i' v_{m+2} \right).$$

LEMMA 7.3. $\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma} \exists_{v_{m+1}} \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}} \psi(S, S')$.

PROOF. $\left(\bigwedge_{i\in I} p_i\right) \Leftrightarrow \left(\bigwedge_{i\in I}\left(p_i \wedge \bigwedge_{j\in I\setminus\{i\}} p_j\right)\right)$ is a tautology, so by (A1) and Lemma 5.1,

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma}\left(\left(\exists_{v_{m+1}} \bigwedge_{i<m+1} v_i S_i v_{m+1}\right) \wedge \left(\bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+1}} \bigwedge_{i<m+1} v_i S'_i v_{m+1}\right)\right).$$

By Lemma 5.10, we have

$$\bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+1}} \bigwedge_{i<m+1} v_i S'_i v_{m+1} \equiv_n \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}} \bigwedge_{i<m+1} v_i S'_i v_{m+2}.$$

So by Lemma 5.6, we obtain

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma}\left(\left(\exists_{v_{m+1}} \bigwedge_{i<m+1} v_i S_i v_{m+1}\right) \wedge \left(\bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}} \bigwedge_{i<m+1} v_i S'_i v_{m+2}\right)\right).$$

Distribute $\exists_{v_{m+1}}$ over the conjuncts in which v_{m+1} does not appear free (Lemmas 5.9 and 5.6):

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma} \exists_{v_{m+1}}\left(\left(\bigwedge_{i<m+1} v_i S_i v_{m+1}\right) \wedge \left(\bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}} \bigwedge_{i<m+1} v_i S'_i v_{m+2}\right)\right).$$

Distribute conjunctions over conjunctions, using (A1) and Lemmas 5.1 and 5.3:

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma} \exists_{v_{m+1}}\left(\bigwedge_{S'\in\Sigma\setminus\{S\}}\left(\left(\bigwedge_{i<m+1} v_i S_i v_{m+1}\right) \wedge \left(\exists_{v_{m+2}} \bigwedge_{i<m+1} v_i S'_i v_{m+2}\right)\right)\right).$$

Now distribute $\exists_{v_{m+2}}$ over the conjuncts in which v_{m+2} does not appear free, using Lemmas 5.9 and 5.6:

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma} \exists_{v_{m+1}} \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}}\left(\left(\bigwedge_{i<m+1} v_i S_i v_{m+1}\right) \wedge \left(\bigwedge_{i<m+1} v_i S'_i v_{m+2}\right)\right).$$

Use the associativity and commutativity of conjunction ((A1), Lemma 5.1, Lemma 5.6). Conclude that

$$\alpha_\Sigma \equiv_n \bigwedge_{S\in\Sigma} \exists_{v_{m+1}} \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}} \underbrace{\left(\bigwedge_{i<m+1}\left(v_i S_i v_{m+1} \wedge v_i S'_i v_{m+2}\right)\right)}_{\psi(S,S')},$$

as required. $\dashv$

Let Σ have color sequence $\mu : \{1,\ldots,m\} \to \{1,\ldots,n-3\}$, and let

$$\mathcal{T}_\mu = \{\mathsf{P}_j : 1 \le j \le n-3,\ j \notin \mathrm{rng}(\mu)\}, \tag{15}$$

where $\mathrm{rng}(\mu)$ is the range of μ. Note that $\mathcal{T}_\mu = \emptyset$ if $m = n-3$.

LEMMA 7.4. *For all distinct $S, S' \in \Sigma$,*

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \psi(S, S') \Rightarrow \bigvee_{R \in \mathcal{T}_\mu} \bigwedge_{i<m+1} \left(v_i S'_i v_{m+2} \wedge v_{m+1} R v_{m+2}\right).$$

PROOF. Let $S, S' \in \Sigma$ with $S \neq S'$. It follows by Lemma 5.10 that $\alpha_1 \equiv_n \forall_{v_{m+1}} \forall_{v_{m+2}} \bigvee_{R \in \mathsf{Rel}} v_{m+1} R v_{m+2}$. So by axiom (A3) and Lemma 5.3,

$$\vdash_n \alpha_1 \wedge \psi(S, S') \Rightarrow \Big(\bigvee_{R \in \mathsf{Rel}} v_{m+1} R v_{m+2} \Big) \wedge \bigwedge_{i<m+1} \left(v_i S_i v_{m+1} \wedge v_i S'_i v_{m+2}\right).$$

By Lemma 5.3,

(16)
$$\vdash_n \alpha_1 \wedge \psi(S, S') \Rightarrow \bigvee_{R \in \mathsf{Rel}} \bigwedge_{i<m+1} \left(v_i S'_i v_{m+2} \wedge v_i S_i v_{m+1} \wedge v_{m+1} R v_{m+2}\right).$$

Now using Lemma 5.10 to change the variables of α_2, and then (A3) and Lemma 5.3 to remove the outer quantifiers and irrelevant conjuncts, we obtain

(17) $\quad \vdash_n \alpha_2 \Rightarrow \neg(v_i P v_{m+2} \wedge v_i Q v_{m+1} \wedge v_{m+1} R v_{m+2})$
for any $i < m+1$, $\langle P, Q, R\rangle \in \mathsf{Forb}$.

From these last two statements we see that every conjunct in (16) of the form $v_i S'_i v_{m+2} \wedge v_i S_i v_{m+1} \wedge v_{m+1} R v_{m+2}$ is contradicted by α_2 according to (17) whenever $\langle S'_i, S_i, R\rangle \in \mathsf{Forb}$. Since $S_0, S'_0 \in \{\mathsf{Q}_i : i < \beta,\ i \text{ even}\}$ and $S_0 \neq S'_0$, the definition of Forb implies that $\langle S_0, S'_0, R\rangle \in \mathsf{Forb}$ whenever $R \in \{\mathsf{Id}, \mathsf{E}, \mathsf{Q}_i : i < \beta\}$. Also, for $1 \leq i \leq m$, $S_i = S'_i = \mathsf{P}_{\mu(i)}$ so $\langle S_i, S'_i, R\rangle \in \mathsf{Forb}$ if $R = \mathsf{P}_{\mu(i)}$. So (15), (17), and Lemma 5.3 give

(18) if $R \in \mathsf{Rel} \setminus \mathcal{T}_\mu$ then
$$\vdash_n \alpha_2 \Rightarrow \bigwedge_{i<m+1} \neg \left(v_i S'_i v_{m+2} \wedge v_i S_i v_{m+1} \wedge v_{m+1} R v_{m+2}\right).$$

Combining (16) and (18) by Lemma 5.3 gives

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \psi(S, S') \Rightarrow \bigvee_{R \in \mathcal{T}_\mu} \bigwedge_{i<m+1} \left(v_i S'_i v_{m+2} \wedge v_i S_i v_{m+1} \wedge v_{m+1} R v_{m+2}\right),$$

and Lemma 5.8 (or 5.3) now yields

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \psi(S, S') \Rightarrow \bigvee_{R \in \mathcal{T}_\mu} \bigwedge_{i<m+1} \left(v_i S'_i v_{m+2} \wedge v_{m+1} R v_{m+2}\right),$$

as required. If $\mathcal{T}_\mu = \emptyset$, this is $\vdash_n \alpha_1 \wedge \alpha_2 \wedge \psi(S, S') \Rightarrow \bot$. ⊣

The proof of Proposition 7.2 now proceeds by induction on $n - 3 - m$. The base case is $n - 3 - m = 0$. In the base case we have $n \geq 4$, $m = n - 3 \geq 1$, μ is surjective, $\rho(n - 3 - m) = 1$, and $|\Sigma| \geq 2$. It follows that there are distinct $S, S' \in \Sigma$ giving non-empty conjunctions in Lemma 7.3, and the disjunction over $\mathcal{T}_\mu$ in Lemma 7.4 is empty, so Lemmas 5.8 and 5.3 give us

$$\vdash_n \alpha_\Sigma \Rightarrow \exists_{v_{m+1}} \exists_{v_{m+2}} \psi(S, S'), \tag{19}$$

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \psi(S, S') \Rightarrow \bot. \tag{20}$$

Hence, in the usual way,

$$\begin{array}{ll}
\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\psi(S, S') & \text{by (20) and Lemma 5.3,} \\
\vdash_n \forall_{v_{m+2}}(\alpha_1 \wedge \alpha_2 \Rightarrow \neg\psi(S, S')) & \text{by Lemma 5.2,} \\
\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \forall_{v_{m+2}} \neg\psi(S, S') & \text{by (A2) and Lemma 5.3,} \\
\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\exists_{v_{m+2}} \psi(S, S') & \text{by Lemma 5.3,} \\
\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\exists_{v_{m+1}} \exists_{v_{m+2}} \psi(S, S') & \text{similarly,} \\
\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma & \text{by (19) and Lemma 5.3.}
\end{array}$$

This completes the proof of the base case.

Assume that $n - 3 - m > 0$. The inductive hypothesis is that $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma$ whenever Σ is an $(m + 1)$-spectrum of size larger than $\rho(n - 3 - (m + 1)) = \rho(n - 4 - m)$. We will show from this assumption that $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma$ whenever Σ is an m-spectrum of size greater than $\rho(n - 3 - m)$.

Fix $S \in \Sigma$. By Lemmas 7.3 and 5.3,

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \alpha_1 \wedge \alpha_2 \wedge \exists_{v_{m+1}} \bigwedge_{S' \in \Sigma \setminus \{S\}} \exists_{v_{m+2}} \psi(S, S').$$

Using Lemmas 5.3, 5.6, and 5.9, we obtain

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \exists_{v_{m+1}} \bigwedge_{S' \in \Sigma \setminus \{S\}} \exists_{v_{m+2}} (\alpha_1 \wedge \alpha_2 \wedge \psi(S, S')).$$

By Lemmas 7.4 and 5.8,

$$\begin{aligned}
&\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \\
&\qquad \exists_{v_{m+1}} \bigwedge_{S' \in \Sigma \setminus \{S\}} \exists_{v_{m+2}} \Big(\bigvee_{R \in \mathcal{T}_\mu} \bigwedge_{i < m+1} (v_i S'_i v_{m+2} \wedge v_{m+1} R v_{m+2}) \Big).
\end{aligned}$$

Distribute the existential quantifiers $\exists_{v_{m+2}}$ over disjunctions (Lemmas 5.9 and 5.6):

$$\begin{aligned}
&\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \\
&\qquad \exists_{v_{m+1}} \bigwedge_{S' \in \Sigma \setminus \{S\}} \bigvee_{R \in \mathcal{T}_\mu} \exists_{v_{m+2}} \Big(\bigwedge_{i < m+1} \Big(v_i S'_i v_{m+2} \wedge v_{m+1} R v_{m+2} \Big) \Big).
\end{aligned}$$

Then, in the critical step, distribute conjunctions over disjunctions using Lemma 5.6 and (A1) with the tautology

$$\Big(\bigwedge_{i\in I}\bigvee_{j\in J} p_{ij}\Big) \Leftrightarrow \Big(\bigvee_{g:I\to J}\bigwedge_{i\in I} p_{i,g(i)}\Big),$$

exponentially expanding the length of the formula:

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow$$
$$\exists_{v_{m+1}} \bigvee_{g:\Sigma\setminus\{S\}\to\mathcal{T}_\mu}\ \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}}\Big(\bigwedge_{i<m+1} v_i S'_i v_{m+2} \wedge v_{m+1} g(S') v_{m+2}\Big).$$

Finally, distribute $\exists_{v_{m+1}}$ over the resulting disjunctions (Lemmas 5.9 and 5.6):

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow$$
$$\bigvee_{g:\Sigma\setminus\{S\}\to\mathcal{T}_\mu} \exists_{v_{m+1}} \bigwedge_{S'\in\Sigma\setminus\{S\}} \exists_{v_{m+2}}\Big(\bigwedge_{i<m+1} v_i S'_i v_{m+2} \wedge v_{m+1} g(S') v_{m+2}\Big).$$

Now each function $g : \Sigma \setminus \{S\} \to \mathcal{T}_\mu$ induces a partition of $\Sigma \setminus \{S\}$ whose pieces are the preimages of the symbols in $\mathcal{T}_\mu$. The number of pieces is at most $|\mathcal{T}_\mu| = n - 3 - m$. For each g, let $\mathcal{P}(g)$ be one of the largest pieces. Restricting the conjunctions over $\Sigma \setminus \{S\}$ to $\mathcal{P}(g)$ we obtain, by Lemma 5.8,

(21) $$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow$$
$$\bigvee_{g:\Sigma\setminus\{S\}\to\mathcal{T}_\mu} \exists_{v_{m+1}} \bigwedge_{S'\in\mathcal{P}(g)} \exists_{v_{m+2}}\Big(\bigwedge_{i<m+1} v_i S'_i v_{m+2} \wedge v_{m+1} g(S') v_{m+2}\Big).$$

For each $g : \Sigma\setminus\{S\} \to \mathcal{T}_\mu$, let $\Sigma(g) = \{\langle S'_0, S'_1, \ldots, S'_m, g(S')\rangle : S' \in \mathcal{P}(g)\}$. Since g is constant on $\mathcal{P}(g)$, $\Sigma(g)$ is an $(m+1)$-spectrum. So with this notation, we can rewrite (21):

$$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \bigvee_{g:\Sigma\setminus\{S\}\to\mathcal{T}_\mu} \exists_{v_{m+1}} \underbrace{\bigwedge_{S^+\in\Sigma(g)} \exists_{v_{m+2}}\Big(\bigwedge_{i<m+2} v_i S^+_i v_{m+2}\Big)}_{\alpha_{\Sigma(g)}}$$

—that is,

(22) $$\vdash_n \alpha_1 \wedge \alpha_2 \wedge \alpha_\Sigma \Rightarrow \bigvee_{g:\Sigma\setminus\{S\}\to\mathcal{T}_\mu} \exists_{v_{m+1}} \alpha_{\Sigma(g)}.$$

Now a partitioned set cannot have more elements than the number of pieces times the size of a largest piece, so for any $g : \Sigma \setminus \{S\} \to \mathcal{T}_\mu$ we have

(23) $$(n - 3 - m)|\mathcal{P}(g)| \geq |\Sigma| - 1.$$

Our assumption on the size of Σ is that

$$|\Sigma| > \rho(n-3-m). \tag{24}$$

Since $|\Sigma(g)| = |\mathcal{P}(g)|$, the definition of ρ and inequalities (23) and (24) give

$$\begin{aligned} 1 + (n-3-m)|\Sigma(g)| \geq |\Sigma| &> \rho(n-3-m) \\ &= 1 + (n-3-m) \cdot \rho(n-4-m), \end{aligned}$$

so $|\Sigma(g)| > \rho(n-3-(m+1))$. Thus, each $\Sigma(g)$ is a sufficiently large $(m+1)$-spectrum that, by the inductive hypothesis, $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_{\Sigma(g)}$. As in the base case, we deduce from this that

$$\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\exists_{v_{m+1}} \alpha_{\Sigma(g)}.$$

This holds for all g, so by (22) and Lemma 5.3, $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma$. This completes the induction and the proof of the proposition. ⊣

Theorem 7.5. $\vdash_n \neg\alpha$.

Proof. Let Σ be the 0-spectrum $\{\mathsf{Q}_i : i < \beta,\ i \text{ even}\}$. By Proposition 7.1, $\vdash_n \alpha_3 \wedge \alpha_4 \Rightarrow \alpha_\Sigma$. By Proposition 7.2 since $|\Sigma| > \rho(n-3)$, $\vdash_n \alpha_1 \wedge \alpha_2 \Rightarrow \neg\alpha_\Sigma$. Also, by Lemma 6.1, $\vdash_n \alpha_1 \wedge \alpha_3 \Rightarrow \alpha_4$. So by Lemma 5.3, we have $\vdash_n \neg(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)$, or $\vdash_n \neg\alpha$, as required. ⊣

Thus the 3-sentence $\neg\alpha$ is n-provable without using (A5). In the next section we show that $\neg\alpha$ is not $(n-1)$-provable.

§8. Alternative semantics. To show that $\neg\alpha$ is not $(n-1)$-provable, we will use non-classical semantics for which $\vdash_{n-1}$ is sound and α is valid. Suppose A is an $(n-1)$-by-$(n-1)$ matrix of relation symbols in Rel. For $i, j < n-1$, the symbol in row i and column j is denoted by A_{ij} or $A_{i,j}$. Thus,

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0,n-2} \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1,n-2} \\ A_{20} & A_{21} & A_{22} & \cdots & A_{2,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{n-2,0} & A_{n-2,1} & A_{n-2,2} & \cdots & A_{n-2,n-2} \end{bmatrix}.$$

Think of A as a list of binary relations that hold among n objects, that is, A is just a special kind of quantifier-free n-type having the same 'meaning' as the conjunction of the atomic formulæ $v_i A_{ij} v_j$:

$$\text{`}A \Leftrightarrow \bigwedge_{i,j<n-1} v_i A_{ij} v_j\text{'}.$$

Let $\mathcal{M}$ be the set of $(n-1)$-by-$(n-1)$ matrices A of relation symbols in Rel that satisfy the following conditions:

(M$_1$) $A_{ii} = \mathsf{Id}$ for all $i < n-1$, (M$_2$) $\langle A_{ij}, A_{ik}, A_{kj}\rangle \in \mathsf{Mand}$ for all $i, j, k < n-1$.

The matrices in $\mathcal{M}$ are called **atomic matrices**.

LEMMA 8.1. *If $i, j, k < n-1$, $A \in \mathcal{M}$, and $A_{ij} = \mathsf{Id}$, then $A_{ik} = A_{kj}$. Hence, any atomic matrix $A \in \mathcal{M}$ is symmetric: it satisfies*

(M$_3$) *$A_{ij} = A_{ji}$ for all $i, j < n-1$.*

PROOF. By (M$_2$), $\langle A_{ij}, A_{ik}, A_{kj}\rangle = \langle \mathsf{Id}, A_{ik}, A_{kj}\rangle \in \mathsf{Mand}$. From the definition of Mand we see that $A_{ik} = A_{kj}$. By (M$_1$), $A_{ii} = \mathsf{Id}$ for any $A \in \mathcal{M}$ and $i, j < n-1$, so by the foregoing, $A_{ij} = A_{ji}$, giving (M$_3$). ⊣

For any $i, j < n-1$, let $[i, j]$ and $[i/j]$ be the functions mapping $n-1$ to $n-1$ defined by the following conditions for all $k < n-1$:

$$\begin{array}{ll} [i,j]i = j & [i/j]i = j \\ [i,j]j = i & \\ [i,j]k = k \text{ if } k \neq i, j & [i/j]k = k \text{ if } k \neq i \end{array}$$

$[i, j]$ is a **transposition** and $[i/j]$ is a **replacement**. If A is an atomic matrix and g is any function mapping $n-1$ into $n-1$, then Ag is the matrix of relation symbols defined by $(Ag)_{ij} = A_{g(i),g(j)}$ for all $i, j < n-1$. It is easy to check that Ag is an atomic matrix. In particular, $A[i, j]$ and $A[i/j]$ are atomic matrices. Let $k, l < n-1$. Two atomic matrices A, B are said to **agree up to** k, written $A \cong_k B$, if $A_{ij} = B_{ij}$ whenever $i, j < n-1$ and $i, j \neq k$, and they **agree up to** k, l, written $A \cong_{kl} B$, if $A_{ij} = B_{ij}$ whenever $i, j < n-1$ and $i, j \notin \{k, l\}$. We will need to know that $\mathcal{M}$ has the following properties.

LEMMA 8.2.

(C$_1$) *if $A, C \in \mathcal{M}$, $i, j < n-1$, $i \neq j$, and $A \cong_{ij} C$, then there is some $B \in \mathcal{M}$ such that $A \cong_i B$ and $B \cong_j C$,*

(C$_2$) *if $A \in \mathcal{M}$ and $i, j < n-1$ then $A[i/j] \in \mathcal{M}$,*

(C$_3$) *if $A \in \mathcal{M}$ and $i, j < n-1$ then $A[i, j] \in \mathcal{M}$.*

PROOF. (C$_2$) and (C$_3$) were discussed above. The hard part is (C$_1$). Assume $A, C \in \mathcal{M}$ and $A \cong_{ij} C$. We wish to find some $B \in \mathcal{M}$ such that $A \cong_i B$ and $B \cong_j C$. The requirements $A \cong_i B$, $B \cong_j C$ determine B_{kl} whenever $\{k, l\} \neq \{i, j\}$, so we only need to choose $B_{ij} = B_{ji}$ so that $\langle B_{ji}, A_{jk}, C_{ki}\rangle \in \mathsf{Mand}$ for every $k < n-1$, $k \neq i, j$. It is easily seen that this is sufficient to ensure that $B \in \mathcal{M}$.

Let $K = \{k : i, j \neq k < n-1\}$. If $A_{jk} = \mathsf{Id}$ for some $k \in K$ then we let $B_{ij} = B_{ji} = C_{ik}$. Then by Lemma 8.1, $\langle B_{ji}, A_{jl}, C_{li}\rangle = \langle C_{ki}, C_{kl}, C_{li}\rangle \in \mathsf{Mand}$ for all $l \in K$. Similarly, if $C_{ik} = \mathsf{Id}$ for some $k \in K$ then we may let $B_{ij} = B_{ji} = A_{kj}$. We may therefore assume that $A_{jk} \neq \mathsf{Id} \neq C_{ik}$ for all $k \in K$. Suppose that for every $k \in K$ there is some $l \in \{1, \ldots, n-3\}$ such that $\{A_{jk}, C_{ki}\} \subseteq \{\mathsf{P}_l, \mathsf{Q}_l\}$. Then $\langle \mathsf{E}, A_{jk}, C_{ki}\rangle$ is mandatory whenever $k \in K$, so we may let $B_{ij} = B_{ji} = \mathsf{E}$. If not, then because $|K| = n-3$

and there are $n-3$ pairs of the form $\{\mathsf{P}_l, \mathsf{Q}_l\}$, $1 \le l \le n-3$, there must be some $l \in \{1, \ldots, n-3\}$ such that for every $k \in K$, $\{A_{jk}, C_{ki}\} \not\subseteq \{\mathsf{P}_l, \mathsf{Q}_l\}$. Then since $A_{jk}, C_{ki} \neq \mathsf{Id}$, $\langle \mathsf{P}_l, A_{jk}, C_{ki}\rangle$ is mandatory for every $k \in K$, so let $B_{ij} = B_{ji} = \mathsf{P}_l$. ⊣

The **satisfaction relation** $\models$ associated with $\mathcal{M}$ is the unique binary relation between matrices and $(n-1)$-formulæ that satisfies the following conditions for all $A \in \mathcal{M}$, $\varphi, \psi \in \mathsf{Fm}_{n-1}$, $i, j < n-1$, and $R \in \mathsf{Rel}$:

$$\begin{array}{lll} A \models v_i{=}v_j & \text{iff} & A_{ij} = \mathsf{Id}, \\ A \models v_i R v_j & \text{iff} & A_{ij} = R, \\ A \models \neg\varphi & \text{iff} & A \not\models \varphi, \\ A \models \varphi \Rightarrow \psi & \text{iff} & \text{either } A \not\models \varphi \text{ or } A \models \psi \text{ or both}, \\ A \models \forall_{v_i}\varphi & \text{iff} & B \models \varphi \text{ whenever } B \in \mathcal{M} \text{ and } B \cong_i A. \end{array}$$

A formula that is $\mathcal{M}$-satisfied by every atomic matrix is said to be **$\mathcal{M}$-valid**.

LEMMA 8.3. *If $\varphi \in \mathsf{Fm}_{n-1}$, $v_i \notin \mathsf{free}(\varphi)$, and $A \cong_i B$, then $A \models \varphi$ iff $B \models \varphi$.*

PROOF. The proof is by induction on φ. Suppose $v_i \notin \mathsf{free}(\varphi)$ and $A \cong_i B$. If φ is atomic, then φ is $v_j{=}v_k$ or $v_j R v_k$ with $R \in \mathsf{Rel}$, and $i \neq j, k$. Then $A_{jk} = B_{jk}$ since $A \cong_i B$, hence $A \models \varphi$ iff $B \models \varphi$. Assume the lemma holds for ψ, and let $\varphi = \neg\psi$. Then $v_i \notin \mathsf{free}(\varphi) = \mathsf{free}(\psi)$, so inductively, $A \models \varphi$ iff $A \not\models \psi$ iff $B \not\models \psi$ iff $B \models \varphi$. It is equally easy to show that if the lemma holds for ψ and χ, then it holds for $\psi \Rightarrow \chi$. Assume the lemma holds for ψ, and let $\varphi = \forall_{v_j}\psi$. We only show that $B \models \varphi$ whenever $A \models \varphi$ and $A \cong_i B$, since the converse follows by interchanging A and B. Assume $A \models \varphi$ and $A \cong_i B$. To show $B \models \varphi$, it suffices to assume $B \cong_j C$ and show $C \models \psi$. If $j = i$, then $C \models \psi$ because $A \cong_i C$, so assume $j \neq i$. From $A \cong_i B$ and $B \cong_j C$ we conclude that $A \cong_{ij} C$. Since $\mathcal{M}$ has property (C_1), there must be some $D \in \mathcal{M}$ such that $A \cong_j D$ and $D \cong_i C$. Then $D \models \psi$ since $A \models \forall_{v_j}\psi$ and $A \cong_j D$. Now $v_i \notin \mathsf{free}(\psi)$ since $j \neq i$, so it follows from the inductive hypothesis and $D \cong_i C$ that $C \models \psi$, as desired. ⊣

LEMMA 8.4. *For all $i, j < n-1$, $A \in \mathcal{M}$, and $\varphi \in \mathsf{Fm}_{n-1}$, we have $A \models \mathsf{S}_{ij}\varphi$ iff $A[i,j] \models \varphi$.*

PROOF. The proof is again by induction. If $\varphi = xRy$, where $x, y \in \mathsf{Var}_{n-1}$ and $R \in \mathsf{Rel}$, then there are several cases to be distinguished according to whether x or y is either v_i, v_j, or neither. We will treat only one case, as an example. Suppose $x = v_i$, $y = v_k$, and $k \neq i, j$. Then, since $\mathsf{S}_{ij}\varphi = \mathsf{S}_{ij}v_i R v_k = v_j R v_k$, and $A[i,j]_{ik} = A_{jk}$, we have $A \models \mathsf{S}_{ij}\varphi$ iff $A \models v_j R v_k$ iff $A_{jk} = R$ iff $A[i,j]_{ik} = R$ iff $A[i,j] \models v_i R v_k$ iff $A[i,j] \models \varphi$. (Notice that (C_3) of Lemma 8.2, giving $A[i,j] \in \mathcal{M}$, is needed here.)

Suppose the lemma holds for ψ, and let $\varphi = \neg\psi$. Then, since $\mathsf{S}_{ij}\varphi = \mathsf{S}_{ij}\neg\psi = \neg\mathsf{S}_{ij}\psi$, we have $A \models \mathsf{S}_{ij}\varphi$ iff $A \not\models \mathsf{S}_{ij}\psi$ iff $A[i,j] \not\models \psi$ iff $A[i,j] \models \neg\psi$ iff $A[i,j] \models \varphi$. It is equally easy to treat the case $\varphi = (\psi \Rightarrow \chi)$.

Finally, suppose the lemma holds for ψ, and let $\varphi = \forall_{v_k}\psi$. We first assume $A \models \mathsf{S}_{ij}\forall_{v_k}\psi$ and prove $A[i,j] \models \forall_{v_k}\psi$. To get this conclusion, we assume $B \cong_k A[i,j]$ and show $B \models \psi$. Let $l = [i,j]k$. Then $k = [i,j]l$, and it is easy to check that $B[i,j] \cong_l A$. Now $A \models \mathsf{S}_{ij}\forall_{v_k}\psi = \forall_{v_l}\mathsf{S}_{ij}\psi$, so $B[i,j] \models \mathsf{S}_{ij}\psi$. But then $B = B[i,j][i,j] \models \psi$ by the inductive hypothesis. Thus $A[i,j] \models \forall_{v_k}\psi$ whenever $A \models \mathsf{S}_{ij}\forall_{v_k}\psi$. Conversely, assume $A[i,j] \models \forall_{v_k}\psi$. To show that $A \models \mathsf{S}_{ij}\forall_{v_k}\psi = \forall_{v_l}\mathsf{S}_{ij}\psi$, we suppose $B \cong_l A$ and show $B \models \mathsf{S}_{ij}\psi$. From $B \cong_l A$ we get $B[i,j] \cong_k A[i,j]$, so $B[i,j] \models \psi$, and hence $B \models \mathsf{S}_{ij}\psi$ by the inductive hypothesis. $\dashv$

LEMMA 8.5. *If $A \in \mathcal{M}$ and $\varphi \in \mathsf{Ax}_{n-1}$ then $A \models \varphi$. Thus, every axiom is $\mathcal{M}$-valid.*

PROOF. Both the alternative and the usual semantics treat the connectives $\neg$ and $\Rightarrow$ in the same way, so the proof that every tautology is semantically valid also shows that every tautology is $\mathcal{M}$-valid. Hence the lemma holds when φ is an instance of (A1).

Suppose φ is an instance $\forall_{v_i}(\psi \Rightarrow \chi) \Rightarrow (\psi \Rightarrow \forall_{v_i}\chi)$ of (A2), where $v_i \notin \mathsf{free}(\psi)$. To show that $A \models \varphi$, it suffices to assume $A \models \forall_{v_i}(\psi \Rightarrow \chi)$, $A \models \psi$, and $B \cong_i A$, and then show $B \models \chi$. From the first and third of these assumptions we know that $B \models \psi \Rightarrow \chi$, *i.e.* either $B \not\models \psi$ or else $B \models \chi$. However, the second and third assumptions imply, by Lemma 8.3 and $v_i \notin \mathsf{free}(\psi)$, that $B \models \psi$. Hence $B \models \chi$.

Suppose φ is an instance $\forall_{v_i}\psi \Rightarrow \psi$ of (A3). If $A \models \forall_{v_i}\psi$, then $A \models \psi$ since $A \cong_i A$. Hence $A \models \varphi$.

If φ is an instance of (A4), then φ is $\exists_{v_i}(v_i{=}v_j)$ for some distinct $i, j < n-1$. Now $A \models \exists_{v_i}(v_i{=}v_j)$ iff there is some $B \in \mathcal{M}$ such that $B \cong_i A$ and $B \models v_i{=}v_j$. Let $B = A[i/j]$. Then $B \in \mathcal{M}$ since $\mathcal{M}$ has property (C_2). Furthermore, $A \cong_i B$, for if $k, l \neq i$ then $[i/j]k = k$ and $[i/j]l = l$ so $A_{kl} = A[i/j]_{kl}$. Finally, $B \models v_i{=}v_j$, since $B_{ij} = A[i/j]_{ij} = A_{jj} = \mathsf{Id}$.

We will treat only one type of instance of (A5), namely $(v_i{=}v_j) \Rightarrow (v_i R v_k \Rightarrow v_j R v_k)$. Suppose $A \models v_i{=}v_j$ and $A \models v_i R v_k$. Then $A_{ij} = \mathsf{Id}$ and $A_{ik} = R$. By Lemma 8.1, $A_{jk} = R$, hence $A \models v_j R v_k$. The other cases are similar.

Finally, let φ be $(v_i{=}v_j) \Rightarrow (\psi \Rightarrow \mathsf{S}_{ij}\psi)$, an instance of (A6). Suppose $A \models v_i{=}v_j$. Then $A_{ij} = \mathsf{Id}$, and it is easy to prove from this that $A = A[i,j]$. (For example, if $k \neq i, j$ then by Lemma 8.1, $A_{ik} = A_{jk} = A[i,j]_{ik}$.) If also $A \models \psi$, then $A[i,j] \models \psi$, so $A \models \mathsf{S}_{ij}\psi$ by Lemma 8.4. Thus $A \models \varphi$ for every $A \in \mathcal{M}$. $\dashv$

LEMMA 8.6. *If φ, $\varphi \Rightarrow \psi \in \mathsf{Fm}_{n-1}$ are $\mathcal{M}$-valid, then so are ψ and $\forall_{v_i}\varphi$ for all $i < n-1$.*

PROOF. Let $A \in \mathcal{M}$. Then $A \models \varphi$ and $A \models \varphi \Rightarrow \psi$ since φ and $\varphi \Rightarrow \psi$ are $\mathcal{M}$-valid, and it follows that $A \models \psi$. We get $A \models \forall_{v_i}\varphi$ because if $A \cong_i B$ then by the $\mathcal{M}$-validity of φ, $B \models \varphi$. ⊣

LEMMA 8.7. *If* $\vdash_{n-1} \varphi$ *then* φ *is* $\mathcal{M}$*-valid.*

PROOF. The set of $\mathcal{M}$-valid formulæ contains Ax_{n-1} by Lemma 8.5, and is closed under *modus ponens* and $(n-1)$-generalization by Lemma 8.6, so it must therefore include all $(n-1)$-provable formulæ. ⊣

LEMMA 8.8. α *is* $\mathcal{M}$*-valid.*

PROOF. We use the second alternative form (13) of α:

$$\alpha \equiv_3 \forall_{v_0}\forall_{v_1} \bigvee_{R\in\mathsf{Rel}} v_0Rv_1 \ \wedge$$
$$\bigwedge_{S,T\in\mathsf{Rel}} \forall_{v_0}\forall_{v_1}\Big(\Big(\exists_{v_2}\,(v_0Sv_2 \wedge v_2Tv_1)\Big) \Leftrightarrow \bigvee_{\langle R,S,T\rangle\in\mathsf{Mand}} v_0Rv_1\Big).$$

Certainly, $\forall_{v_0}\forall_{v_1}\bigvee_{R\in\mathsf{Rel}} v_0Rv_1$ is $\mathcal{M}$-valid. Suppose $S,T \in \mathsf{Rel}$. Let $A \in \mathcal{M}$. If $A \models \exists_{v_2}\,(v_0Sv_2 \wedge v_2Tv_1)$ then for some $B \in \mathcal{M}$ with $B \cong_2 A$, we have $B \models v_0Sv_2 \wedge v_2Tv_1$. By (M_2), $B \models \bigvee_{\langle R,S,T\rangle\in\mathsf{Mand}} v_0Rv_1$, so by Lemma 8.3, $A \models \bigvee_{\langle R,S,T\rangle\in\mathsf{Mand}} v_0Rv_1$. Conversely, suppose that $R \in \mathsf{Rel}$, $\langle R,S,T\rangle \in \mathsf{Mand}$, and $A \models v_0Rv_1$. Define $\pi = [3/0][4/0]\cdots[n-2/0] : (n-1) \to \{0,1,2\}$, and define B by

$$B_{lm} = \begin{cases} \mathsf{Id} & \text{if } \pi(l) = \pi(m), \\ R & \text{if } \{\pi(l),\pi(m)\} = \{0,1\}, \\ S & \text{if } \{\pi(l),\pi(m)\} = \{0,2\}, \\ T & \text{if } \{\pi(l),\pi(m)\} = \{2,1\}, \end{cases}$$

where $l, m < n-1$. It is easy to check that $B \in \mathcal{M}$. Now evidently,

$$A \cong_3 A[3/0] \cong_4 A[3/0][4/0] \cong_5 \cdots \cong_{n-2} A[3/0]\cdots[n-2/0] = A\pi \cong_2 B. \tag{25}$$

By (C_2), all matrices shown are in $\mathcal{M}$. As $B_{02} = S$ and $B_{21} = T$, we have $B \models \exists_{v_2}\,(v_0Sv_2 \wedge v_2Tv_1)$. But $\mathsf{free}(\exists_{v_2}(v_0Sv_2 \wedge v_2Tv_1)) = \{v_0, v_1\}$, so by repeated use of Lemma 8.3 on (25), we obtain $A \models \exists_{v_2}(v_0Sv_2 \wedge v_2Tv_1)$, as required. ⊣

THEOREM 8.9. $\nvdash_{n-1} \neg\alpha$.

PROOF. By Lemma 8.8, α is $\mathcal{M}$-valid. Hence $\neg\alpha$ is not $\mathcal{M}$-valid. By Lemma 8.7, $\nvdash_{n-1} \neg\alpha$. ⊣

We have shown that $\neg\alpha$ is n-provable without (A5), but is not $(n-1)$-provable, even using (A5).

§9. Reduction to one binary relation symbol. Finally, we obtain from α a 3-sentence, using only the binary relation symbol E and without equality, that requires n variables to prove. For every $\varphi \in \mathsf{Fm}_n$, define a formula φ^δ next. The definition of φ^δ formalises the fact that the relation symbol E 'generates' all the others. Below, $i, j < n$, $i \neq j$, k is the least element of $n \setminus \{i, j\}$, $1 \leq l < \beta - 2$, and $1 \leq m \leq n - 3$.

$$
\begin{aligned}
(v_i{=}v_i)^\delta &= (v_i{=}v_i)\,, \\
(v_i \mathsf{Id} v_i)^\delta &= (v_i{=}v_i)\,, \\
(v_i R v_i)^\delta &= \neg\,(v_i{=}v_i) \text{ if } R \in \mathsf{Rel} \setminus \{\mathsf{Id}\}, \\
(v_i{=}v_j)^\delta &= (v_i{=}v_j)\,, \\
(v_i \mathsf{E} v_j)^\delta &= v_i \mathsf{E} v_j\,, \\
(v_i \mathsf{Q}_0 v_j)^\delta &= \neg\exists_{v_k}\,(v_i \mathsf{E} v_k \wedge v_k \mathsf{E} v_j) \wedge \neg v_i \mathsf{E} v_j\,, \\
(v_i \mathsf{Id} v_j)^\delta &= \exists_{v_k}\left((v_i \mathsf{Q}_0 v_k)^\delta \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right) \wedge \neg\exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right), \\
(v_i \mathsf{Q}_1 v_j)^\delta &= \exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right) \wedge \neg\exists_{v_k}\left((v_i \mathsf{Q}_0 v_k)^\delta \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right), \\
(v_i \mathsf{Q}_2 v_j)^\delta &= \exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_1 v_j)^\delta\right) \wedge \neg\exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right) \\
&\qquad \wedge \exists_{v_k}\,(v_i \mathsf{E} v_k \wedge v_k \mathsf{E} v_j)\,, \\
(v_i \mathsf{Q}_{l+2} v_j)^\delta &= \exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_{l+1} v_j)^\delta\right) \wedge \neg\exists_{v_k}\left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_l v_j)^\delta\right), \\
(v_i \mathsf{P}_m v_j)^\delta &= \exists_{v_k}\left((v_i \mathsf{Q}_0 v_k)^\delta \wedge (v_k \mathsf{Q}_0 v_j)^\delta\right) \\
&\qquad\qquad \wedge \neg\exists_{v_k}\left((v_i \mathsf{Q}_m v_k)^\delta \wedge (v_k \mathsf{Q}_m v_j)^\delta\right), \\
(\varphi \Rightarrow \psi)^\delta &= \left(\varphi^\delta \Rightarrow \psi^\delta\right), \\
(\neg\varphi)^\delta &= \neg\varphi^\delta, \\
(\forall_{v_i}\varphi)^\delta &= \forall_{v_i}\varphi^\delta.
\end{aligned}
$$

It is easy to prove by induction that $\mathsf{free}(\varphi) = \mathsf{free}(\varphi^\delta)$ for all $\varphi \in \mathsf{Fm}_n$. Note that if $3 \leq l \leq n$ and $\varphi \in \mathsf{Fm}_l$ then $\varphi^\delta \in \mathsf{Fm}_l$. Because no atomic subformula of α is an equality or of the form $v_i R v_i$ for $R \in \mathsf{Rel}$, it follows that $\neg\alpha^\delta$ is a 3-sentence with only one binary relation symbol in it, namely E. It does not involve equality.

Lemma 9.1. $\nvdash_{n-1} \neg\alpha^\delta$.

Proof. We use the notation and results of the preceding section. First, we show by induction on φ that for all $\varphi \in \mathsf{Fm}_{n-1}$, $\varphi \Leftrightarrow \varphi^\delta$ is $\mathcal{M}$-valid. Let $\varphi \in \mathsf{Fm}_{n-1}$ be atomic. If it is $v_i{=}v_i$ or $v_i{=}v_j$ or $v_i E v_j$ there is nothing to prove. Suppose it is $v_i R v_i$ for some $R \in \mathsf{Rel}$. For every $A \in \mathcal{M}$,

$A \models v_i R v_i$ iff $A_{ii} = R$, so by (M_1), $A \models v_i R v_i$ iff $R = \mathsf{Id}$. We have $A \models v_i \mathsf{Id} v_i$ and $A \models (v_i \mathsf{Id} v_i)^\delta$, and $A \not\models \neg (v_i R v_i)$ and $A \not\models (v_i R v_i)^\delta$ whenever $R \in \mathsf{Rel} \setminus \{\mathsf{Id}\}$. So for all $R \in \mathsf{Rel}$, $A \models v_i R v_i$ iff $A \models (v_i R v_i)^\delta$, and it follows that $v_i R v_i \Leftrightarrow (v_i R v_i)^\delta$ is $\mathcal{M}$-valid.

The remaining cases $v_i R v_j$ for $i \neq j$ and $R = \mathsf{Q}_0, \mathsf{Id}, \mathsf{Q}_1, \ldots, \mathsf{P}_1, \ldots$ are established by investigating instances and consequences of the second alternative form (13) of α for various choices of S and T, with $k = \min(n \setminus \{i,j\}) < n-1$. By Lemmas 8.7 and 8.8, this alternative form is $\mathcal{M}$-valid, and by symmetry of $\mathcal{M}$ (or by Lemmas 5.10 and 8.7), we see that

$$\bigvee_{\mathsf{R} \in \mathsf{Rel}} v_i \mathsf{R} v_j \;\wedge\; \bigwedge_{S,T \in \mathsf{Rel}} \forall_{v_i} \forall_{v_j} \Big(\Big(\exists_{v_k} (v_i S v_k \wedge v_k T v_j) \Big) \Leftrightarrow \bigvee_{\langle R,S,T\rangle \in \mathsf{Mand}} v_i R v_j \Big)$$

is also $\mathcal{M}$-valid. Consider first E with itself:

$$\begin{aligned} \exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{E} v_j) \Leftrightarrow (v_i \mathsf{Id} v_j \vee v_i \mathsf{Q}_1 v_j \vee \cdots \vee v_i \mathsf{Q}_{\beta-1} v_j \\ \vee v_i \mathsf{P}_1 v_j \vee \cdots \vee v_i \mathsf{P}_{n-3} v_j) \text{ is } \mathcal{M}\text{-valid.} \end{aligned}$$

Since $\bigvee_{\mathsf{R} \in \mathsf{Rel}} v_i \mathsf{R} v_j$ is $\mathcal{M}$-valid, we get $\mathcal{M}$-validity of

$$\neg \exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{E} v_j) \Leftrightarrow (v_i \mathsf{E} v_j \vee v_i \mathsf{Q}_0 v_j) \tag{26}$$

and so of

$$v_i \mathsf{Q}_0 v_j \Leftrightarrow \neg \exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{E} v_j) \wedge \neg v_i \mathsf{E} v_j .$$

By the definition of ${}^\delta$, this says that

$$v_i \mathsf{Q}_0 v_j \Leftrightarrow (v_i \mathsf{Q}_0 v_j)^\delta \text{ is } \mathcal{M}\text{-valid.} \tag{27}$$

Next consider E with Q_0, and Q_0 with Q_0. We obtain $\mathcal{M}$-validity of

$$\exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{Q}_0 v_j) \Leftrightarrow (v_i \mathsf{Q}_1 v_j \vee v_i \mathsf{P}_1 v_j \vee \cdots \vee v_i \mathsf{P}_{n-3} v_j), \tag{28}$$

and of

$$\exists_{v_k} (v_i \mathsf{Q}_0 v_k \wedge v_k \mathsf{Q}_0 v_j) \Leftrightarrow (v_i \mathsf{Id} v_j \vee v_i \mathsf{P}_1 v_j \vee \cdots \vee v_i \mathsf{P}_{n-3} v_j). \tag{29}$$

Hence,

$$v_i \mathsf{Id} v_j \Leftrightarrow \exists_{v_k} (v_i \mathsf{Q}_0 v_k \wedge v_k \mathsf{Q}_0 v_j) \wedge \neg \exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{Q}_0 v_j)$$

and

$$v_i \mathsf{Q}_1 v_j \Leftrightarrow \exists_{v_k} (v_i \mathsf{E} v_k \wedge v_k \mathsf{Q}_0 v_j) \wedge \neg \exists_{v_k} (v_i \mathsf{Q}_0 v_k \wedge v_k \mathsf{Q}_0 v_j)$$

are $\mathcal{M}$-valid. Using these last two statements and (27), we get $\mathcal{M}$-validity of

$$v_i \mathsf{Id} v_j \Leftrightarrow \exists_{v_k} \left((v_i \mathsf{Q}_0 v_k)^\delta \wedge (v_k \mathsf{Q}_0 v_j)^\delta \right) \wedge \neg \exists_{v_k} \left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_0 v_j)^\delta \right)$$

and

$$v_i \mathsf{Q}_1 v_j \Leftrightarrow \exists_{v_k} \left(v_i \mathsf{E} v_k \wedge (v_k \mathsf{Q}_0 v_j)^\delta \right) \wedge \neg \exists_{v_k} \left((v_i \mathsf{Q}_0 v_k)^\delta \wedge (v_k \mathsf{Q}_0 v_j)^\delta \right),$$

and hence, by the definition of ${}^{\delta}$,

(30) $\quad v_i\mathsf{Id}v_j \Leftrightarrow (v_i\mathsf{Id}v_j)^{\delta}$ and $v_i\mathsf{Q}_1v_j \Leftrightarrow (v_i\mathsf{Q}_1v_j)^{\delta}$ are $\mathcal{M}$-valid.

Next, considering E with Q_1, we get the $\mathcal{M}$-validity of

$$\begin{aligned}(31)\quad \exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_1v_j\right) \Leftrightarrow (&v_i\mathsf{E}v_j \vee v_i\mathsf{Q}_0v_j \vee v_i\mathsf{Q}_1v_j \vee v_i\mathsf{Q}_2v_j\\ &\vee v_i\mathsf{P}_1v_j \vee \cdots \vee v_i\mathsf{P}_{n-3}v_j)\,,\end{aligned}$$

which implies, by (26) and (28), the $\mathcal{M}$-validity of

$$\begin{aligned}v_i\mathsf{Q}_2v_j \Leftrightarrow\ &\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_1v_j\right) \wedge \neg\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_0v_j\right)\\ &\wedge \exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{E}v_j\right),\end{aligned}$$

and, by (27) and (30), the $\mathcal{M}$-validity of

$$\begin{aligned}v_i\mathsf{Q}_2v_j \Leftrightarrow\ &\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge (v_k\mathsf{Q}_1v_j)^{\delta}\right) \wedge \neg\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge (v_k\mathsf{Q}_0v_j)^{\delta}\right)\\ &\wedge \exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{E}v_j\right),\end{aligned}$$

hence $v_i\mathsf{Q}_2v_j \Leftrightarrow (v_i\mathsf{Q}_2v_j)^{\delta}$ is $\mathcal{M}$-valid. The argument for $v_i\mathsf{Q}_3v_j$ is similar, using E and Q_2. We have the following $\mathcal{M}$-validities:

$$\begin{aligned}\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_2v_j\right) \Leftrightarrow (&v_i\mathsf{E}v_j \vee v_i\mathsf{Q}_1v_j \vee v_i\mathsf{Q}_2v_j \vee v_i\mathsf{Q}_3v_j\\ &\vee v_i\mathsf{P}_1v_j \vee \cdots \vee v_i\mathsf{P}_{n-3}v_j)\end{aligned}$$

by definition of Mand, so by (31),

$$v_i\mathsf{Q}_3v_j \Leftrightarrow \exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_2v_j\right) \wedge \neg\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge v_k\mathsf{Q}_1v_j\right),$$

so by the result for Q_1 and Q_2,

$$v_i\mathsf{Q}_3v_j \Leftrightarrow \exists_{v_k}\left(v_i\mathsf{E}v_k \wedge (v_k\mathsf{Q}_2v_j)^{\delta}\right) \wedge \neg\exists_{v_k}\left(v_i\mathsf{E}v_k \wedge (v_k\mathsf{Q}_1v_j)^{\delta}\right),$$

whence $v_i\mathsf{Q}_3v_j \Leftrightarrow (v_i\mathsf{Q}_3v_j)^{\delta}$ is $\mathcal{M}$-valid. This pattern continues, and $v_i\mathsf{Q}_lv_j \Leftrightarrow (v_i\mathsf{Q}_lv_j)^{\delta}$ is $\mathcal{M}$-valid whenever $3 < l \leq \beta - 1$. To complete the proof, it suffices to continue in a similar way for P_m $(1 \leq m \leq n-3)$, using the definition of Mand and (29) to obtain $\mathcal{M}$-validity of

$$\exists_{v_k}\left(v_i\mathsf{Q}_mv_k \wedge v_k\mathsf{Q}_mv_j\right) \Leftrightarrow v_i\mathsf{Id}v_j \vee v_i\mathsf{E}v_j \vee \bigvee_{l\neq m} v_i\mathsf{P}_lv_j,$$

$$\begin{aligned}v_i\mathsf{P}_mv_j \Leftrightarrow\ &\exists_{v_k}\left(v_i\mathsf{Q}_0v_k \wedge v_k\mathsf{Q}_0v_j\right)\\ &\wedge \neg\exists_{v_k}\left(v_i\mathsf{Q}_mv_k \wedge v_k\mathsf{Q}_mv_j\right),\\ v_i\mathsf{P}_mv_j \Leftrightarrow\ &\exists_{v_k}\left((v_i\mathsf{Q}_0v_k)^{\delta} \wedge (v_k\mathsf{Q}_0v_j)^{\delta}\right)\\ &\wedge \neg\exists_{v_k}\left((v_i\mathsf{Q}_mv_k)^{\delta} \wedge (v_k\mathsf{Q}_mv_j)^{\delta}\right),\end{aligned}$$

and $v_i\mathsf{P}_mv_j \Leftrightarrow (v_i\mathsf{P}_mv_j)^{\delta}$. Thus, $\varphi \Leftrightarrow \varphi^{\delta}$ is $\mathcal{M}$-valid for all atomic $\varphi \in \mathsf{Fm}_{n-1}$. To show the same for all $\varphi \in \mathsf{Fm}_{n-1}$ is now a straightforward induction on φ, left to the reader.

By Lemma 8.8, α is $\mathcal{M}$-valid. We have shown $\alpha \Leftrightarrow \alpha^\delta$ to be $\mathcal{M}$-valid, so α^δ is also $\mathcal{M}$-valid. It now follows that $\nvdash_{n-1} \neg\alpha^\delta$, exactly as in Theorem 8.9. ⊣

After two more lemmata, we can show the main result of this paper, that $\neg\alpha^\delta$ requires n variables to prove.

LEMMA 9.2. *Let $\psi \in \mathsf{Fm}_n$ and $l, m < n$. Then $\mathsf{S}_{lm}\left(\psi^\delta\right) \equiv_n (\mathsf{S}_{lm}\psi)^\delta$.*

PROOF. The proof is by induction. First, consider atomic ψ, of the form $v_i R v_j$, for $i, j < n$ and $R \in \mathsf{Rel} \cup \{=\}$. If R is $=$ or E, or if $i = j$, the result holds trivially. For example,

$$\begin{aligned}\mathsf{S}_{lm}\left((v_i \mathsf{E} v_j)^\delta\right) &= \mathsf{S}_{lm}\left(v_i \mathsf{E} v_j\right) = v_{[l,m](i)} \mathsf{E} v_{[l,m](j)} \\ &= \left(v_{[l,m](i)} \mathsf{E} v_{[l,m](j)}\right)^\delta = \left(\mathsf{S}_{lm}\left(v_i \mathsf{E} v_j\right)\right)^\delta.\end{aligned}$$

Now assume that $i \neq j$. Let k be the least element of $n \setminus \{i, j\}$, and let h be the least element of $n \setminus \{[l,m](i), [l,m](j)\}$. Let $i' = [l,m](i)$, and similarly define j', k'. Then by Lemmas 5.6 and 5.10,

$$\begin{aligned}\mathsf{S}_{lm}\left((v_i \mathsf{Q}_0 v_j)^\delta\right) &= \neg\exists_{v_{k'}}\left(v_{i'} \mathsf{E} v_{k'} \wedge v_{k'} \mathsf{E} v_{j'}\right) \wedge \neg v_{i'} \mathsf{E} v_{j'} \\ &\equiv_n \mathsf{S}_{hk'}\left(\neg\exists_{v_{k'}}\left(v_{i'} \mathsf{E} v_{k'} \wedge v_{k'} \mathsf{E} v_{j'}\right) \wedge \neg v_{i'} \mathsf{E} v_{j'}\right) \\ &= \neg\exists_{v_h}\left(v_{i'} \mathsf{E} v_h \wedge v_h \mathsf{E} v_{j'}\right) \wedge \neg v_{i'} \mathsf{E} v_{j'} \\ &= \left(v_{i'} \mathsf{Q}_0 v_{j'}\right)^\delta = \left(\mathsf{S}_{lm}(v_i \mathsf{Q}_0 v_j)\right)^\delta,\end{aligned}$$

so the lemma holds for Q_0. Next we prove the lemma for Id. We use Lemma 5.10, the definition of $(v_i \mathsf{Id} v_j)^\delta$, the fact that the lemma holds for Q_0, Lemma 5.6, and the definition of $(v_{i'} \mathsf{Id} v_{j'})^\delta$, in that order. Note that

$h, k' \notin \{i', j'\} = \mathsf{free}(\mathsf{S}_{lm}(v_i \mathsf{Id} v_j)) = \mathsf{free}(\mathsf{S}_{lm}((v_i \mathsf{Id} v_j)^\delta))$.

$$\begin{aligned}
&\mathsf{S}_{lm}\left((v_i \mathsf{Id} v_j)^\delta\right)\\
&\equiv_n \mathsf{S}_{hk'}\mathsf{S}_{lm}\left((v_i \mathsf{Id} v_j)^\delta\right)\\
&= \exists_{v_h}\left(\mathsf{S}_{hk'}\mathsf{S}_{lm}\left((v_i \mathsf{Q}_0 v_k)^\delta\right) \wedge \mathsf{S}_{hk'}\mathsf{S}_{lm}\left((v_k \mathsf{Q}_0 v_j)^\delta\right)\right)\\
&\quad \wedge \neg\exists_{v_h}\left(\mathsf{S}_{hk'}\mathsf{S}_{lm}\,(v_i \mathsf{E} v_k) \wedge \mathsf{S}_{hk'}\mathsf{S}_{lm}\left((v_k \mathsf{Q}_0 v_j)^\delta\right)\right)\\
&\equiv_n \exists_{v_h}\left(\left(\mathsf{S}_{hk'}\mathsf{S}_{lm}\,(v_i \mathsf{Q}_0 v_k)\right)^\delta \wedge \left(\mathsf{S}_{hk'}\mathsf{S}_{lm}\,(v_k \mathsf{Q}_0 v_j)\right)^\delta\right)\\
&\quad \wedge \neg\exists_{v_h}\left(\mathsf{S}_{hk'}\mathsf{S}_{lm}\,(v_i \mathsf{E} v_k) \wedge \left(\mathsf{S}_{hk'}\mathsf{S}_{lm}(v_k \mathsf{Q}_0 v_j)\right)^\delta\right)\\
&= \exists_{v_h}\left((v_{i'} \mathsf{Q}_0 v_h)^\delta \wedge (v_h \mathsf{Q}_0 v_{j'})^\delta\right)\\
&\quad \wedge \neg\exists_{v_h}\left(v_{i'} \mathsf{E} v_h \wedge (v_h \mathsf{Q}_0 v_{j'})^\delta\right)\\
&= (v_{i'} \mathsf{Id} v_{j'})^\delta\\
&= (\mathsf{S}_{lm}(v_i \mathsf{Id} v_j))^\delta,
\end{aligned}$$

so the lemma holds for Id. Proceed similarly through the remaining relation symbols to complete the proof for atomic ψ. By Lemma 5.6, the same holds for arbitrary $\psi \in \mathsf{Fm}_n$, completing the proof of the lemma. ⊣

LEMMA 9.3. *Suppose that $\varphi \in \mathsf{Fm}_n$ has an n-proof not using axiom* (A5). *Then $\vdash_n \varphi^\delta$.*

PROOF. Let $\pi_0, \ldots, \pi_l = \varphi$ be an n-proof of φ not using (A5). We show by induction on $m = 0, \ldots, l$ that $\vdash_n \pi_m^\delta$. Let $m \le l$ and assume inductively that $\vdash_n \pi_s^\delta$ for $s < m$. If π_m is any instance of (A1)–(A4), it is clear that π_m^δ is also an instance of the same axiom, so that $\vdash_n \pi_m^\delta$. Assume that π_m is an instance $v_i = v_j \Rightarrow (\psi \Rightarrow \mathsf{S}_{ij}\psi)$ of (A6). The previous lemma shows that for any ψ and $i, j < n$, $(\mathsf{S}_{ij}\psi)^\delta \equiv_n \mathsf{S}_{ij}(\psi^\delta)$. Then π_m^δ is

$$v_i = v_j \Rightarrow (\psi^\delta \Rightarrow (\mathsf{S}_{ij}\psi)^\delta),$$

which by the above and Lemma 5.6 is $\equiv_n$-equivalent to

$$v_i = v_j \Rightarrow (\psi^\delta \Rightarrow \mathsf{S}_{ij}(\psi^\delta)),$$

another instance of (A6). So $\vdash_n \pi_m^\delta$. Finally, if π_m follows from earlier π_s by a rule of inference, it is clear by definition of δ that π_m^δ follows from the earlier π_s^δ by the same rule. This completes the induction, and shows that $\vdash_n \varphi^\delta$. ⊣

THEOREM 9.4. *$\neg\alpha^\delta$ requires n variables to prove.*

PROOF. We proved $\not\vdash_{n-1} \neg\alpha^\delta$ in Lemma 9.1. It follows from the preceding lemma that $\vdash_n \neg\alpha^\delta$, because (A5) was never used in the n-proof of $\neg\alpha$ given in Theorem 7.5. ⊣

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