Complete Representations in Algebraic Logic

Robin Hirsch & Ian Hodkinson*

March 1, 1996

Abstract

A boolean algebra is shown to be completely representable if and only if it is atomic, whereas it is shown that neither the class of completely representable relation algebras nor the class of completely representable cylindric algebras of any fixed dimension are elementary.

1 Introduction

There are several types of representation for boolean algebras with operators, and of interest in this paper are the 'ordinary' and 'complete' representations [8, 10, 11, 17, 12, 13, 5, 1, 21, 19, 6]. Ordinary representations, or just 'representations', are isomorphisms from a boolean algebra with operators to a more concrete structure whose elements are sets, in which the boolean operators $\vee$ and $-$ are interpreted as $\cup$ and $\setminus$ and the other operators have certain set-theoretically definable interpretations. For example, in Tarski's relation algebras, the binary operation ';' gets interpreted as composition of binary relations.

Complete representations have the additional property that they preserve arbitrary disjunctions (hence conjunctions too), wherever these are defined. Historically, there has been a certain confusion here, with complete and arbitrary representations at times being mistaken for each other (e.g., [10]).

Representation classes — i.e., classes of all structures of the appropriate type that possess a representation — form *varieties* in almost every case in algebraic logic. This means that they can be characterised by a set of equational axioms, although in most cases it has been shown that infinitely many axioms are required.

Consider instead the class of *completely representable* structures of some type of algebraic logic. These are the ones that have a complete representation. In most cases, the class is not closed under taking substructures, and therefore cannot be characterised by universal axioms and cannot be a variety. In this paper, we investigate whether the classes of completely representable boolean algebras, relation algebras, and cylindric algebras form *elementary classes* — that is, in each case we ask whether there is *any* set of first-order sentences that characterises the class. Now this might sound rather unlikely: after all, the idea of a complete representation is an essentially second-order one, so why should the completely representable structures be characterised by first-order axioms? Corollary 6 is a warning against drawing this conclusion too rapidly: the class of completely representable boolean algebras is finitely axiomatisable, and in fact, a boolean algebra has a complete representation if and only if it is atomic.

For relation algebras and cylindric algebras, we do get a negative answer: the class of completely representable relation algebras is not elementary. The proof uses ultrapowers and a combinatorial construction. The results are presented in a game-theoretic framework, and this is for a number of reasons. Firstly, the game-theoretic method has been used effectively to simplify and clarify quite difficult problems in the field [7]. This works largely because the goal of the second player is to build a representation, and it turns out that the algebra is representable if and only if the second

*Hirsch was supported by SERC grant reference GR/H46343. Thanks to Roger Maddux, Mark Reynolds, and the referee for valuable remarks on the text.

player has a winning strategy. This makes the proof natural. Secondly, we believe that techniques in game theory should become standard scientific methods, just like proof by induction. Having said that, we should reassure the reader that absolutely no prior knowledge of game theory is required here. The definitions and results concerning games used here are very simple, and are all explained in the text.

Plan of paper In the next section, on boolean algebra, we introduce the notions of *complete* and *atomic* representations. These are shown to be equivalent (theorem 5) which gives us the corollary that a boolean algebra is atomic if and only if it has a complete representation.

Section 3 gives the basic definitions and results for relation algebras. The central result (theorem 24) is that the class of completely representable relation algebras is not an elementary class.

This result is obtained by defining certain two-player games played on 'networks' over an atomic relation algebra, $\mathcal{A}$. These networks are effectively *forcing conditions* for the existence of a complete representation of $\mathcal{A}$, and it is shown, for countable $\mathcal{A}$, that the second, 'existential' player can win the game of length ω if and only if $\mathcal{A}$ has a complete representation (proposition 13). We then show (proposition 15) that if the existential player can win the games of all finite lengths then $\mathcal{A}$ is *elementarily equivalent* to a completely representable relation algebra. The proof uses ultrapowers (or saturation) to construct an algebra elementarily equivalent to $\mathcal{A}$ for which the existential player can win the ω-length game. This algebra will have a countable elementary subalgebra with the same property, which will therefore be completely representable. Finally, in theorem 21 we construct a relation algebra $\mathcal{A}$ for which the existential player can win all the finite-length games but not the game of length ω. Putting these results together, we have that $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra but has no complete representation. So the class of completely representable relation algebras is not closed under elementary equivalence, and so cannot be elementary.

As an aside, the existence of a winning strategy for the existential player in all finite-length games over a relation algebra $\mathcal{A}$ is a property of $\mathcal{A}$ that turns out to be expressible by a first-order theory, equivalent to the Lyndon conditions of [10]. It is now a short step to prove (theorem 20) that an atomic relation algebra satisfies the Lyndon conditions if and only if it is elementarily equivalent to a completely representable relation algebra. In the light of [11], this is perhaps of interest in its own right.

In section 4 we show that the class of completely representable cylindric algebras is not elementary, either (theorem 34). In outline, the argument for finite dimensional cylindric algebras is similar to the one for relation algebra. The infinite dimensional case is handled by a cardinality argument.

Notation for the paper Throughout, $\mathcal{P}(X)$ denotes the power set of X. As is standard practice, we will usually use the same notation for a structure as for its domain.

2 Boolean Algebra

Here we recall some basic facts about boolean algebra, and discuss the representation theory. We will prove that a boolean algebra has a complete representation if and only if it is atomic.

DEFINITIONS 1 *Let $\mathcal{B} = (B, 0, 1, -, \vee)$ be a boolean algebra.*

- *The symbols $\wedge$ and $\leq$ are the usual abbreviations: $x \wedge y = -(-x \vee -y)$, and $x \leq y \iff x \vee y = y$.*
- *An* atom *is a minimal non-zero element of $\mathcal{B}$. We say that $\mathcal{B}$ is* atomic *if for every non-zero element $b \in \mathcal{B}$ there is an atom a of $\mathcal{B}$ with $a \leq b$. The set of all atoms of $\mathcal{B}$ is denoted $At(\mathcal{B})$.*

- *If $S \subseteq \mathcal{B}$, we write $\bigvee S$ for the supremum (or least upper bound) of S in $\mathcal{B}$, if it exists; otherwise, $\bigvee S$ is undefined. Formally, $\bigvee S$, if it exists, is the unique element satisfying*

$$\forall x \in \mathcal{B}(x \geq s \text{ for all } s \in S \text{ if and only if } x \geq \bigvee S).$$

  If it does exist, it is necessarily unique. Similarly, we define $\bigwedge S$ to be the infimum (greatest lower bound) of S, if it exists.

- *An* ultrafilter *of $\mathcal{B}$ is a subset λ of $\mathcal{B}$ satisfying:*
  - $a, b \in \lambda \Rightarrow (a \wedge b) \in \lambda$,
  - $a \in \lambda, b \geq a \Rightarrow b \in \lambda$, *and*
  - $a \notin \lambda \Leftrightarrow (-a) \in \lambda$,

  for all $a, b \in \mathcal{B}$. An ultrafilter λ is said to be principal *if it is of the form $\{b \in \mathcal{B} : b \geq a\}$ for some $a \in \mathcal{B}$. In this case it is clear that $a = \bigwedge \lambda$, and that a is an atom of $\mathcal{B}$.*

DEFINITION 2 *A* representation *of a boolean algebra $\mathcal{B}$ is an injective map*

$$h : \mathcal{B} \to \mathcal{P}(X),$$

for some non-empty set X, such that

$$\begin{aligned} h(0) &= \emptyset \\ h(1) &= X \\ h(a \vee b) &= h(a) \cup h(b) \\ h(-a) &= X \setminus h(a) \end{aligned}$$

for all $a, b \in \mathcal{B}$.

THEOREM 3 *Every boolean algebra is representable [20].*

PROOF:

Given the boolean algebra $\mathcal{B}$, let X be the set of all ultrafilters over $\mathcal{B}$. Represent $\mathcal{B}$ by

$$h(b) = \{x \in X : b \in x\}.$$

$\square$

Notation Let $\mathcal{B}$ be a boolean algebra, and $h : \mathcal{B} \to \mathcal{P}(X)$ be a representation of $\mathcal{B}$. If $x \in X$, we let

$$h^{-1}(x) = \{b \in \mathcal{B} : x \in h(b)\}.$$

Clearly, $h^{-1}(x)$ is an ultrafilter of $\mathcal{B}$, though not necessarily a principal one.

DEFINITIONS 4

- *An* atomic representation *$h : \mathcal{B} \to \mathcal{P}(X)$ is a representation such that for each $x \in X$, the ultrafilter $h^{-1}(x)$ is principal. Equivalently, x is in the range of some atom of $\mathcal{B}$.*

- *A* complete representation *of $\mathcal{B}$ is a representation h satisfying*

$$h(\bigvee S) = \bigcup h[S]$$

  wherever $S \subseteq \mathcal{B}$ and $\bigvee S$ is defined. (By $h[S]$, we mean $\{h(s) : s \in S\}$.) One can check that a complete representation also preserves infinite conjunctions.

THEOREM 5 *Let $\mathcal{B}$ be a boolean algebra. A representation h of $\mathcal{B}$ is atomic if and only if it is complete.*

PROOF:

Let $h : \mathcal{B} \to \mathcal{P}(X)$ be a representation of $\mathcal{B}$. First, suppose that h is an atomic representation. Note that for any $b \in \mathcal{B}$, we have

$$h(b) = \bigcup \{h(\beta) : \text{atoms } \beta \leq b\}.$$

To show that h is a complete representation, let S be a subset of $\mathcal{B}$, and suppose that the supremum $\bigvee S$ exists. Then

$$\begin{array}{rcll} x \in h(\bigvee S) & \Leftrightarrow & x \in h(\beta) & \text{some atom } \beta \leq \bigvee S \\ & \Leftrightarrow & x \in h(\beta) & \text{some atom } \beta \text{ some } s \text{ with } \beta \leq s \in S \\ & \Leftrightarrow & x \in h(s) & \text{some } s \in S \\ & \Leftrightarrow & x \in \bigcup h[S]. & \end{array}$$

Conversely, suppose that h is a complete representation. Let $x \in X$. In order to show that h is an atomic representation, we must show that the ultrafilter $h^{-1}(x)$ is principal. Now 0 is a lower bound of $h^{-1}(x)$. If 0 is the greatest lower bound ($0 = \bigwedge h^{-1}(x)$), then since h is complete,

$$\emptyset = h(0) = h(\bigwedge h^{-1}(x)) = \bigcap h[h^{-1}(x)].$$

But $x \in \bigcap h[h^{-1}(x)]$, which gives a contradiction. Therefore, there is a non-zero lower bound b of $h^{-1}(x)$. Since $b \not\leq -b$, we have $-b \notin h^{-1}(x)$. As $h^{-1}(x)$ is an ultrafilter, $b \in h^{-1}(x)$. It follows that $h^{-1}(x)$ is principal. Thus, h is an atomic representation. □

Note If a Boolean algebra $\mathcal{B}$ admits an atomic representation, then it is atomic. Moreover, a representation $h : \mathcal{B} \to \mathcal{P}(X)$ is atomic if and only if $\langle h(\beta) : \beta \in At(\mathcal{B})\rangle$ is a partition of X. So every non-atomic representation $h : \mathcal{B} \to \mathcal{P}(X)$ of an atomic Boolean algebra can be turned into an atomic representation by restricting X to the interpretations of atoms, as follows. Let $Y = \bigcup h[At(\mathcal{B})]$, and let $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ be the map defined by $f(Z) = Z \cap Y$. The composition $f \circ h$ is an atomic representation of $\mathcal{B}$. By theorems 3 and 5, this gives us the following, which we will prove explicitly.

COROLLARY 6 *A boolean algebra $\mathcal{B}$ has a complete representation if and only if it is an atomic boolean algebra.*

PROOF:

If $\mathcal{B}$ is atomic then let $X = At(\mathcal{B})$. The representation $h : \mathcal{B} \to \mathcal{P}(X)$ defined by

$$h(b) = \{\beta \in At(\mathcal{B}) : \beta \leq b\}$$

is an atomic representation, and hence, by theorem 5, a complete representation.

Conversely, suppose that $\mathcal{B}$ has a complete representation, h. By theorem 5, h must be an atomic representation. For any non-zero $b \in \mathcal{B}$, pick any point $x \in h(b)$. So there is an atom β of $\mathcal{B}$ with $x \in h(\beta)$. Hence $\beta \wedge b \neq 0$, so, as β is an atom, $\beta \leq b$. It follows that $\mathcal{B}$ is atomic. □

3 Relation Algebra

In this section, which takes up about half the paper, we will consider complete representations of relation algebras. In section 3.1, we will recall the elementary definitions of relation algebras and their representations. We illustrate that relation algebras behave differently from boolean algebras as regards complete representability. Then, in section 3.3, we define a two-player game for building complete representations from 'networks' (section 3.2). In section 3.4 we show that for any an atomic relation algebra, if the second player has a winning strategy in all the games of finite length then it is elementarily equivalent to a completely representable relation algebra. Finally, in section 3.6, we construct a relation algebra that satisfies this condition but has no complete representation. This will prove that the class of completely representable relation algebras is not an elementary class.

3.1 Basics

- A *relation algebra* $\mathcal{A} = (A, 0, 1, -, \vee, Id, ^{\smile}, ;)$ is a structure satisfying

  1. The reduct $(A, 0, 1, -, \vee)$ is a boolean algebra.
  2. $Id; a = a; Id = a$
  3. $(a^{\smile})^{\smile} = a$
  4. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
  5. ; is an associative binary operator on $\mathcal{A}$
  6. $a; (b \vee c) = a; b \vee a; c$
  7. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
  8. $(a; b) \wedge c^{\smile} = 0 \Leftrightarrow (b; c) \wedge a^{\smile} = 0$,

  where a, b are arbitrary elements of $\mathcal{A}$.

- A relation algebra $\mathcal{A}$ is *atomic* if its boolean reduct is an atomic boolean algebra. We write $At(\mathcal{A})$ for the set of atoms of $\mathcal{A}$.

- A relation algebra is *simple* if every non-zero element r satisfies $1; r; 1 = 1$. Every relation algebra $\mathcal{A}$ can be decomposed as a subalgebra of a direct product of simple relation algebras $\mathcal{A}_i$ $(i \in I)$ such that each $\mathcal{A}_i$ is a homomorphic image of $\mathcal{A}$ (theorem 4.15 of [9]). The $\mathcal{A}_i$ are called the *simple components* of $\mathcal{A}$.

- An *integral* relation algebra satisfies $a; b = 0 \Rightarrow (a = 0) \vee (b = 0)$. A relation algebra is integral if and only if the identity is an atom.

- In a *symmetric* relation algebra every element is self-converse.

- A relation algebra is *commutative* if the composition operator ';' is commutative. Every symmetric relation algebra is commutative.

The canonical example of a relation algebra is a set $\mathcal{A}$ of binary relations on a non-empty set X. The elements of $\mathcal{A}$ are subsets of $\mathcal{P}(X \times X)$, $\vee$ is interpreted as $\cup$, '$-$' as complement relative to the largest relation in $\mathcal{A}$ — the interpretation of 1 — and 0 is interpreted as $\emptyset$. Id is interpreted as equality, namely $\{(x, x) : x \in X\}$, '$^{\smile}$' as *converse*, namely $a^{\smile} = \{(y, x) : (x, y) \in a\}$, and ';' as *composition of relations:* $a; b = \{(x, y) : \exists z((x, z) \in a \wedge (z, y) \in b)\}$. To be an algebra, $\mathcal{A}$ should be closed under these operations. We call such an $\mathcal{A}$ a *proper relation algebra.*

A relation algebra is representable if it is isomorphic to a proper relation algebra. We use the following, equivalent definition.

- A *representation* of a relation algebra $\mathcal{A}$ is a map $h : \mathcal{A} \to \mathcal{P}(X \times X)$, for some non-empty set X, that respects all the operations. More precisely:

- h acts as a boolean representation from the boolean reduct of $\mathcal{A}$ to $\mathcal{P}(h(1))$ — hence h is 1–1,
- $(x, y) \in h(Id)$ if and only if $x = y$,
- $(x, y) \in h(a)$ if and only if $(y, x) \in h(a^{\smile})$, and
- $(x, z) \in h(a; b)$ if and only if there is $y \in X$ such that $(x, y) \in h(a)$ and $(y, z) \in h(b)$,

where a, b are arbitrary elements of $\mathcal{A}$, and $x, y, z \in X$.

- A relation algebra is said to be *representable* if it has a representation.
- It can be shown that the class of representable relation algebras is a variety, and that a relation algebra is representable if and only if all its simple components are [9].

Many of the definitions and results of the previous section carry over to the boolean reduct of a relation algebra. In particular, we can define a *complete representation* to be a representation that preserves arbitrary disjunctions wherever they are defined, and an *atomic representation* to be a representation such that for any pair $(x, y) \in h(1)$ there is a (unique) atom $\beta \in \mathcal{A}$ with $(x, y) \in h(\beta)$. A relation algebra is said to be *completely representable* if it has a complete representation. The proof of the following theorem is unchanged from theorem 5.

THEOREM 7 *A representation of a relation algebra is a complete representation if and only if it is an atomic representation.*

One direction of corollary 6 carries over to relation algebras: if a relation algebra $\mathcal{A}$ has a complete representation then it must be an atomic algebra. However, the other direction of the corollary is not true for relation algebras. There are atomic (in fact finite) relation algebras with no representation at all (e.g. [17] page 286); the so-called *point algebra,* with three atoms, $<, >, =$, has only infinite representations [22]; and there are atomic, representable relation algebras possessing no complete representations (proved by Maddux, [12] pages 154–173; also see theorem 21 herein). So, in contrast to boolean algebra, it does not seem likely that we can build complete representations out of the atoms of an atomic relation algebra. Indeed, we will show that neither atomicity nor any other elementary condition characterises the completely representable relation algebras.

3.2 Networks

Fix, for this section and the next, an atomic relation algebra, $\mathcal{A}$.

DEFINITION 8 *Define an atomic $\mathcal{A}$-network (N) to be a function $N : \alpha \times \alpha \to At(\mathcal{A})$, for some set of 'nodes', α, such that for all nodes $l, m, n \in \alpha$, we have*

- $N(l, m) \leq Id$ *if and only if* $l = m$,
- $N(m, n) = N(n, m)^{\smile}$, *and*
- $N(l, n) \leq N(l, m); N(m, n)$.

The network is said to be non-empty if $\alpha \neq \emptyset$.

Maddux [13] uses the term 'labelling' instead of 'network', in the case where α is an ordinal. Networks will be used as 'forcing conditions', and an atomic network can be thought of as an approximation to a complete representation of a relation algebra.

Frequently, we do not have a special name for the set of nodes of the network N, but abuse the notation by letting N stand for the network and the set of nodes of the network too. Thus, $n \in N$ means that n is a node of the network N. We write $|N|$ for the cardinality of the set of nodes of N. We will often refer to a pair of nodes (m, n) as an 'edge', and the atom $N(m, n)$ as the 'label' on the edge. $Anets(\mathcal{A})$ (or simply $Anets$ if the context is clear) denotes the class of atomic $\mathcal{A}$-networks.

DEFINITION 9 *Let N be an atomic $\mathcal{A}$-network. Define a map $N^* : \mathcal{A} \to \mathcal{P}(N \times N)$ by*

$$N^*(a) = \{(m,n) \in N \times N : N(m,n) \leq a\}, \qquad \textit{for } a \in \mathcal{A}.$$

LEMMA 10 *Assume that $\mathcal{A}$ is simple. Let N be a non-empty atomic $\mathcal{A}$-network. Then N^* is a complete representation of $\mathcal{A}$ if and only if for all $m, n \in N$ and for all $r, s \in At(\mathcal{A})$ with $r;s \geq N(m,n)$, there is a node $l \in N$ such that $N(m,l) = r$ and $N(l,n) = s$.*

PROOF:

Since N is an atomic network, N^* is an 'atomic map', meaning that for every pair of nodes m, n in N, there is an atom $a = N(m,n)$ such that $(m,n) \in N^*(a)$. So, by theorem 7, if N^* is a representation of $\mathcal{A}$, then it is automatically a complete one.

Hence we only need ascertain that N^* is a representation of $\mathcal{A}$. From the definition of an atomic network, we can see that N^* preserves the identity and converse. Let us check that composition is preserved. This will be so if and only if for all $r, s \in At(\mathcal{A})$ and $m, n \in N$,

$$(m,n) \in N^*(r;s) \iff (\exists l \in N)\ [(m,l) \in N^*(r) \wedge (l,n) \in N^*(s)].$$

This is equivalent to $(\forall m, n \in N)(\forall r, s \in At(\mathcal{A}))$

$$N(m,n) \leq r;s \iff (\exists l \in N)\ [N(m,l) = r\ \wedge\ N(l,n) = s]. \tag{1}$$

The implication from right to left in formula (1) holds automatically because of the final part of the definition of an atomic network, so the formula is equivalent to the condition in the lemma.

Clearly, $N^*(0) = \emptyset$, $N^*(a \vee b) = N^*(a) \cup N^*(b)$, and $N^*(-a) = N^*(1) \setminus N^*(a)$, for all $a, b \in \mathcal{A}$. Moreover, N^* is 1–1. For if a is a non-zero element of $\mathcal{A}$, then as $\mathcal{A}$ is simple, $1;a;1 = 1$. Since $N^*(1) \neq \emptyset$ and N^* preserves ';', it follows that $N^*(a) \neq \emptyset$, so that N^* is 1–1, as claimed. This completes the proof that N^* is a representation of $\mathcal{A}$. $\square$

REMARK 11 A nearly identical result (lemma 5.15) was proved by Maddux [13]. The lemma also illustrates the relation between networks and the 'bounded morphisms' of Goldblatt [3]. Let N be a non-empty atomic $\mathcal{A}$-network. Let $\mathcal{B}$ be the proper relation algebra whose elements are all of the possible binary relations on N (i.e., the universe of $\mathcal{B}$ is $\mathcal{P}(N \times N)$). Then $\mathcal{A}$ and $\mathcal{B}$ are atomic relation algebras, and we may endow $At(\mathcal{A})$ and $At(\mathcal{B})$ with relational structure by introducing three relation symbols, I, S, C, of arities 1, 2, 3, respectively, interpreting them as follows:

$$\begin{array}{rcl} At(\mathcal{A}) \models I(r) & \iff & \mathcal{A} \models r \leq Id \\ At(\mathcal{A}) \models S(r,s) & \iff & \mathcal{A} \models r^{\smile} = s \\ At(\mathcal{A}) \models C(r,s,t) & \iff & \mathcal{A} \models r;s \geq t, \end{array}$$

and similarly for $\mathcal{B}$.

The atoms of $\mathcal{B}$ are essentially edges of N, so we can view N as a map $: At(\mathcal{B}) \to At(\mathcal{A})$. Then to say that N is a network simply says (in the notation of [3], section 2.3) that N meets the universal ($\forall$) conditions of the definition of bounded morphism. And by lemma 10, an atomic network N yields a complete representation N^* if and only if N meets the universal–existential ($\forall\exists$) conditions of 'bounded morphism', too. In fact, a bounded epimorphism $: At(\mathcal{B}) \to At(\mathcal{A})$ yields a complete representation of $\mathcal{A}$ even if $\mathcal{A}$ is not simple (cf. [3], theorem 2.3.1); but we will not require that degree of sophistication here.

Our succeeding work with games can in part be viewed as a 'step-by-step' construction of a $\mathcal{B}$ and a bounded morphism from $At(\mathcal{B})$ into $At(\mathcal{A})$, in the style of [12, 13], and readers may prefer this view.

3.3 Games

Definition of the Game We define a game, $G_\omega(N, \mathcal{A})$, with two players, $\forall$ (male) and $\exists$ (female), who construct a countable (length ω) sequence of finite, atomic networks

$$N = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$$

(Here, $N_{i-1} \subseteq N_i$ means that the set of nodes of N_i extends that of N_{i-1}, and we have $N_i(n,m) = N_{i-1}(n,m)$ whenever both are defined.) For each $i > 0$, the atomic network N_i extends N_{i-1} by at most a single node.

In the ith round ($i = 1, 2, \ldots$), the play so far is $N = N_0 \subseteq \ldots \subseteq N_{i-1}$. $\forall$ picks any two nodes m, n from N_{i-1}, and any two atoms $r, s \in \mathcal{A}$, such that $r;s \geq N_{i-1}(m,n)$. $\exists$ responds (if possible) by extending N_{i-1} to some atomic $\mathcal{A}$-network N_i, with at most one extra node, such that there is a node $l \in N_i$ with $N_i(m,l) = r$ and $N_i(l,n) = s$. (Possibly such a node already exists in N_{i-1}, in which case $\exists$ does not need to extend the network and can set $N_i = N_{i-1}$).

If $\exists$ can do this in each round, she wins the play. Otherwise, $\forall$ wins.

The game $G_\omega(\emptyset, \mathcal{A})$ is a special case, with a 'zeroth round' in which $\forall$ picks any atom a of $\mathcal{A}$, and $\exists$ responds with any atomic network N_0 with an edge $e \in N_0^2$ such that $N_0(e) = a$. Without loss, $\exists$ can choose $|N_0| \leq 2$, and we will always assume that she does so. The remaining rounds of a play of the game are as before.

The *limit* of a play $N = N_0 \subseteq N_1 \subseteq \cdots$ of $G_\omega(N, \mathcal{A})$ is the network $N_\omega =$ '$\bigcup_{i<\omega} N_i$', defined in the obvious way: the nodes of N_ω are those occurring as nodes of some N_i, and $N_\omega(n,m)$ is defined to be $N_i(n,m)$ for any N_i containing n, m (any $i < \omega$; the value is independent of the choice of i). It is clear that N_ω is indeed an atomic $\mathcal{A}$-network.

The games $G_n(N, \mathcal{A}), G_n(\emptyset, \mathcal{A})$ ($n < \omega$) are similar to $G_\omega(N, \mathcal{A}), G_\omega(\emptyset, \mathcal{A})$, but there are only n rounds. If $\exists$ is able to survive to the end of the nth round, she has won the play. The final network in a play of $G_n(\emptyset, \mathcal{A})$ is N_n, and has at most $n + 2$ nodes.

For notational reasons, let us assume that each node is a natural number.

Notation If S is any set, let $S^{<\omega}$ denote the set of all finite sequences from S.

Winning Strategy Let $\sigma : Anets^{<\omega} \times \omega \times \omega \times At(\mathcal{A}) \times At(\mathcal{A}) \to Anets$ be such that always $\sigma((N_0, \ldots, N_n), m, n, r, s) \supseteq N_n$. In the ith round, $i > 0$, let $\forall$ pick nodes m, n and atoms r, s. $\exists$ can use σ to respond (possibly illegally, in which case she loses the play immediately) with the atomic network $\sigma((N_0, N_1, \ldots, N_{i-1}), m, n, r, s)$. If there is a 0th round, $\exists$ can use σ to respond to $\forall$'s atom r with $\sigma((), 0, 0, r, r)$. We say that σ is a *winning strategy* for $\exists$ if in any play of $G_\omega(N, \mathcal{A})$ where $\exists$ always uses σ, she is guaranteed to win the play.

DEFINITION 12 *Let N be an atomic $\mathcal{A}$-network, and $h : \mathcal{A} \to \mathcal{P}(X \times X)$ a representation of $\mathcal{A}$. We say that N* embeds *in h if there is a map $\Xi : N \to X$ such that for all nodes $m, n \in N$,*

$$(\Xi(m), \Xi(n)) \in h(N(m,n)).$$

Observe that since an embedding Ξ preserves identity, Ξ must be one-one. Of course, the empty network embeds in any representation.

PROPOSITION 13 *Let $\mathcal{A}$ be an atomic relation algebra, and N a finite atomic $\mathcal{A}$-network.*

1. *If N embeds in a complete representation of $\mathcal{A}$, then $\exists$ has a winning strategy in the game $G_\omega(N, \mathcal{A})$.*

2. *Conversely, assume that $\mathcal{A}$ is simple and has at most countably many atoms, and that $\exists$ has a winning strategy in the game $G_\omega(N, \mathcal{A})$. Then N embeds in a complete representation of $\mathcal{A}$.*

3. *Assume that $\mathcal{A}$, not necessarily simple, has at most countably many atoms. Then $\exists$ has a winning strategy in the game $G_\omega(\emptyset, \mathcal{A})$ if and only if $\mathcal{A}$ has a complete representation.*

PROOF:

1. Suppose that $h : \mathcal{A} \to \mathcal{P}(X \times X)$ is a complete representation of $\mathcal{A}$ in which the network N embeds. Let $\Xi : N \to X$ be as in the definition. $\exists$ can use the representation to give her a winning strategy in $G_\omega(N, \mathcal{A})$, as follows. In round 1, let $\forall$ pick the nodes $m, n \in N$ and atoms r, s. The points $\Xi(m), \Xi(n)$ are in X; and $(\Xi(m), \Xi(n)) \in h(N(m,n)) \subseteq h(r;s)$, since, according to the rules of the game, $r;s \geq N(m,n)$. Therefore there is a point $x \in X$ with $(\Xi(m), x) \in h(r)$ and $(x, \Xi(n)) \in h(s)$. Since h is a complete representation, it is an atomic representation: every two points are related by an atom. So the points $\{x\} \cup \{\Xi(l) : l \in N\}$ define an atomic network N_1 extending N_0, as required. If $\exists$ responds with N_1, it is still the case that N_1 embeds in h, and so she can continue in this way forever.

   For the special case of round 0 of the game when N is empty, suppose that $\forall$ picks the atom a of $\mathcal{A}$. $\exists$ may pick $(x, y) \in h(a)$, and then play the atomic network N_0 with nodes x, y, where for each edge $e \in \{x, y\}^2$, $N_0(e)$ is the atom $\bigwedge h^{-1}(e)$. She also defines Ξ to be the identity map on the nodes of N_0.

2. Assume the hypotheses. Consider a play of the game in which $\forall$ eventually picks every possible pair of nodes m, n that appear in the play and every legitimate pair of atoms r, s. He can do this because there are only countably many nodes that appear in the play and countably many atoms in $\mathcal{A}$. If $\exists$ uses her winning strategy, the limit will be an atomic network $N_\omega \supseteq N$ satisfying the conditions of lemma 10. Because we are assuming that $\mathcal{A}$ is simple, the lemma implies that N_ω^* will be a complete representation of $\mathcal{A}$, in which N clearly embeds.

3. Assume the hypotheses. The implication '$\Leftarrow$' follows from (1). We pass to the converse implication. Consider the set of elements $\{1;a;1 : a \in At(\mathcal{A})\}$. A simple component $\mathcal{S}_a$ of $\mathcal{A}$ can then be obtained by *relativizing* to $1;a;1$, for any atom a. We let $\mathcal{S}_a = \langle S_a, 0,\ 1;a;1,\ -_a,\ \vee,\ Id \wedge 1;a;1,\ \smile,\ ;\rangle$ where $S_a = \{x \in \mathcal{A} : x \leq 1;a;1\}$ and $-_a(x) = (-x) \wedge 1;a;1$.

   Consider a play of $G_\omega(\emptyset, \mathcal{A})$ where $\forall$ chooses the atom a for his first turn. Then, as in part 2, it can be arranged that the limit of the play is a complete representation of $\mathcal{S}_a$. So for each $a \in At(\mathcal{A})$, $\mathcal{S}_a$ has a complete representation, h_a, say.

   Now we can define a complete representation h of $\mathcal{A}$. The domain of h will be the disjoint union of the domains of the h_a, and we define, for any $\alpha \in \mathcal{A}$,

$$h(\alpha) = \bigcup_{a \in At(\mathcal{A})} \{h_a(\alpha \wedge (1;a;1))\}.$$

   Evidently, this is a representation of $\mathcal{A}$. It is an atomic representation because any two points x, y in its domain are either taken from the domains of distinct h_as — in which case $(x, y) \notin h(1)$ — or they both lie in the domain of h_a (some atom a). In the latter case there is an atom $b \in \mathcal{S}_a$ with $(x, y) \in h_a(b)$ (since h_a is complete). But then b is also an atom of $\mathcal{A}$, and it follows that $(x, y) \in h(b)$. Hence, h is a complete representation.

□

REMARK 14 It is not necessary to assume in the preceding lemma that $\mathcal{A}$ is a relation algebra. The result still holds under the weaker conditions that $\mathcal{A}$ is an atomic boolean algebra enriched with constant Id, a unary operator $\smile$ and a binary ; such that $\smile$ and ; are *completely additive operators* — i.e., for all $a, b \in \mathcal{A}$, $S \subseteq \mathcal{A}$,

- If $b = \bigvee S$ then $b^\smile = \bigvee\{s^\smile : s \in S\}$,
- If $b = \bigvee S$ then $a;b = \bigvee\{a;s : s \in S\}$, and $b;a = \bigvee\{s;a : s \in S\}$.

A structure $\mathcal{A}$ like this obeys axioms 1, 4, and 6 in the axiomatisation of relation algebras in section 3.1.

It can be checked that an algebra $\mathcal{A}$ with these properties is a relation algebra if and only if $\exists$ has a winning strategy in $G_2(\mathcal{A}, \emptyset)$. Also, $\mathcal{A}$ is a relation algebra if it has, or embeds in, a complete representation. Hence, given such an algebra $\mathcal{A}$, each of the hypotheses of the lemma implies that it is a relation algebra.

3.4 Games and complete representations

Our aim now is to show that the completely representable relation algebras do not form an elementary class. Our proof falls into two parts. The first part connects the games to complete representability. We will prove the following.

PROPOSITION 15 *Let $\mathcal{A}$ be any atomic relation algebra. Suppose that $\exists$ has a winning strategy in the games $G_i(\emptyset, \mathcal{A})$ for all $i < \omega$. Then $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra.*

PROOF:

The proof uses elementary extensions and ultraproducts. Let $\mathcal{B}$ be any non-principal ultraproduct of atomic relation algebras $\mathcal{A}_i$ $(i < \omega)$ over ω (in symbols, $\mathcal{B} = \Pi_D \mathcal{A}_i$, where D is some non-principal ultrafilter on ω). By Łoś' theorem (see [2], for example), $\mathcal{B}$ is also an atomic relation algebra. Let N be a finite, atomic $\mathcal{B}$-network. Now, for each $i < \omega$, we may pick an $\mathcal{A}_i$-graph N_i (an $\mathcal{A}_i$-graph is just a completely labelled graph with each edge labelled by an atom of $\mathcal{A}_i$), on the same set of nodes, such that

$$N = [\langle N_i : i < \omega \rangle]_D.$$

This notation means that $N(l,m)$ is the atom $\langle N_i(l,m) : i < \omega \rangle / D$ of $\mathcal{B}$, for all nodes l, m of N. Since N is an atomic network, for any three nodes $l, m, n \in N$, we have $N(l,m); N(m,n) \geq N(l,n)$. By the definition of ultraproduct, this implies that $\{i < \omega : N_i(l,m); N_i(m,n) \geq N_i(l,n)\} \in D$. Similarly, $\{i < \omega : (N_i(l,m) \leq Id) \Leftrightarrow (l = m)\} \in D$, and $\{i < \omega : N_i(l,m) = N_i(m,l)^{\smile}\} \in D$. So, regardless of the choice of N_i, the set

$$\eta = \{i < \omega : N_i \text{ is an atomic } \mathcal{A}_i\text{-network}\} \in D,$$

as N is finite and η is a finite intersection of members of D.

LEMMA 16 *In this notation, if there is a natural number k such that*

$$\{i \in \eta : i \geq k, \exists \textit{ has a winning strategy for } G_{i-k}(N_i, \mathcal{A}_i)\} \in D,$$

then $\exists$ has a winning strategy for $G_\omega(N, \mathcal{B})$.

PROOF:

We assume for simplicity that N is non-empty; the special round 0 that takes place if not is handled similarly. In the first round of the game $G_\omega(N, \mathcal{B})$, let $\forall$ play (e, a, b). So e is an edge of N, and a, b are atoms of $\mathcal{B}$ such that $a; b \geq N(e)$. Let $a = [\langle a_i : i \in \omega \rangle]_D$, $b = [\langle b_i : i \in \omega \rangle]_D$, where $a_i, b_i \in At(\mathcal{A}_i)$ for each $i < \omega$. As above, it is clear that $\{i \in \eta : N_i(e) \leq a_i; b_i\} \in D$. Let k be as in the theorem. It follows that

$$\xi = \{i \in \eta : i > k, (e, a_i, b_i) \text{ is a valid } \forall\text{-move in } G_{i-k}(N_i, \mathcal{A}_i), \text{ and } \exists \text{ has a winning strategy}\} \in D.$$

For each $i \in \xi$, let $\exists$ respond to the move (e, a_i, b_i) with her winning strategy in $G_{i-k}(N_i, \mathcal{A}_i)$. Call the resulting atomic $\mathcal{A}_i$-network N_i^+. Then

$N_i^+ \supseteq N_i$, and we may assume that any extra node of $N_i^+ \setminus N_i$ added by $\exists$ is always the same, for all i. For $i \in \omega \setminus \xi$, let N_i^+ be arbitrary: say, $N_i^+ = N_i$. Then the N_i^+ extend N_i, and they have the same nodes (either $|N|$ or $|N|+1$ in number), for 'many' i. Hence,

$$N^+ = [\langle N_i^+ : i \in \omega \rangle]_D$$

is well-defined. It is easily checked that N^+ is an atomic $\mathcal{B}$-network extending N. Let $\exists$ respond to $\forall$'s move in the game $G_\omega(N, \mathcal{B})$ with N^+ — a legal response, as can be checked.

But now, for each $i \in \eta, i > k$, if $\exists$ has a winning strategy for $G_{i-k}(N_i, \mathcal{A}_i)$ then she also has a winning strategy for $G_{i-(k+1)}(N_i^+, \mathcal{A}_i)$. So, replacing k by $k+1$,

$$\begin{array}{ll} & \{i \in \eta : i \geq k+1, \exists \text{ has a winning strategy for } G_{i-(k+1)}(N_i^+, \mathcal{A}_i)\} \\ \supseteq & \{i \in \eta : i \geq k+1, \exists \text{ has a winning strategy for } G_{i-k}(N_i, \mathcal{A}_i)\}. \end{array}$$

The latter set differs from

$$\{i \in \eta : i \geq k, \exists \text{ has a winning strategy for } G_{i-k}(N_i, \mathcal{A}_i)\}$$

by at most a single point (k), and therefore belongs to the non-principal ultrafilter D. Thus, the conditions of the theorem are maintained for the next round of the game, and in this way $\exists$ can survive forever. □

COROLLARY 17 *Let $\mathcal{A}$ be an atomic relation algebra. If $\exists$ has a winning strategy for $G_i(\emptyset, \mathcal{A})$ (each $i < \omega$), then there is an elementary extension $\mathcal{B}$ of $\mathcal{A}$ such that $\exists$ has a winning strategy for $G_\omega(\emptyset, \mathcal{B})$.*

PROOF:

Take $\mathcal{B} = \Pi_D \mathcal{A}$, any non-principal ultrapower of $\mathcal{A}$ over ω. By Łoś' theorem, we have $\mathcal{B} \succeq \mathcal{A}$ up to isomorphism. Now let $k = 0$ in the preceding theorem. □

REMARK 18 We can avoid the use of ultrapowers in the proof of corollary 17, using saturation instead. Indeed, suppose that $\mathcal{A}$ is an atomic relation algebra, and $\exists$ has a winning strategy in $G_i(\emptyset, \mathcal{A})$, for each $i < \omega$. Then $\exists$ has a winning strategy in $G_\omega(\emptyset, \mathcal{B})$, for any ω-saturated relation algebra $\mathcal{B}$ elementarily equivalent to $\mathcal{A}$.

We sketch a proof of this. Let N be a finite $\mathcal{B}$-network, and suppose that $\exists$ has a winning strategy σ_i in $G_i(N, \mathcal{B})$ for all $i < \omega$. We show that she has a winning strategy in $G_\omega(N, \mathcal{B})$.

Again, we assume for simplicity that N is non-empty. Let $\forall$ play (e, a, b) in the first round of $G_\omega(N, \mathcal{B})$. Suppose that $\exists$'s response, using σ_i, is the network $N_i^+ \supseteq N$. We can assume that all the N_i^+ have the same set S of nodes.

Now one can construct, by induction on $j < \omega$, a first-order formula $\psi_j = \psi_j^S$, with free variables v_{nm} $(n, m \in S)$, having the following property: for any i, $\exists$ has a winning strategy in $G_j(N_i^+, \mathcal{B})$ if and only if $\mathcal{B} \models \psi_j(v_{nm}/N_i^+(n,m) : n, m \in S)$. (See section 3.5 and [7] for more on this.) Since $\exists$ clearly has a winning strategy in $G_j(N_i^+, \mathcal{B})$ for all $j < i$, it follows that the type

$$\{\psi_j(v_{nm}/N(n,m) : n, m \in N) : j < \omega\},$$

with free variables $v_{nm} : (n,m) \in S^2 \setminus N^2$ and finitely many parameters $N(n,m) : n, m \in N$, is consistent. As $\mathcal{B}$ is ω-saturated, it is realised in $\mathcal{B}$, so we can find what is evidently a network $N_\omega^+ \supseteq N$, with nodes S, such that $\mathcal{B} \models \psi_j(v_{nm}/N_\omega^+(n,m) : n, m \in S)$ for all $j < \omega$. If $\exists$ plays N_ω^+ in response to $\forall$'s move, she recovers a situation where she can win $G_i(N^+_\omega, \mathcal{B})$ for all $i < \omega$. So she can repeat the process. If she continues like this, she will win $G_\omega(N, \mathcal{B})$.

Now for each $i < \omega$, $\psi^\emptyset_i$ is a sentence; and if $\mathcal{C}$ is any atomic relation algebra, $\mathcal{C} \models \psi^\emptyset_i$ if and only if $\exists$ has a winning strategy in the game $G_i(\emptyset, \mathcal{C})$. Since $\mathcal{B} \equiv \mathcal{A}$, it follows that $\exists$ does have a winning strategy in $G_i(\emptyset, \mathcal{B})$ for all $i < \omega$. So she has a winning strategy in $G_\omega(\emptyset, \mathcal{B})$, as required.

We have nearly constructed a relation algebra, elementarily equivalent to $\mathcal{A}$, with a complete representation. The only remaining problem is that the ultrapower $\mathcal{B}$ may be uncountable.

LEMMA 19 *If $\mathcal{A}$ is an atomic relation algebra, and $\exists$ has a winning strategy for $G_i(\emptyset, \mathcal{A})$ (for each $i < \omega$) then $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra.*

PROOF:

Let $\mathcal{B}$ be any non-principal ultrapower of $\mathcal{A}$ over ω. We know (corollary 17) that $\exists$ has a winning strategy in $G_\omega(\emptyset, \mathcal{B})$. But $\mathcal{B}$ may be uncountable, so we cannot conclude using proposition 13 that $\mathcal{B}$ has a complete representation. The job of this proof is to find a countable, elementary subalgebra $\mathcal{A}'$ of $\mathcal{B}$ such that $\exists$ has a winning strategy for $G_\omega(\emptyset, \mathcal{A}')$.[1] Proposition 13 then does imply that $\mathcal{A}'$ has a complete representation.

To construct $\mathcal{A}'$, we build a countable chain of countable, elementary subalgebras,

$$\mathcal{B}_0 \preceq \mathcal{B}_1 \preceq \dots \qquad (\preceq \mathcal{B}),$$

and define $\mathcal{A}'$ to be the union of the chain. The chain is defined inductively. Let $\mathcal{B}_0$ be any countable elementary subalgebra of $\mathcal{B}$. Suppose that we have defined the countable relation algebra $\mathcal{B}_j \preceq \mathcal{B}$ (some $j < \omega$). We know that $\exists$ has a winning strategy, say σ, in the game $G_\omega(\emptyset, \mathcal{B})$. σ is a function which takes an initial play of the game $\emptyset \subseteq N_1 \subseteq \cdots \subseteq N_k$, and a move by $\forall$ (say (e, a, b), where e is an edge of N_k and $a, b \in \mathcal{B}$), and gives some extension N_{k+1} of N_k. But σ can be defined alternatively as a function giving the atomic network that $\exists$ should respond with, given that $\forall$'s first k moves are $\langle (e_0, a_0, b_0), \dots (e_k, a_k, b_k)\rangle$. This is because $\exists$'s previous moves are *determined* by $\forall$'s moves and σ. Defining σ in this way, consider $\sigma|_{\mathcal{B}_j}$, the 'restriction of σ to $\mathcal{B}_j$' — that is, restrict $\forall$'s moves so that the atoms $a_0, b_0, \dots, a_k, b_k$ can only be selected from $\mathcal{B}_j$ (all k). The range of $\sigma|_{\mathcal{B}_j}$ will be a countable set of atomic $\mathcal{B}$-networks, and the set S of atoms that occur in some atomic network from the range of $\sigma|_{\mathcal{B}_j}$ will be countable. Let $\mathcal{B}_{j+1}$ be a countable, elementary subalgebra of $\mathcal{B}$ containing S and $\mathcal{B}_j$ (existence follows from the downward Löwenheim–Skolem–Tarski theorem, or see [2], theorem 3.1.6). This completes the construction of the chain.

By construction, if $\forall$ plays only atoms from $\mathcal{B}_j$, then $\exists$ can win the play while only choosing atomic $\mathcal{B}_{j+1}$-networks.

Now let $\mathcal{A}'$ be the union of the chain: $\mathcal{A}' = \bigcup_{j<\omega} \mathcal{B}_j$. Then $\mathcal{A}'$ is an elementary extension of each $\mathcal{B}_j$ (elementary chain theorem, [2] theorem 3.1.13), and hence is elementarily equivalent to $\mathcal{B}$ and $\mathcal{A}$. Furthermore, it is clear that $\exists$ has a winning strategy for the game $G_\omega(\emptyset, \mathcal{A}')$. Since $\mathcal{A}'$ is countable, we can use proposition 13, and conclude that $\mathcal{A}'$ has a complete representation. □

Lemma 19 completes the proof of proposition 15. □

[1] In fact, the set of such $\mathcal{A}'$ forms a club (closed unbounded set) in $\mathcal{P}_{\leq\omega}(\mathcal{B})$, the set of countable subsets of $\mathcal{B}$.

3.5 Lyndon conditions and complete representations

This section is not essential to the main results of the paper, and the reader may skip it without loss. We simply wish to point out a connection between the games (above) and some well-known work of Lyndon in the 1950s. This allows us to extend proposition 15.

In [10], §5, Lyndon gave an infinite (recursive) set **C** of first-order sentences in the language of relation algebras, which we will call the *Lyndon conditions.* They are related to the 'n-dimensional bases' of [12]. Although Lyndon only defined the conditions in the context of simple atomic relation algebras, they make sense in any atomic relation algebra. He proved that a finite (simple) relation algebra satisfies the conditions if and only if it is representable.

There is a close association between the Lyndon conditions on an atomic relation algebra $\mathcal{A}$, and the existence of a winning strategy for $\exists$ in the games $G_i(\emptyset, \mathcal{A})$ for $i < \omega$. Hints of this can be seen in [10], especially p. 711 and the 'graphical interpretation' on p. 712. See also [7]. Formally, we have ($1 \Leftrightarrow 2$) of the following theorem.

THEOREM 20 *Let $\mathcal{A}$ be any atomic relation algebra. The following are equivalent.*

1. *$\mathcal{A}$ satisfies the Lyndon conditions.*
2. *$\exists$ has a winning strategy in the games $G_i(\emptyset, \mathcal{A})$ for all $i < \omega$.*
3. *$\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra.*

Proof of theorem 20 ($1 \Leftrightarrow 2$). The argument involves a change of notation to handle games on '*loose-networks*'. The definition of these loose-networks is the same as for networks except that the '$\Leftrightarrow$' in the condition '$N(l, m) \leq Id \Leftrightarrow l = m$' is replaced by '$\Leftarrow$'. In the revised game each N_{i+1} of the play has is an atomic loose-network with exactly one more node than the preceding N_i, and (most interestingly) $\forall$ must declare his choice of edges in advance, before play starts. (There is no such restriction on his choice of atoms.) The Lyndon conditions on an atomic $\mathcal{A}$ simply express that $\exists$ has a winning strategy in such games (of all finite lengths) using atomic $\mathcal{A}$-loose-networks and starting from any atomic loose-network of size three. Straightforward game-theoretic arguments show that this is equivalent to her having a winning strategy in $G_n(\emptyset, \mathcal{A})$ for all finite n, as required. (For example, let N be a non-empty atomic network — so also a loose-network — of size at most three. Suppose that $\forall$ promises to choose his edges in n 'blocks', in the following manner. In the first block, he chooses in succession all the edges of N, in some arbitrary order; if the loose-networks that results at the end of the block is N_1, then in the second block $\forall$ chooses all edges of N_1; and so on, for n blocks altogether. Suppose that $\exists$ has a winning strategy in this game. Then she can win $G_n(N, \mathcal{A})$.) We will leave the details of the proof as an exercise for the interested reader. The equivalence will not be used here, and a full proof would be too long a diversion.

Proof of theorem 20 ($3 \Rightarrow 1$). Assume (3) for $\mathcal{A}$. Let $\mathcal{B} \equiv \mathcal{A}$ be a relation algebra that has a complete representation. Clearly, $\mathcal{B}$ is atomic. By proposition 13, $\exists$ has a winning strategy in $G_\omega(\emptyset, \mathcal{B})$, and so also in $G_i(\emptyset, \mathcal{B})$ for all $i < \omega$. It follows from ($2 \Rightarrow 1$) of the theorem that $\mathcal{B}$ satisfies the Lyndon conditions. But these conditions are first-order, so $\mathcal{A}$ must satisfy them too. Hence ($3 \Rightarrow 1$) holds.

Proof of theorem 20 ($2 \Rightarrow 3$). This is proposition 15.

Note Any finite relation algebra is atomic, and is representable if and only if it is completely representable; and two finite structures are elementarily equivalent if and only if they are isomorphic. Hence, by the theorem, a finite relation algebra satisfies the Lyndon conditions if and only if it is representable — this gives Lyndon's result, mentioned above.

3.6 The Rainbow Relation Algebra

We return to our main purpose. We have now seen that if $\exists$ has a winning strategy in $G_n(\emptyset, \mathcal{A})$ for all finite n, then $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra, while if $\mathcal{A}$ itself is completely representable then $\exists$ has a winning strategy in $G_\omega(\emptyset, \mathcal{A})$. So to complete the proof that 'completely representable' is not an elementary property, it suffices to prove the following.

THEOREM 21 *There is an atomic relation algebra $\mathcal{A}$ such that*

- *$\exists$ has a winning strategy in the game $G_n(\emptyset, \mathcal{A})$ for each $n < \omega$*
- *$\exists$ has no winning strategy in the game $G_\omega(\emptyset, \mathcal{A})$.*

This will be shown in propositions 23 and 22 below.

Let us define $\mathcal{A}$ by listing its atoms, stating their converses and then defining composition on the atoms. Any algebra based on this 'atom structure' will then do. For example, we can take the full power set of this set of atoms, and define converse and composition by an infinitary distribution rule. Or, if we wish to keep the cardinality of $\mathcal{A}$ countable, we can just take the subalgebra of the full power set algebra *generated* by the atoms. In proposition 23 we will show that $\exists$ has a winning strategy in the game $G_2(\emptyset, \mathcal{A})$, and it follows from remark 14 that $\mathcal{A}$ will actually be a relation algebra.

$\mathcal{A}$ will be an integral, symmetric relation algebra — the identity is an atom and all elements are self-converse. We will regard the atoms as coloured. The identity atom has no colour. Non-identity atoms are of five different colours: red, green, yellow, black, and white,[2] and here they are:

$$\begin{aligned}
\text{Red} &= \{r^n_i : 0 \leq i < n < \omega\} \\
\text{Green} &= \{g_i : i < \omega\} \\
\text{Yellow} &= \{y\} \\
\text{Black} &= \{b\} \\
\text{White} &= \{w\} \cup \{w_{ij} : i, j < \omega\}.
\end{aligned}$$

The composition of an atomic relation algebra is defined by listing all triples of atoms (x, y, z) such that $x; y \geq z^\smile$ (compare §4 of [10]). We shall call such triples *consistent triples.* If (x, y, z) is a consistent triple, then so are (y, z, x) and (z, x, y) and, since a symmetric relation algebra is commutative, the triples $(x, z, y), (z, y, z)$, and (y, z, x) must also be consistent. We define the consistent triples of atoms of $\mathcal{A}$ to consist of all triples *except for* permutations of the following.

(Id, r, s)	unless $r = s$ (any atoms r, s)
(g_i, g_j, g_k)	any $i, j, k < \omega$
(g_i, g_j, b)	any $i, j < \omega$
(y, y, y)	
(y, y, W)	for any white atom W
(w_{ij}, g_k, y)	unless $k = i$ or j
(r^l_i, r^m_j, r^n_k)	unless $l = m = n$ and $\lvert\{i, j, k\}\rvert = 3$.

Note Let N be an atomic $\mathcal{A}$-network with each edge $(k, l) \in N$ $(k \neq l)$ labelled by a red atom. Then since on each triangle all three edges are red, the upper index must be the same throughout the network. Let the common upper index be n. Now fix a single node $k \in N$. Each pair of edges (k, l) and (k, m) (with $l \neq m$) must be labelled by *distinct* atoms, say r^n_i and r^n_j $(i \neq j)$. This

[2] There is a heated debate taking place in the school playground of the first author's seven year old daughter as to whether black and white should be regarded as colours or not. The majority view is that white and black can be treated as colours for certain purposes and that is the view we take in this article. For those who are unhappy about this position, please substitute other 'colours', say, purple and orange.

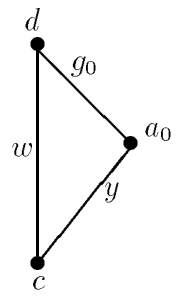


Figure 1:

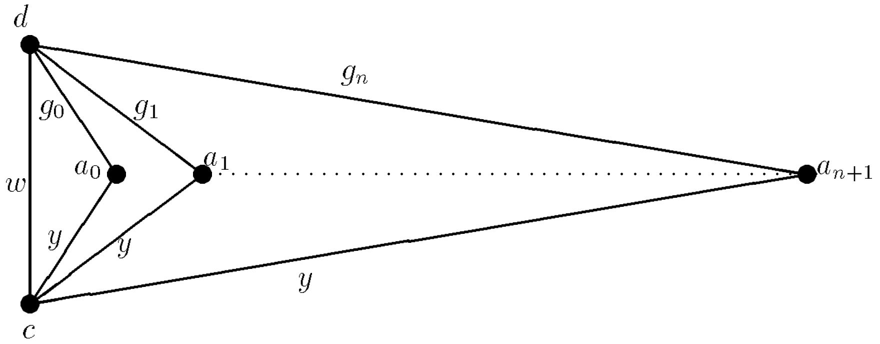


Figure 2:

is because the lower indices of the atoms on an all-red triangle must be distinct. Since there are only n different red atoms with upper index n, it follows that the size of the network N cannot exceed $n+1$.

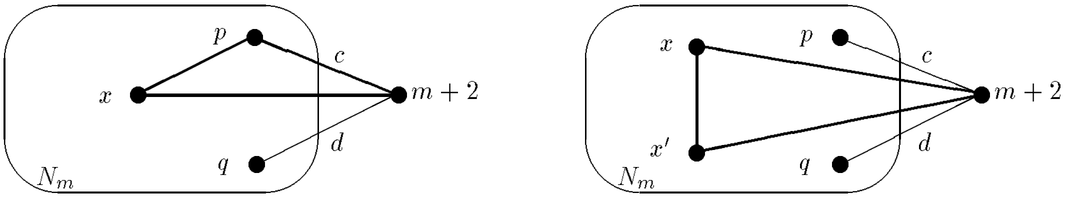
PROPOSITION 22 *$\exists$ has no winning strategy for $G_\omega(\emptyset, \mathcal{A})$.*

By proposition 13, this implies that $\mathcal{A}$ has no complete representation.
PROOF:

Let $\forall$ start with the following sequence of moves.

1. Let $\forall$'s first move be the atom w, and let $\exists$ respond with the network N_0 consisting of two nodes, d, c, with $N_0(d, c) = w$. In his second turn, $\forall$ picks the edge (d, c) and the atoms g_0 and y. $\exists$ responds by adding one extra node, a_0, and labelling the edges as in figure 1.
2. Next, $\forall$ picks the edge (d, c) again, and chooses the atoms (g_1, y). $\exists$ must add a new node a_1, and label the missing edge, (a_0, a_1). The label must be an atom x such that the triples (x, g_0, g_1) and (x, y, y) are both consistent. Thus, x can only be a red atom — say, $x = r^n_k$, for some $k < n < \omega$. Now $\exists$ can choose n as large as she likes, but it will be finite. In the following n moves, $\forall$ will be able to defeat her (see figure 2).
3. The next n moves by $\forall$ each use the same edge (d, c) and the pair of atoms (g_i, y) (for $i = 2, 3, \ldots, n+1$). Each time, $\exists$ must respond by adding a new node a_i (since g_i is new), and labelling the missing edges (a_i, a_j) $(j < i)$, and in each case they can only be labelled by red atoms with upper index n.

By the foregoing remark, an all-red atomic network where the upper index is n cannot have size greater than $n+1$. However, if $\exists$ were still winning after a_{n+1} was



Primary triangle $(x, p, m+2)$ Secondary triangle $(x, x', m+2)$

Figure 3:

included, the atomic network on the nodes $a_0, \ldots, a_{n+1}$ would be an all-red atomic network of size $n+2$, with upper index n. This is impossible, so $\forall$ must have won. $\square$

PROPOSITION 23 $\exists$ *has a winning strategy for the game* $G_n(\emptyset, \mathcal{A})$, *for each* $n < \omega$.

By proposition 15, this implies that $\mathcal{A}$ is elementarily equivalent to a completely representable relation algebra. Since the representable relation algebras form a variety, we can conclude that $\mathcal{A}$ is a representable relation algebra.

PROOF:

The idea behind the proof is not hard: if $\forall$ plays the strategy used in proposition 22, above, then $\exists$ can win any game of finite length by choosing red atoms of sufficiently high superscript (say $(n+2)^2$ in a game of length n) so that she is never forced to build an inconsistent all-red network. The proof gets complicated somewhat, however, because we cannot assume that $\forall$ plays in this way. We have to provide a winning $\exists$-strategy to cover every eventuality.

For each turn in a play of $G_n(\emptyset, \mathcal{A})$, $\exists$ plays using the following strategy. In round 0, let $\forall$ play the atom a. If $a = Id$, $\exists$ responds with the network N_0 with a single node, m, with $N_0(m, m) = Id$. If not, she plays the network N_0 with two nodes, l, m, with $N_0(l, l) = N_0(m, m) = Id$ and $N_0(l, m) = N_0(m, l) = a$.

At the start of the $(m+1)$th round, let the play so far be

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m,$$

where $|N_k| \leq k+2$ for each $k \leq m$. Without loss, we may assume that the nodes of N_k are a subset of $\{0, 1, \ldots, k+1\}$. Suppose that $\forall$ picks the edge $(p, q) \in N_m^2$ and the pair of atoms $c, d \in \mathcal{A}$. So the triple $(c, d, N_m(p, q))$ is consistent.

If there is already a node $z \in N_m$ such that $N_m(p, z) = c$ and $N_m(z, q) = d$, then $\exists$ does nothing but sets

$$N_{m+1} = N_m. \tag{2}$$

Otherwise, $\exists$ adds a new node, '$m+2$', defining $N_{m+1}(m+2, m+2)$ to be Id. She now has to define the atomic labels $N_{m+1}(x, m+2)$, $N_{m+1}(m+2, x)$ on each missing edge $(x, m+2), (m+2, x)$ for $x \in N_m \setminus \{p, q\}$ in such a way that the resulting graph forms an atomic network, N_{m+1}. The label $N_{m+1}(m+2, x)$ will always be the same as $N_{m+1}(x, m+2)$, so we need only specify $N_{m+1}(x, m+2)$.

For the resulting N_{m+1} to be an atomic network, it is sufficient that every triangle $(x, x', m+2)$, for distinct $x, x' \in N_m$, is consistent. There are three kinds of triangle of this form: the triangle $(p, q, m+2)$ (if $p \neq q$), which $\forall$ chose and so is guaranteed consistent; 'primary' triangles, which involve p or q, so that $\exists$ has to define only one of the labels on the edges of the triangle; and finally 'secondary' ones, in which she has to define two. See figure 3. Her strategy selects an atom for $N_{m+1}(x, m+2)$ according to the rule: the *colour* of the atom is always white if this makes the *primary* triangles

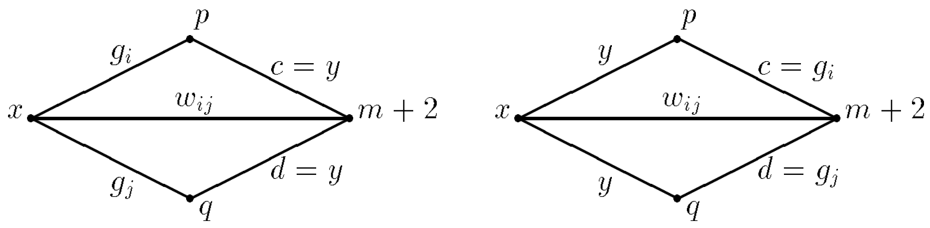


Figure 4:

involving x consistent, otherwise black if possible, or finally red. (She may add *indices* too.) Formally, her choice of colour is as follows.

1. If it is not the case that (a) the colours of c and $N_m(x,p)$ are both yellow, nor is it the case that (b) the colours of d and $N_m(x,q)$ are both yellow, then she chooses the colour white for $N_{m+1}(x, m+2)$.

   This only might produce an inconsistent primary triangle if $\exists$ chooses w_{ij}, and a triangle (w_{ij}, g_k, y) with $k \neq i, k \neq j$ results — but her choice of indices in (4), below, will ensure this doesn't happen.
2. If (a) or (b) does hold (so $\exists$ cannot choose white) but neither (a') the colours of c and $N_m(x,p)$ are both green nor (b') the colours of d and $N_m(x,q)$ are both green, then she chooses the colour black for $N_{m+1}(x, m+2)$. Clearly, all primary triangles created are consistent.
3. If either (a) or (b) holds and either (a') or (b') holds (which means either (a) and (b') holds, or (a') and (b)) then both white and black are illegal, so she chooses red. Regardless of her choice of indices (below), all primary triangles created are consistent.

This much already ensures that all secondary triangles — with the critical exception of all-red triangles — are consistent. For any secondary triangle has two edges coloured by $\exists$, hence two edges labelled by white, black, or red atoms. Inspection of the original definition of consistent triples of atoms of $\mathcal{A}$ shows that in such a case, inconsistency is only possible if all three edges are red.

Now the indices.

4. For case 1, $\exists$ gives subscripts in some cases (see figure 4). She adds the subscript ij to w if either
   (a) $N_m(x,p) = g_i, N_m(x,q) = g_j$ and $\forall$ chose y and y for $(p, m+2)$ and $(m+2, q)$;
   (b) $N_m(x,p) = N_m(x,q) = y$, and $\forall$ chose g_i and g_j for $(p, m+2)$ and $(m+2, q)$. (This is (4a) with $x, m+2$ reversed.)
   (c) Otherwise there is no subscript: the label is just w

   See figure 4. It is now certain that all primary triangles created when $\exists$ chooses white are consistent.
5. If $\exists$ is forced to choose a red colour (case 3) then the atom she chooses is $r^{(n+2)^2}_{(n+2).x+m+1}$. The idea here is that in a game of length n, there will not be more than $n+2$ distinct nodes and $(n+2)^2$ edges. By choosing $(n+2)^2$ as the superscript, there will be 'enough room' for $\exists$ to choose distinct red colours with the same superscript, for all the red edges she chooses.

We now know that all primary triangles created are consistent. It only remains to check that any secondary all-red triangles are consistent. This is the key step of the proof. So suppose that there are two nodes $x, x' \in N_m$ where $N_m(x, x')$ is already a red atom, and $\exists$ sets

$$N_{m+1}(x, m+2) = r^{(n+2)^2}_{(n+2).x\ +m+1} \text{ and } N_{m+1}(x', m+2) = r^{(n+2)^2}_{(n+2).x'\ +m+1}.$$

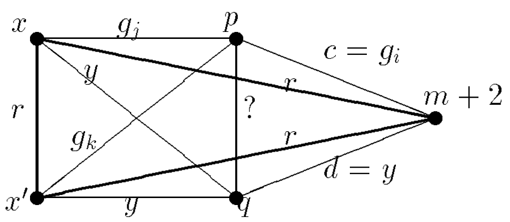


Figure 5:

There is no loss of generality if we assume that $x' > x$.

The only reason $\exists$ chooses red for the edge $(x, m+2)$ (rather than white or black) is that (a) and (b') hold, i.e., $N_m(x,p) = c = y$ and $N_m(x,q) = g_j, d = g_i$ (some j, i), or (a') and (b) hold, i.e., $N_m(x,p) = g_j, c = g_i$ (some j, i) and $N_m(x,q) = d = y$. Without loss of generality, assume the latter.

Since we now have $d = y$, the only reason that $\exists$ chose red for the edge $(x', m+2)$ is that $N_m(x', p) = g_k$ (some k), $c = g_i$ (already assumed) and $N_m(x', q) = d = y$. Thus we have the situation represented in figure 5.

Now if the label on (x, x') was chosen by $\exists$ in a previous round, her strategy would have selected the atom $r^{(n+2)^2}_{(n+2).x+x'-1}$. In this case, the triangle $(x, x', m+2)$ is consistent, as $(n+2).x + x' - 1 < (n+2).x\ + m + 1 < (n+2).x'\ + m + 1$, and the definition of the atom structure permits all-red triangles with equal upper indices provided that the lower indices are all distinct.

But what if the label on (x, x') were chosen by $\forall$? In fact, this is not possible. To see this, first note the following claim.

Claim. If N is an atomic network constructed in any play of $G_n(\emptyset, \mathcal{A})$, then for any quadrangle $Q \subseteq N$, at least one of the six edges of Q was coloured by $\exists$.

The claim is not hard to check. Now returning to the case in hand, consider the quadrangle in N_m with nodes x, x', p, q. Assume for contradiction that the edge (x, x') was labelled by $\forall$. By the claim, $\exists$ must have labelled one of its other edges. We know (figure 5) that the edges $(x,p), (x,q), (x',p)$ and (x',q) are labelled by yellow or green atoms; $\exists$'s strategy never selects those colours, so none of these could have been labelled by her.

That leaves the edge (p, q). $\exists$ must have labelled this edge. But we can see by their colours that (x,p,q) and (x',p,q) were both primary triangles in the round when $\exists$ labelled (p,q), so her strategy would have led her to pick the atom w_{jk} or w_{kj} (by case 4). The consistency of $(c, d, N_m(p,q))$, it being $\forall$'s choice, now forces $i = j$ or $i = k$. This makes the new node $m+2$ into a duplicate of either x or x'. In either case, $\exists$ could respond in the current round using equation 2 — i.e., by doing nothing, and setting $N_{m+1} = N_m$. Since we assumed at the outset that she could not do this, we have a contradiction.

We have shown that $\exists$'s strategy always provides her with a legal move. Hence, it is a winning strategy for $G_n(\emptyset, \mathcal{A})$. This proves the proposition. $\square$

To sum up, we have shown the following.

THEOREM 24 *The class of all completely representable relation algebras is not elementary.*

PROOF:

It follows from proposition 23 and remark 14 that the algebra $\mathcal{A}$ just constructed is actually a relation algebra. It is clearly atomic. By propositions 22 and 13(1), it has no complete representation. But by propositions 23 and 15, it is elementarily equivalent to a completely representable relation algebra.

Hence, the class of completely representable relation algebras cannot be elementary.

$\square$

Note This example answers an open question of Maddux, by constructing an integral, representable relation algebra with no complete representation. The example has the further property of being symmetric.

4 Cylindric Algebra

Here, we will show that for any fixed dimension $\alpha \geq 3$, the α-dimensional cylindric algebras that have a complete representation do not form an elementary class. It is clearly of interest to do this. Cylindric algebras are the accepted generalisation of relation algebras to relations of arity larger than two, so this result rounds out our investigation to cover a natural class of algebras. Unfortunately, proving it is not quite a matter of duplicating the earlier arguments for relation algebra. Though much of the procedure is similar, the analogue of proposition 23 for cylindric algebras involves some important technical variations that are perhaps not obvious, and it would be disingenuous to say simply that the result for relation algebras generalises easily to cylindric algebras. Therefore, we hope the reader will bear with us if the work appears to be repeating that of earlier sections.

One might ask if we may simply obtain the result for cylindric algebras directly from the one for relation algebras. We already have two elementarily equivalent relation algebras $\mathcal{A}, \mathcal{A}'$, with the latter completely representable, and the former, not. It would be nice to construct from them two elementarily equivalent n-dimensional cylindric algebras $\mathcal{C}_n, \mathcal{C}'_n$, with $\mathcal{C}'_n$ completely representable but $\mathcal{C}_n$ not. There *is* a connection between n-dimensional cylindric algebras and relation algebras in which representations of the first give rise to representations of the second (through the so-called *neat reduct* to two variables); but, unfortunately, a representation of a relation algebra does not in general give rise to a representation of a corresponding n-dimensional cylindric algebra, unless the relation algebra has what Maddux calls an *n-dimensional cylindric basis* [14, 15]. Sadly, our relation algebras $\mathcal{A}, \mathcal{A}'$ do not possess such a basis. So instead, we construct $\mathcal{C}_n, \mathcal{C}'_n$ afresh. Most of the other results required from relation algebra do carry over to cylindric algebras, at least in the finite dimensional case. Surprisingly perhaps, we can also prove that for any infinite ordinal α, the class of all α-dimensional cylindric algebras with complete representations is not elementary. The proof of this turns out to be quite simple and is dealt with separately as it is not based on the same argument.

4.1 Basics

We assume some familiarity with cylindric algebras and their representations, though we will give complete definitions. The principal references are [4, 5].

Let $\alpha \geq 3$ be an ordinal.

- A *cylindric algebra of dimension* α is a structure $\mathcal{C} = \langle C, 0, 1, -, \vee, d_{\lambda\mu}, c_\lambda \rangle_{\lambda,\mu<\alpha}$ obeying the following axioms.
  - The reduct $\langle C, 0, 1, -, \vee \rangle$ is a boolean algebra.
  - $c_\lambda 0 = 0$
  - $x \leq c_\lambda x$
  - $c_\lambda(x \wedge c_\lambda y) = c_\lambda x \wedge c_\lambda y$
  - $c_\lambda c_\mu x = c_\mu c_\lambda x$

- $d_{\lambda\lambda} = 1$
- if $\lambda \neq \mu, \kappa$ then $d_{\mu\kappa} = c_\lambda(d_{\mu\lambda} \wedge d_{\lambda\kappa})$
- if $\lambda \neq \mu$ then $c_\lambda(d_{\lambda\mu} \wedge x) \wedge c_\lambda(d_{\lambda\mu} \wedge -x) = 0$,

for any $x, y \in C$ and any $\kappa, \lambda, \mu < \alpha$.

- A *representation* of $\mathcal{C}$ over the non-empty set X is a map $h : \mathcal{C} \to \mathcal{P}(X^\alpha)$ such that:

  1. $h(1) = \bigcup_{i \in I} X_i^\alpha$, for some pairwise disjoint subsets $X_i \subseteq X$ $(i \in I)$
  2. $h : \mathcal{C} \to \mathcal{P}(h(1))$ is a representation of the boolean reduct (hence h is 1–1)
  3. for all $\lambda, \mu < \alpha$, $\bar{x} \in h(d_{\lambda\mu})$ if and only if $x_\lambda = x_\mu$
  4. for all $\lambda < \alpha$, $\bar{x} \in h(c_\lambda(c))$ if and only if there is a point $y \in X$ with $\bar{x}[\lambda \to y] \in h(c)$.

  Here, $\bar{x}$ is an arbitrary α-sequence in $h(1)$, and $\bar{x}[\lambda \to y]$ is the sequence obtained from $\bar{x}$ by replacing the λth term, x_λ, by y.

- A *complete representation* is a representation respecting arbitrary disjunctions (and conjunctions), wherever they are defined. As before, a representation is complete if and only if it is *atomic* (i.e., for every $\bar{x} \in h(1)$ there is an atom $\gamma \in \mathcal{C}$ with $\bar{x} \in h(\gamma)$).

- Let us call a structure $\mathcal{C}$ an *α-dimensional cylindric-type algebra* if it is a boolean algebra enriched with constants $d_{\lambda\mu}$ $(\lambda, \mu < \alpha)$ and unary operators c_λ $(\lambda < \alpha)$, such that each operator c_λ is idempotent ($c_\lambda(c_\lambda(x)) = c_\lambda(x)$ for all $x \in \mathcal{C}$) and completely additive: if $x \in \mathcal{C}$ and $S \subseteq \mathcal{C}$ then $x = \bigvee S$ implies $c_\lambda(x) = \bigvee_{s \in S} c_\lambda(s)$. This implies normality ($c_\lambda(0) = 0$).

4.2 The Infinite Dimensional Case

We show by a simple cardinality argument that the class of all cylindric algebras of any fixed infinite dimension that have a complete representation is not elementary.

Let α be a fixed infinite ordinal, the dimension. Below, $|\alpha|$ denotes the cardinality of α. Write $\mathbf{CCA}_\alpha$ for the class of all α-dimensional cylindric algebras $\mathcal{C}$ such that $\mathcal{C}$ has a complete representation. If $\mathcal{C} \in \mathbf{CCA}_\alpha$ then $\mathcal{C}$ must be atomic. We write $At(\mathcal{C})$ for the set of all atoms of $\mathcal{C}$.

LEMMA 25 *Let $\mathcal{C} \in \mathbf{CCA}_\alpha$ be such that $\mathcal{C} \models d_{01} < 1$. Then $|At(\mathcal{C})| \geq 2^{|\alpha|}$.*

PROOF:

Let $h : \mathcal{C} \to \mathcal{P}(X^\alpha)$ be a complete representation of $\mathcal{C}$. Since $\mathcal{C} \models d_{01} < 1$, there is $\bar{a} \in h(-d_{01})$. So if $x = a_0, y = a_1$, then $x \neq y$.

Let $S \subseteq \alpha$ be arbitrary such that $0 \in S$, and define $\bar{a}_S \in X^\alpha$ to be the sequence whose ith coordinate is x, if $i \in S$, and y, if $i \in \alpha \setminus S$. By definition of 'complete representation', every $\bar{a}_S$ is in $h(1)$ and so in $h(\gamma)$ for some unique atom γ of $\mathcal{C}$; we can say that γ *labels* $\bar{a}_S$. Let $S, S' \subseteq \alpha$ be any distinct sets containing 0. Then, without loss of generality, there is $i < \alpha$ with $i \in S, i \notin S'$. So $\bar{a}_S \in h(d_{0i})$ and $\bar{a}_{S'} \in h(-d_{0i})$. It follows that the atom of $\mathcal{C}$ that labels $\bar{a}_S$ is different from the atom labelling $\bar{a}_{S'}$. Hence, the number of atoms in $\mathcal{C}$ is at least the number of subsets of α that contain 0 — i.e., at least $2^{|\alpha|}$. □

COROLLARY 26 *The class $\mathbf{CCA}_\alpha$ is not elementary.*

PROOF:

Take any $\mathcal{C} \in \mathbf{CCA}_\alpha$ such that $\mathcal{C} \models d_{01} < 1$. (There exist such $\mathcal{C}$ — for example, the full power set algebra $\mathcal{P}(X^\alpha)$, regarded as a cylindric algebra in the natural way, where X is any set with at least two elements.) Since the cardinality of the language of α-dimensional cylindric algebras is $|\alpha|$, we may use the downward Löwenheim–Skolem–Tarski theorem to take $\mathcal{B} \preceq \mathcal{C}$ with $|\mathcal{B}| \leq |\alpha|$. Then $\mathcal{B} \models d_{01} < 1$, since $\mathcal{B}$ is an elementary substructure of $\mathcal{C}$. But $\mathcal{B}$ has at most $|\alpha|$ atoms. Hence, by the lemma, $\mathcal{B} \notin \mathbf{CCA}_\alpha$. So the class $\mathbf{CCA}_\alpha$ is not closed under elementary equivalence, and so cannot be elementary. □

4.3 The Finite Dimensional Case

Let us turn now to the finite dimensional case. Fix a finite dimension $n < \omega$, with $n \geq 3$. As with relation algebras, we can characterise the completely representable n-dimensional cylindric algebras by a game played on networks. In fact, provided the algebra has at most countably many atoms, we show that a cylindric algebra has a complete representation if and only if the existential player has a winning strategy in the game of length ω. We then construct an n-dimensional cylindric algebra $\mathcal{C}_n$ so that the existential player can win any game of finite length, but she has no winning strategy for the game of length ω. So $\mathcal{C}_n$ does not have a complete representation.

Then, as with relation algebras, we can show that for any non-principal ultrapower of $\mathcal{C}_n$ the existential player can win the game of length ω, and therefore there is a countable, elementary subalgebra of the ultrapower which does have a complete representation. This suffices to show that the class of n-dimensional cylindric algebras with complete representations is not closed under elementary equivalence, and is not elementary.

In outline, then, the argument is very similar to the one we used for relation algebras. The construction of the required $\mathcal{C}_n$, where the existential player can win games of finite but not infinite length, turns out to be rather more complex than the example we constructed for relation algebras. The reason for this seems to be that $\mathcal{C}_n$ possesses elements that are *essentially* $(n-1)$-*ary,* by which we mean that they are not equal to boolean combinations of lower dimensional elements.

4.3.1 Networks and Games

DEFINITION 27 *Let $\mathcal{C}$ be an atomic n-dimensional cylindric-type algebra.*

1. *An* atomic network *N is a set of nodes Δ and a total function $N : \Delta^n \to At(\mathcal{C})$ such that*
   - $N(\bar\delta) \leq d_{ij}$ *if and only if* $\delta_i = \delta_j$ *(for any $\bar\delta \in \Delta^n$ and any $i, j < n$)*
   - $N(\bar\delta[i \to d]) \leq c_i N(\bar\delta)$ *(for any $i < n, \bar\delta \in \Delta^n, d \in \Delta$)*

   As before, we let N stand for the set of nodes (Δ) as well as for the function and the network itself.
2. *An atomic network N* extends *an atomic network M, and we write $N \supseteq M$, if*
   - *the nodes of M form a subset of the nodes of N,*
   - *if $\bar\mu \in M^n$ then $M(\bar\mu) = N(\bar\mu)$.*
3. *The game $G_k(\mathcal{C})$ has k rounds, $k \leq \omega$, and is defined as follows. In the zero'th round, $\forall$ picks any atom a of $\mathcal{C}$. $\exists$ has to respond with a finite atomic network N_0 such that $N_0(\bar d) = a$ for some n-tuple of nodes $\bar d \in N_0^n$. Without loss, $|N_0| \leq n$. In any further round, let the last network played be N. $\forall$ picks an index $i < n$, a 'face' $F = \langle f_0, f_1, \ldots, f_{n-1}\rangle \in N^{n-1}$, and an atom $b \leq c_i N(f_0, \ldots, f_{i-1}, x, f_i, \ldots, f_{n-1})$ (the choice of $x \in N$ is arbitrary, as the second part of the definition of an atomic network together with the fact that $c_i(c_i(x)) = c_i(x)$ for all $x \in \mathcal{C}$ ensures that the right-hand side does not depend on it). $\exists$ must respond, if possible, with a network $N^+ \supseteq N$ with at most one more node, such that there is a node $d \in N^+$ with $N^+(f_0, \ldots, f_{i-1}, d, f_i, \ldots, f_{n-1}) = b$. If she can do this in every round, she has won the play.*

4. *The definition of a winning strategy is similar to the one for relation algebra.*

The following results (from proposition 13, corollary 17, and lemma 19) go over to n-dimensional cylindric algebras unaltered and are not proved here.

THEOREM 28 *Let $\mathcal{C}_n$ be an n-dimensional cylindric-type algebra ($n < \omega$).*

1. *If $\mathcal{C}_n$ has at most countably many atoms, then it has a complete representation (and is therefore certainly a cylindric algebra) if and only if $\exists$ has a winning strategy in the game $G_\omega(\mathcal{C}_n)$.*
2. *If $\exists$ has a winning strategy in the games $G_k(\mathcal{C})$ (all $k < \omega$) and if $\mathcal{D}$ is any non-principal ultrapower of $\mathcal{C}_n$ over ω, then $\exists$ has a winning strategy for $G_\omega(\mathcal{D})$.*
3. *If $\exists$ has a winning strategy for $G_\omega(\mathcal{D})$, then $\mathcal{D}$ has a countable, elementary subalgebra $\mathcal{B}_n$ which is completely representable.*

Thus, as for relation algebras, if we can construct an n-dimensional cylindric algebra $\mathcal{C}_n$ with countably many atoms such that $\exists$ has a winning strategy for each game $G_k(\mathcal{C}_n)$ ($k < \omega$), but not for the game $G_\omega(\mathcal{C}_n)$, then we will have shown that the class of all completely representable n-dimensional cylindric algebras is not elementary. This follows from the previous theorem because $\mathcal{C}_n \equiv \mathcal{B}_n$, $\mathcal{C}_n$ is not completely representable, but $\mathcal{B}_n$ is.

4.3.2 The 'rainbow' construction for cylindric algebras

We turn now to the construction of the required cylindric algebra $\mathcal{C}_n$, but first we need to define a class of graphs. As mentioned before, the atoms of the required cylindric algebra cannot be coded simply as binary relations. This makes them hard to picture. To deal with this we have coded the atoms as certain coloured graphs, where almost all the information is stored in the colours of binary edges but one range of colours (yellow) is reserved to label $(n-1)$-tuples from the graphs. This allows us to keep much of the work similar to that for relation algebras and confines the essentially $(n-1)$-ary coding to just one part of the construction.

4.3.3 Coloured Graphs

A *coloured graph* is an undirected graph Γ such that every edge of Γ is coloured by a unique *edge colour* (below), and some $(n-1)$-tuples have unique colours, too. The edge colours are:

- greens: g_i ($i = 1, \dots n-2$) and g_0^i ($i < \omega$);
- whites: w_i ($i = 0, \dots, n-2$);
- reds: r_i^m ($i < m < \omega$).

The colours for $(n-1)$-tuples are:

- yellows: y_S ($S \subseteq \omega$, $S = \omega$ or S finite).

We will sometimes write $\Gamma(x, y)$ for the colour of an edge (x, y), and $\Gamma(a_1, \dots, a_{n-1})$ for the colour of an $(n-1)$-tuple $a_1, \dots, a_{n-1}$, in the coloured graph Γ. Note that these may not always be defined: for example, $\Gamma(x, x)$ is not.

Let Γ, Δ be coloured graphs, and $\psi : \Gamma \to \Delta$ be a map. ψ is said to be a *coloured graph embedding,* or simply an embedding, if it is 1–1 and preserves all edges, and all colours, where defined, in both directions.

Let $i < \omega$ and let Γ be a coloured graph with n nodes, $x_0, \dots, x_{n-2}, y$, such that (x_j, y) is an edge of Γ for each $j < n-1$. We call Γ an *i-cone* if for each $j < n-1$, the edge (x_j, y) is coloured g_j if $j > 0$, and g_0^i if $j = 0$ (see figure 6), and no other edges of Γ (if any) are coloured green. The *apex* of the cone is y, its *base* $\{x_0, \dots, x_{n-2}\}$. The *tint* of the cone is i. These are well-defined, as any Γ can be viewed as a cone in at most one way. Notice that a cone induces a linear ordering on its base, namely, $x_0, \dots, x_{n-2}$.

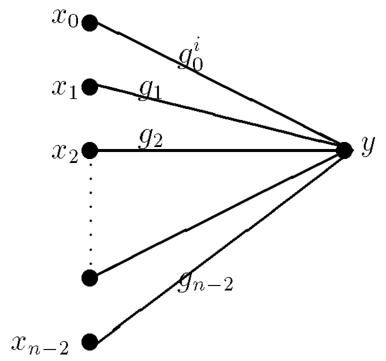


Figure 6: an i-cone

DEFINITION 29 *The class $\mathcal{G}$ consists of all coloured graphs Γ with the following properties.*

1. *Γ is a complete graph (all possible edges are present)*
2. *Γ contains no triangles of the following types:*
   - (g, g', g^*) *any green colours g, g', g^**
   - (g_i, g_i, w_i) *any $i = 1, \ldots n-2$*
   - (g_0^j, g_0^k, w_0) *any $j, k < \omega$*
   - $(r_i^m, r_j^{m'}, r_k^{m^*})$ *unless $m = m' = m^*$ and $|\{i, j, k\}| = 3$.*
3. *If $a_0, \ldots, a_{n-2} \in \Gamma$ are distinct, and no edge (a_i, a_j) $(i < j < n-1)$ is coloured green, then the sequence $\langle a_0, \ldots, a_{n-2}\rangle$ is coloured a unique shade of yellow. No other $(n-1)$-tuples are coloured yellow.*
4. *If $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq \Gamma$, and $\Gamma \restriction_D$ (the coloured graph induced on D) is an i-cone with apex δ, inducing the ordering $d_0, \ldots, d_{n-2}$ on its base, and the tuple $\langle d_0, \ldots, d_{n-2}\rangle$ is coloured y_S, then $i \in S$.*

Clearly, $\mathcal{G}$ is closed under isomorphism and under induced subgraphs. $\mathcal{G}$ depends on n.

4.3.4 Definition of $\mathcal{C}_n$

We are now ready to construct the required cylindric-type algebra $\mathcal{C}_n$, which turns out to be a cylindric algebra. Consider the class of surjective maps $a : n\ (= \{0, \ldots, n-1\}) \to \Gamma_a$, any $\Gamma_a \in \mathcal{G}$. Many of these maps, though formally distinct, will differ only because the nodes in the image graphs will not be the same. So we define an equivalence relation $\sim$ on these surjective maps:

$$\begin{array}{rcl} a \sim b & \iff & a(i) = a(j) \Leftrightarrow b(i) = b(j) \\ & & \text{and } \Gamma_a(a(i), a(j)) = \Gamma_b(b(i), b(j)), \text{ if defined,} \\ & & \text{and } \Gamma_a(a(k_0), \ldots, a(k_{n-2})) = \Gamma_b(b(k_0), \ldots, b(k_{n-2})), \text{ if defined,} \end{array}$$

for all $i, j \in n$ and $(n-1)$-tuples $\bar{k} \in n$. In other words, a and b define isomorphic coloured graphs. Write $[a]$ for the $\sim$-equivalence class of a.

An atom of $\mathcal{C}_n$ is any equivalence class $[a]$ for some surjective map $a : n \to \Gamma_a \in \mathcal{G}$. We can define the diagonal d_{ij} to be $\{[a] : \text{atoms } [a], a(i) = a(j)\}$. We also define $c_i([a])$ as the set of all atoms $[b]$ such that for some $c \sim a$, $b(j) = c(j)$ for all $j \neq i$. In other words, $a[n \setminus \{i\}]$ and $b[n \setminus \{i\}]$ define isomorphic coloured graphs. Clearly, this does not depend on the choice of representatives a, b. $\mathcal{C}_n$ can now be defined to be the n-dimensional cylindric-type algebra whose universe is the full power set of the set of atoms, each $[a]$ being identified with the singleton $\{[a]\}$, and the operations on non-atomic elements being defined by the distribution rules. Or we can define $\mathcal{C}_n$ to be any subalgebra of this that contains all the atoms.

4.3.5 Networks and Games over $\mathcal{C}_n$

Let N be an atomic $\mathcal{C}_n$-network. Let x, y be any two distinct nodes occurring in the n-tuple $\bar{z} \in N$. $N(\bar{z})$ is some atom of $\mathcal{C}_n$ which defines an edge colour for (x, y). Using the fact that the dimension n is at least 3, it can be shown that this edge colour depends only on x and y (not the other elements of $\bar{z}$, nor the positions of x and y in $\bar{z}$). Similarly, N defines shades of yellow for certain $(n-1)$-tuples. In this way N translates into a member of $\mathcal{G}$. This translation has an inverse, too. More formally, we have the following lemma, whose proof is entirely routine but rather lengthy and omitted here.

LEMMA 30 *1. Let $\Gamma \in \mathcal{G}$ be non-empty. Define N_Γ, whose nodes are those of Γ, as follows. For each $a_0, \ldots, a_{n-1} \in \Gamma$, define $N(a_0, \ldots, a_{n-1}) = [\alpha]$, where $\alpha : n \to \Gamma \restriction_{\{a_0,\ldots,a_{n-1}\}}$ is given by $\alpha(i) = a_i$ for all $i < n$. Then N_Γ is an atomic $\mathcal{C}_n$-network.*

2. *Conversely, let N be any non-empty atomic $\mathcal{C}_n$-network. Define a complete coloured graph Γ_N, whose nodes are the nodes of N, as follows.*

   - *For all distinct $x, y \in \Gamma_N$ and edge colours χ, $\Gamma_N(x, y) = \chi$ if and only if for some $\bar{z} \in N^n$, $i, j < n$, and atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_i = x, z_j = y$, and the edge $(\alpha(i), \alpha(j))$ is coloured χ.*
   - *For all $x_0, \ldots, x_{n-2} \in \Gamma_N^{n-1}$ and all yellows y_S, $\Gamma_N(x_0, \ldots, x_{n-2}) = y_S$ if and only if for some $\bar{z} \in N^n$, $i_0, \ldots, i_{n-2} < n$, and some atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_{i_j} = x_j$ for each $j < n-1$, and the $(n-1)$-tuple $\langle \alpha(i_0), \ldots, \alpha(i_{n-2}) \rangle$ is coloured y_S.*

   Then Γ_N is a well-defined coloured graph, and is in $\mathcal{G}$.

3. *For any non-empty $\Gamma \in \mathcal{G}$, $\Gamma_{N_\Gamma} = \Gamma$, and for any non-empty $\mathcal{C}_n$-network N, $N_{\Gamma_N} = N$.*

This translation allows us to make the following, equivalent formulation of the game[3] $G_k(\mathcal{C}_n)$ $(k \leq \omega)$. It builds a nested sequence of coloured graphs in $\mathcal{G}$: $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_k$, if k is finite, or an ω-sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ in the case $k = \omega$.

In the zeroth round, $\forall$ picks any graph $\Gamma_0 \in \mathcal{G}$ with $|\Gamma_0| = n$. $\exists$ makes no response in this round. In any subsequent round, let the last graph built be Γ_i. $\forall$ picks

- a graph $\Phi \in \mathcal{G}$ with $|\Phi| = n$,
- a single node $\alpha \in \Phi$, and
- a (coloured graph) embedding $\Xi : \Phi \setminus \{\alpha\} \to \Gamma_i$.

Let $F = \Phi \setminus \{\alpha\}$. F is called a *face*. $\exists$ must respond by *amalgamating* Γ_i and Φ with the embedding Ξ. In other words, she must define a graph $\Gamma_{i+1} \in \mathcal{G}$ and embeddings $\lambda : \Gamma_i \to \Gamma_{i+1}$, $\mu : \Phi \to \Gamma_{i+1}$ such that $\lambda \circ \Xi = \mu \restriction_F$.

Let us consider the possibilities, and set up some notation. There may already be a point $z \in \Gamma_i$ such that the map $(\alpha \mapsto z)$ is an isomorphism over F. In this case $\exists$ does not need to make an extension and she can let $\Gamma_{i+1} = \Gamma_i$, $\lambda = Id_{\Gamma_i}$, and $\mu \restriction_F = Id_F$, $\mu(\alpha) = z$.

Otherwise, without loss of generality let $F \subseteq \Gamma_i$, $\alpha \notin \Gamma_i$ (see figure 7). Let Γ_i^* be the coloured graph with nodes $Nodes(\Gamma_i) \cup \{\alpha\}$, whose edges are the combined edges of Γ_i and Φ, such that for any for pair or $(n-1)$-tuple $\bar{x}$ of nodes of Γ_i^*, the colour $\Gamma_i^*(\bar{x})$ is

- $\Gamma_i(\bar{x})$ if the nodes of $\bar{x}$ all lie in Γ_i and $\Gamma_i(\bar{x})$ is defined,
- $\Phi(\bar{x})$ if the nodes of $\bar{x}$ all lie in Φ and $\Phi(\bar{x})$ is defined

[3] The graph game that follows is equivalent to the specific game $G_k(\mathcal{C}_n)$. Obviously, for general cylindric algebras it is still necessary to use the game defined in definition 27.

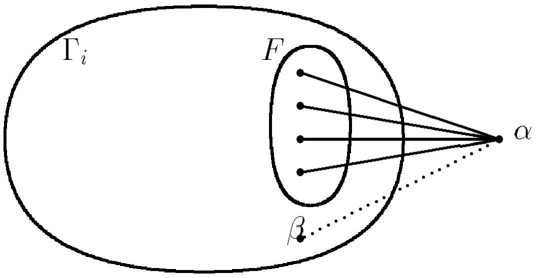


Figure 7: Γ_i^*

- undefined otherwise.

This is well-defined. $\exists$ needs to complete the labelling of Γ_i^* by adding all missing edges to obtain a complete graph, colouring each edge (α, β) for $\beta \in \Gamma_i \setminus \Phi$, and then choosing a shade of yellow for each $(n-1)$-tuple $\bar{a}$ of distinct elements of Γ_i^* not wholly contained in Γ_i or in Φ, if none of the edges in $\bar{a}$ get coloured green. She must do this in such a way that the resulting graph belongs to $\mathcal{G}$.

As usual, if she survives each round, $\exists$ has won the play.

The equivalence between these 'graph games' and the games $G_k(\mathcal{C}_n)$ is stated formally in the next lemma; as before, the proof is omitted. We only remark that the two 'restrictions' on $\forall$ — that the initial graph must be of size exactly n, and every face of size exactly $n-1$ — make it no easier for $\exists$ to win.

LEMMA 31

1. *$\exists$ has a winning strategy in the graph games of all finite lengths if and only if she has a winning strategy in the games $G_k(\mathcal{C}_n)$ for all finite k*
2. *$\exists$ has a winning strategy in the graph game of length ω if and only if she has a winning strategy in the game $G_\omega(\mathcal{C}_n)$.*

4.3.6 The completely representable n-dimensional cylindric algebras are not closed under elementary equivalence

Our final two propositions show that $\mathcal{C}_n$ does not have a complete representation, but that it is elementarily equivalent to a cylindric algebra that does have a complete representation. See theorem 28.

PROPOSITION 32 *$\forall$ has a winning strategy for $G_\omega(\mathcal{C}_n)$.*

PROOF:

In his zeroth move, $\forall$ plays a graph $\Gamma \in \mathcal{G}$ with nodes $0, 1, \ldots, n-1$ and such that $\Gamma(i,j) = w_0$ $(i < j < n-1)$, $\Gamma(i, n-1) = g_i$ $(i = 1, \ldots n-2)$, $\Gamma(0, n-1) = g_0^0$, and $\Gamma(0, 1, \ldots, n-2) = y_\omega$ (see figure 8). This is a 0-cone with base $\{0, \ldots, n-2\}$. In the following moves, $\forall$ repeatedly chooses the face $\langle 0, \ldots, n-2\rangle$ and demands a new node α (without loss, let $n-1 < \alpha < \omega$) with $\Phi(i, \alpha) = g_i$ $(i = 1, \ldots n-2)$ and $\Phi(0, \alpha) = g_0^\alpha$, in the graph notation — i.e., an α-cone on the same base. $\exists$, among other things, has to colour all the edges connecting nodes α and β for $\alpha, \beta \geq n-2$. The only permissible colours are red (by definition 29 part 2). As before, $\forall$ can force a win eventually (cf. the proof of proposition 22). $\square$

PROPOSITION 33 *$\exists$ has a winning strategy for $G_k(\mathcal{C}_n)$ $(k < \omega)$.*

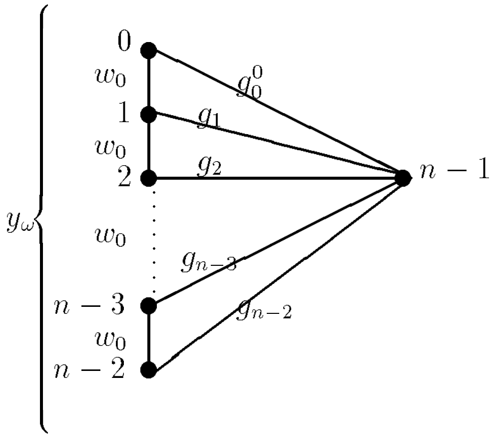


Figure 8:

PROOF:

At any stage during the game, let $\Gamma \in \mathcal{G}$ be the graph constructed so far. We may assume without loss that the nodes of Γ form an initial segment of natural numbers $< k+n$, as there are at most $k+n$ nodes constructed in $G_k(\mathcal{C}_n)$. Let $\forall$ choose the graph $\Phi \in \mathcal{G}$ with distinct nodes $F \cup \{\alpha\}$, where $F \subseteq \Gamma$ has size $n-1$ and $\alpha \notin \Gamma$.

If there is already a node z in Γ such that $\Gamma \restriction_{F \cup \{z\}}$ is isomorphic to Φ over F, then $\exists$ does not need to add a node.

So assume otherwise. Let Γ^* be the coloured graph with nodes $Nodes(\Gamma) \cup \{\alpha\}$, as in section 4.3.5. $\exists$ has to extend Γ^* to a complete graph on the same nodes and complete the colouring, yielding $\Gamma^+ \in \mathcal{G}$. Thus, she has to define $\Gamma^+(\beta, \alpha)$ for all nodes $\beta \in \Gamma \setminus F$, and also select appropriate yellow colours for $(n-1)$-tuples of Γ^* where necessary. She does this as follows.

Case I If α and β are both the apexes of cones on F in Γ^* that induce the same linear ordering on F, then, as in the preceding proposition, $\exists$ has no choice but to pick a red colour for $\Gamma^+(\alpha, \beta)$. The colour she chooses is $r^{(k+n)^2}_{\alpha+(k+n)\beta}$ (we may assume without loss that $\beta < \alpha < k+n$). This colour has a fixed superscript and is unique to α, β.

Case II Otherwise, for some $i < n-1$, there is no $f \in F$ such that $\Gamma^*(\beta, f)$, $\Gamma^*(f, \alpha)$ are both coloured g_i, or — if $i = 0$ — g_0^l and $g_0^{l'}$ for some l, l'. $\exists$ can set the colour $\Gamma^+(\beta, \alpha)$ to be w_i for any such i (say, the least such).

This defines all edge colours of Γ^+. Finally, for each tuple of distinct elements $\bar{a} = a_0, \ldots, a_{n-2} \in (\Gamma^+)^{n-1}$, $\bar{a} \notin \Gamma^{n-1} \cup \Phi^{n-1}$, with no edge (a_i, a_j) coloured green in Γ^+, $\exists$ colours $\bar{a}$ by y_S, where

$$S = \{i < \omega \ : \ \text{there is an } i\text{-cone in } \Gamma^* \text{ with base } a_0, \ldots, a_{n-2}, \text{ in the induced ordering}\}.$$

This completes the definition of Γ^+.

It remains to check that this strategy works — that $\Gamma^+ \in \mathcal{G}$. Notice that whenever $\exists$ makes a proper extension to the previous graph, she only chooses red or white colours for her edges. She never uses green. Let us check the conditions for Γ^+ to belong to $\mathcal{G}$: i.e., that conditions 1, 2 and 3 from the definition of $\mathcal{G}$ (definition 29) are obeyed. For condition 1, about edge colours, no green triangles are created that are not already in Γ or Φ (as $\exists$ never uses green), nor any of the form (g_i, g_i, w_i) or (g_0^j, g_0^k, w_0) (by the condition in case II above). To verify that condition 1 is satisfied, it only remains to consider red triangles, which we will do in a moment.

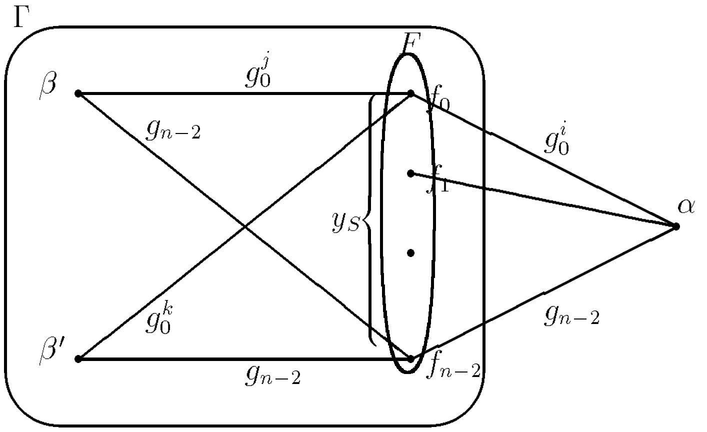


Figure 9:

Let us check that $(n-1)$-tuples are labelled appropriately, by yellow colours. Condition 2, definition 29, is trivially satisfied. Consider condition 3. Let D be a set of n nodes of Γ^+, and suppose that $\Gamma^+ \restriction_D$ is an i-cone with apex δ and base $\{d_0, \ldots, d_{n-2}\}$, and that the tuple $\bar{d} = \langle d_0, \ldots, d_{n-2} \rangle$ (in the induced ordering) is labelled y_S in Γ^+. We must show that $i \in S$. Note first that if $D \subseteq \Gamma$ then (inductively) the graph Γ constructed so far is in $\mathcal{G}$, and therefore $i \in S$. If $D \subseteq \Phi$, then as $\forall$ chose Φ in $\mathcal{G}$ we get $i \in S$ similarly. If neither holds, then D contains α and also some node $\beta \in \Gamma \setminus \Phi$. $\exists$ has just chosen the colour $\Gamma^+(\alpha, \beta)$, and her strategy ensures that it is not green. Hence, neither α nor β can be the apex of the cone $\Gamma^+ \restriction_D$, so they must both lie in the base, $\bar{d}$. This implies that $\bar{d}$ is not yet labelled in Γ^*; so $\exists$ has just applied her strategy to choose the colour y_S to label $\bar{d}$ in Γ^+. But the strategy will have chosen S containing i, since $\Gamma^* \restriction_D$ is already a cone in Γ^* — $\exists$ never chooses a green edge, so all green edges of Γ^+ lie in Γ^*. This is satisfactory, and we are done.

That leaves one hard case, where there are two nodes $\beta, \beta' \in \Gamma$, $\exists$ colours both (β, α) and (β', α) red, and the old edge (β, β') has already been coloured red (earlier in the game).

If (β, β') was coloured by $\exists$, then, as with relation algebras (cf. proposition 23), there is no problem. So suppose, for a contradiction, that (β, β') was coloured by $\forall$. Since $\exists$ chose red colours for (α, β) and (α, β'), it must be the case that there are cones in Γ^* with apexes α, β, β' and the same base, F, each inducing the same linear ordering, $\bar{f} = f_0, \ldots, f_{n-2}$, say, on F. Of course, the tints of these cones may all be different (see figure 9).

Clearly, no edge in F is labelled green, as no cone base can contain green edges. It follows that $\bar{f}$ must be labelled by some yellow colour, y_S, say. Since $\Phi \in \mathcal{G}$, it obeys condition 4, so the tint i (say) of the cone from α to $\bar{f}$ lies in S.

Suppose that λ was the last node of $F \cup \{\beta, \beta'\}$ to be created, as the game proceeded. As $|F \cup \{\beta, \beta'\}| = n + 1$, we see that $\exists$ must have chosen the colour of at least one edge in this: say, (λ, μ). Now all edges from β into F are green, and so chosen by $\forall$, and the edge (β, β') was also chosen by him. The same holds for edges from β' to F. Hence $\lambda, \mu \in F$.

We can now see that it was $\exists$ who chose the colour y_S of $\bar{f}$. For y_S was chosen in the round when F's last node, i.e., λ, was created. It could only have been chosen by $\forall$ if he also picked the colour of every edge in F. This is not so, as the edge (λ, μ) was coloured by $\exists$, and lies in F.

As $i \in S$, it follows from the definition of $\exists$'s strategy that at the time when λ was added, there was already an i-cone with base $\bar{f}$, and apex γ, say.

We claim that $F \cup \{\alpha\}$ and $F \cup \{\gamma\}$ are isomorphic over F. For this, note that the only $(n-1)$-tuples of either $F \cup \{\alpha\}$ or $F \cup \{\gamma\}$ with a yellow colour are in F (since all others involve a green edge).

But this means that $\exists$ could have taken $\alpha = \gamma$ in the current round, and not extended the graph. This is contrary to our original assumption, and completes the proof. □

We now obtain:

THEOREM 34 *The class of all completely representable cylindric algebras of any fixed dimension greater than or equal to 3 is not closed under elementary equivalence, and so is not elementary.*

PROOF:

For the infinite-dimensional case, this is corollary 26. For the finite-dimensional case, let $3 \leq n < \omega$ and consider the n-dimensional cylindric-type algebra $\mathcal{C}_n$. By proposition 33, lemma 31, and theorem 28, there is an n-dimensional cylindric-type algebra $\mathcal{B}_n \equiv \mathcal{C}_n$ which has a complete representation and is therefore a bona fide cylindric algebra. Hence $\mathcal{C}_n$ is also a cylindric algebra, since the axioms defining n-dimensional cylindric algebras are first-order. But by proposition 32, and lemma 31 and theorem 28 again, $\mathcal{C}_n$ itself has no complete representation. This completes the proof. □

5 To Conclude

The class of completely representable boolean algebras is elementary, indeed finitely axiomatisable, but for relation algebras and cylindric algebras the completely representable ones do not form elementary classes. It might be worth making an additional observation: the completely representable, weakly associative algebras [13] form a finitely axiomatisable class. In fact it is not hard to show that *any* weakly associative algebra possesses a complete representation — but note that a representation of such an algebra can be relativized to some reflexive, symmetric binary relation. This suggests some rather vaguely formulated questions: how much additional structure can be added to boolean algebra, such that we still get an elementary class of completely representable algebras? And in the other direction, what requirements on the notion of representation need to be dropped before the completely representable algebras become elementary?

These questions are perhaps of interest to the growing school of people involved in 'taming' logics by relativized representations and by other means; see, for example, the recent Ph.D. theses of Marx [16] and Mikulás [18].

References

[1] H Andréka, J D Monk, and I Németi. *Algebraic Logic.* North-Holland, 1991.

[2] C.C. Chang and H.J. Keisler. *Model Theory.* North-Holland, Amsterdam, 2nd edition, 1973.

[3] R Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44:173–242, 1989.

[4] L Henkin, J D Monk, and A Tarski. *Cylindric Algebras Part I.* North-Holland, 1971.

[5] L Henkin, J D Monk, and A Tarski. *Cylindric Algebras Part II.* North-Holland, 1985.

[6] R Hirsch. Completely representable relation algebras. *Bulletin of the interest group in propositional and predicate logics*, 3(1):77–92, 1995.

[7] R Hirsch and I Hodkinson. Step by step — building representations in algebraic logic. *Journal of Symbolic Logic*, 1996? To appear.

[8] B Jónsson and A Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, 54, 1948.

[9] B Jónsson and A Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 74:127 – 162, 1952.

[10] R Lyndon. The representation of relational algebras. *Annals of Mathematics*, 51(3):707–729, 1950.

[11] R Lyndon. The representation of relation algebras, ii. *Annals of Mathematics*, 63(2):294–307, 1956.

[12] R Maddux. *Topics in Relation Algebra.* PhD thesis, University of California, Berkeley, 1969.

[13] R Maddux. Some varieties containing relation algebras. *Transactions of the AMS*, 272(2):501–526, 1982.

[14] R Maddux. Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections. In *Algebraic Logic*, pages 361–392. North-Holland, 1991.

[15] R D Maddux. Non-finite axiomatizability results for cylindric and relation algebras. *Journal of Symbolic Logic*, 54(3):951–974, 1989.

[16] M Marx. *Algebraic relativization and arrow logic.* PhD thesis, University of Amsterdam, 1995. ILLC Dissertation Series 95-3.

[17] R McKenzie. Representation of integral relation algebras. *Michigan Mathematics Journal*, 17:279–287, 1970.

[18] S Mikulás. *Taming logics.* PhD thesis, University of Amsterdam, 1995. ILLC Dissertation Series 95-12.

[19] J D Monk. Lectures on cylindric set algebras. In *Algebraic Methods in Logic and in Computer Science*, volume 28 of *Institute of Mathematics, Polish Academy of Sciences.* Banach Center publications, 1993.

[20] M Stone. The theory of representations for boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[21] Y Venema. *Many-Dimensional Modal Logic.* PhD thesis, University of Amsterdam, 1992.

[22] M Villain and H Kautz. Constraint propagation algorithms for temporal reasoning. In *Proceedings of the fifth AAAI*, pages 377–382, 1986.