Step by Step — Building Representations in Algebraic Logic

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Abstract

We consider the problem of finding and classifying representations in algebraic logic. This is approached by letting two players build a representation using a game. Homogeneous and universal representations are characterised according to the outcome of certain games. The Lyndon conditions defining representable relation algebras (for the finite case) and a similar schema for cylindric algebras are derived. Countable relation algebras with homogeneous representations are characterised by first order formulas. Equivalence games are defined, and are used to establish whether an algebra is ω-categorical. We have a simple proof that the perfect extension of a representable relation algebra is completely representable.

An important open problem from algebraic logic is addressed by devising another two-player game, and using it to derive equational axiomatisations for the classes of all representable relation algebras and representable cylindric algebras.

Other instances of this approach are looked at, and include the step by step method.

1 Introduction and Outline

This paper may be used as an introduction to algebraic logic, and addresses a common problem from the field: given some algebraic axioms Σ and a class of structures $\mathcal{K}$, it is required to show that every algebra $\mathcal{A}$ satisfying the axioms Σ is isomorphic to some structure in $\mathcal{K}$. This establishes the *completeness* of Σ over $\mathcal{K}$.

The axioms of Σ are generally first-order axioms (or even equations) in a signature consisting of some functional operations and with no non-logical relation symbols. For example, groups — the operations might be identity, inverse, and composition; boolean algebra — here the operations are disjunction and negation; and relation algebra, where the operations are the boolean operations, identity, converse, and composition. The axioms of Σ may simply be the axioms defining groups, boolean algebras, relation algebras, or whatever; alternatively, they may define a special subclass of these structures.

The structures in the class $\mathcal{K}$ are usually of a rather concrete kind, with *set-theoretically definable*[1] operations corresponding to the algebraic operations. So for groups, $\mathcal{K}$ might consist of groups of non-singular matrices, the group operations being represented by the identity matrix, the operation of taking the inverse of a matrix, and matrix multiplication, respectively. For boolean algebra, $\mathcal{K}$ could consist of *fields of sets,* the boolean operations being represented by set-theoretic union and complementation. For relation algebra, it is natural to consider *proper relation algebras* consisting of binary relations, and here the algebraic operations are interpreted as union, complementation, the identity relation, and converse and composition of relations. In

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[1] [N91] includes, along with many other things, a useful discussion about the term "set-theoretically definable".

cylindric algebra, *cylindric set algebras* of n-ary relations are considered; the operations here are projections and equalities (diagonals).

There is a potential ambiguity when we say that an algebra $\mathcal{A} \models \Sigma$ is isomorphic to a structure in $\mathcal{K}$. Commonly in algebraic logic, a structure $M \in \mathcal{K}$ consists of a set of relations over some domain D: say $M = (\mathcal{R}, D)$. There are two distinct types of isomorphism to consider. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras satisfying Σ. An *isomorphism* from $\mathcal{A}$ onto $\mathcal{B}$ is a bijection from the elements of $\mathcal{A}$ to those of $\mathcal{B}$ respecting all the algebraic operations. In this way, an isomorphism from $\mathcal{A}$ to some $M = (\mathcal{R}, D) \in \mathcal{K}$ is a bijection from the elements of $\mathcal{A}$ to the *relations* $\mathcal{R}$ that preserves all the algebraic operations. In this case, the isomorphism is called a *representation* of $\mathcal{A}$. This is the kind of isomorphism we have in mind.

On the other hand, there is also a notion of equivalence between two representations of the same algebra $\mathcal{A}$. Let $h : \mathcal{A} \to (\mathcal{R}, D)$ and $k : \mathcal{A} \to (\mathcal{S}, E)$ be two representations of $\mathcal{A}$. These representations are said to be *base-isomorphic* if there is a bijection $\lambda : D \to E$ such that for all $a \in \mathcal{A}$ and all n-tuples $\bar{d} \in D$, we have $\bar{d} \in h(a)$ if and only if $\lambda(\bar{d}) \in k(a)$. We will be considering base-isomorphisms later, but they will play no role at present.

Whether we can establish completeness or not will depend on Σ and $\mathcal{K}$. An important method of attempting to do this is what H. Andréka calls the *step by step method.* Let $\mathcal{A}$ be any algebra satisfying Σ. Suppose that each structure in $\mathcal{K}$ consists of n-ary relations for some n. Make a first-order language $L = L(\mathcal{A})$ with an n-ary predicate symbol R for each $r \in \mathcal{A}$. In any L-structure M, each $r \in \mathcal{A}$ is interpreted as an n-ary relation R^M on M. Thus, M is a collection of n-ary relations, so is a candidate for being in $\mathcal{K}$. Moreover, we may map $\mathcal{A}$ to this set of relations, by $r \mapsto R^M$; and since the algebraic operations have set-theoretic definitions in $\mathcal{K}$ which we may perhaps apply to the relations R^M, M is also a candidate for being an algebra isomorphic to $\mathcal{A}$.

Now suppose that we can find a certain *universal–existential* L-theory $\Gamma = \Gamma(\mathcal{A})$ whose models are precisely the L-structures M that are in $\mathcal{K}$ and isomorphic to $\mathcal{A}$. This is certainly the case for proper relation algebras and cylindric set algebras. Under these conditions, we can try to build an L-structure satisfying Γ step by step. This structure is isomorphic to $\mathcal{A}$, so completeness is proved.

The first step is to find some countable L-structure M_0 satisfying all the universal formulas in Γ. Next, suppose the formula $\phi = \forall\bar{x}\exists\bar{y}\psi(\bar{x}, \bar{y}) \in \Gamma$ is not satisfied in M_0 (where $\bar{x}, \bar{y}$ are finite tuples of variables and ψ is a quantifier free formula). Then there is some tuple $\bar{m} \in M_0$ such that $\exists\bar{y}\psi(\bar{m}, \bar{y})$ fails in M_0. It may be possible to add new points to M_0 in such a way that the universal axioms of Γ are preserved but $\exists\bar{y}\psi(\bar{m}, \bar{y})$ now holds. Do this for all such $\bar{m} \in M_0$, and for all formulas $\phi \in \Gamma$. Let the resulting $\mathcal{A}$-structure be M_1. This procedure is iterated ω times, to give a limit structure M — still a model of all the universal formulas and now satisfying the universal–existential formulas in Γ, too. We are done.

A further twist is that even if Σ fails to be complete for $\mathcal{K}$, often we can write down additional algebraic axioms Δ that an algebra $\mathcal{A}$ must satisfy in order that the step-by-step construction of a model of $\Gamma(\mathcal{A})$ succeeds. Δ can be added to the original set Σ to obtain a system that is complete for $\mathcal{K}$. For example, we can add a set of axioms to the basic 'Tarski axioms' for relation algebras, and show that a complete axiomatisation of the proper relation algebras is obtained. We will do the same for some other classes, too.

Expressing the construction in the form of a two-player game greatly eases the task of writing these axioms Δ, and simplifies the handling of the step-by-step construction itself. We believe the explicit use of game theory in this setting to be novel, but very natural. It has the advantage that the arguments are given structure, because the goal of one of the players in the game is typically to build the required model — or, in effect, to prove the theorem. The proofs become very simple, and so are easy to adapt and generalize.

Well, that is the idea; to see how it really works, it will be necessary to study some of the actual proofs. A number of examples of this method are given in the text. We will use it to axiomatise the representable relation algebras and cylindric algebras, and the weakly associative relation algebras, following in the footsteps of Lyndon, Maddux, and others. We will go on to axiomatise the relation algebras and cylindric algebras having homogeneous representations. Background knowledge[2] of these algebraic classes is not essential, as the basic definitions are all included and explained in the text.

Outline The following section gives a few examples of the step by step method from different branches of logic. Section 3 is an introduction to the basic results about relation algebras and their representations. In section 4, we define a two player game and characterise the representable relation algebras by the outcome of the game. Theorem 5 is the central theorem of this section, characterising some model-theoretic properties — homogeneity and universality — by the outcome of appropriate games. This is shown to be equivalent to the existence of certain classes of *networks.*

In section 5 we derive a first-order axiom schema which determines whether a finite relation algebra is representable. In a similar way, an axiom schema for finite relation algebras with homogeneous representations is defined. Section 6 gives a number of examples of relation algebras, and includes a relation algebra (one of the Lyndon algebras) that is representable but possesses no homogeneous representation. A second type of game is defined in section 7 to determine when a relation algebra has a countable representation unique up to base-isomorphism. Section 8 includes a proof that the perfect extension of a representable relation algebra is representable.

Section 9 includes the most important result of this paper — an axiom schema for arbitrary representable relation algebras. Section 10 gives a similar schema for countable relation algebras with homogeneous representations. Sections 11 and 12 briefly investigate the corresponding results in weakly associative and cylindric algebras. In particular, we derive an axiom schema that characterises the representable cylindric algebras of any given dimension.

We stress that some of these results have been obtained before. However, we feel that the game-theoretic approach yields a quite considerable simplification of the statement and proof of the main theorems.

Readers with some knowledge of algebraic logic who wish only to read our treatment of the axiomatisation of representable relation algebras and cylindric algebras may turn directly to sections 9 and 12, and follow up the references to earlier results as they please.

2 Examples of the Step by Step method

Examples of this technique are numerous. The Henkin construction for proving the completeness of first-order logic is an important early example.

The method is used by Roger Maddux [Mad69] to show that every *semi-associative relation algebra* has a representation as a subalgebra of a *relativized proper relation algebra.* Maddux went on in [Mad82] to show that every *weakly associative relation algebra* can be represented as a subalgebra of a proper relation algebra relativized to a symmetric, reflexive relation. (See section 11 below.) A considerable generalisation of this work on relativized relation algebra, also based on the step by step method, can be found in [Mar95].

The step by step method appears explicitly in a proof by Andréka of the Resek–Thompson theorem [AT88] to prove that their finite axiom set Σ is complete over a certain class of relativized cylindric set algebras.

An interesting example from temporal logic is the proof in [Bur82] where a set of axioms in the temporal language with *Until* and *Since* is shown to be complete over linear flows of time. The proof differs from the standard Henkin construction in the following way. In a Henkin completeness proof for an **F** (future) **P** (past) logic you take your set of worlds to be *all* maximally consistent sets (MCS). (The operators **G** and **H** are introduced as abbreviations for $\neg\mathbf{F}\neg$ and $\neg\mathbf{P}\neg$ respectively.) For a countable language this gives uncountably many worlds. These are then ordered by the following rule: $\Gamma < \Delta$ if and only if $(\forall\phi)\ \ \mathbf{G}\phi \in \Gamma \rightarrow \phi \in \Delta$. Note that it is not possible to enforce that this construction gives irreflexive models and so completeness proofs over irreflexive flows of

[2] References to relation algebra include [Tar41, JT48, TG87, Mad91, Giv94, N9́5] and to cylindric algebra there is [HMT71, HMT85, Mon93, N9́5]. [AMN91] contains a variety of articles on algebraic logic.

time require an additional section in the proof where reflexive clusters are *bulldozed* to make them irreflexive (see, for example, [Bla89]).

The Burgess proof works in a different way. To prove completeness a consistent set of formulas is extended to a MCS. So we start with just a single point in the structure. New points, labelled by MCSs, are added step by step, producing in the limit a countable model of the original set of formulas. At each stage and for each universal formula $\mathbf{G}\phi$ the formula belongs to a point Γ only if all the MCSs ordered above Γ include the formula ϕ. Other universal-existential formulas require the addition of new points to the structure and this is done, step by step. For example if $\mathbf{F}\phi$ is in one the points it may be necessary to add a new higher point containing the formula ϕ. The odd thing is that a formula $\mathbf{U}(\phi, \psi)$ appears to correspond to a universal-existential-universal formula:

$$(\forall\Gamma)\ \mathbf{U}(\phi, \psi) \in \Gamma \rightarrow [(\exists\Delta > \Gamma)\ \phi \in \Delta \wedge (\forall\Xi)\ \Gamma < \Xi < \Delta \rightarrow \psi \in \Xi].$$

Thus, the Burgess proof appears to fall outside our framework which deals only with universal-existential formulas. However, more careful examination of the proof shows that by moving to a two-dimensional framework the model is built step by step and only universal-existential conditions need be satisfied. Restrictions on space prevent us from giving more details here.

The same step by step method is used in [Rey92] to extend this result to predicate temporal logic by showing the completeness of some axioms over the class of all linear flows with constant domains (in predicate temporal logic each moment in time is associated with a domain and there is quantification over that domain). Another related example of the method is in [Ven93] where a characterisation and completeness theorem are given for the class of so-called 'finite dimensional cube-frames'

This type of construction is closely related to *Robinson forcing* [Hod85, Rob79, BR70]. In fact the idea has a long history, but we have no space to survey it here.

3 Relation algebras

Relation algebras are intended to capture algebraically the study of binary relations. First we define concrete structures of binary relations — *proper relation algebras* — and then go on to define the abstract or algebraic class of all relation algebras. The study of *representations* is really the investigation of the relation between these two; in particular it asks when an abstract relation algebra is isomorphic to a proper relation algebra. The corresponding classes for unary relations are *fields of sets* and *boolean algebras*. For boolean algebra Stone's theorem tells us that every boolean algebra can be represented as a field of sets but for relation algebra this problem turns out to be much harder.

Definition Let D be a non-empty set (domain).

- Converse and composition are natural operations on binary relations. Let $r, s \subseteq D \times D$ be any binary relations on D. The *converse* ($r^{\smile}$) of r, and the *composition* ($r;s$) of r, s, are defined as follows.

$$\begin{aligned} (d_1, d_2) \in r^{\smile} &\Leftrightarrow (d_2, d_1) \in r \\ (d_1, d_2) \in r;s &\Leftrightarrow (\exists d_3 \in D)\ [(d_1, d_3) \in r \wedge (d_3, d_2) \in s]. \end{aligned}$$

- The identity relation on D is defined by $Id_D = \{(d, d) : d \in D\}$.

- Let $r, 1$ be binary relations over D with $r \subseteq 1$ (1 will be the top element of a boolean algebra). We define $-r$, the *complement* of r (relative to 1), by

$$(d_1, d_2) \in (-r) \Leftrightarrow (d_1, d_2) \in 1 \wedge (d_1, d_2) \notin r.$$

- A *proper relation algebra (***PRA***) with domain D* is a set of binary relations on D, including the empty relation $\varnothing$, a biggest relation 1, the identity relation Id and closed under union, complement relative to 1, and converse and composition of relations. More formally, it is an algebra of the form $(B, I, \varnothing, \backslash, \cup, Id, -, ^\smile, ;),$, where $\varnothing, Id_D, I \in B \subseteq \wp(D \times D)$, '$-$' is complement relative to I, and B is closed under the named operations.
- **PRA** will also denote the *class* of all proper relation algebras.
- The simplest example of a **PRA** is the full power set algebra on D:

$$\mathcal{P}(D) =_{\text{def.}} (\wp(D \times D), D \times D, \varnothing, \backslash, \cup, Id_D, {}^\smile, ;).$$

3.1 Abstract Relation Algebras

Definition An (abstract) relation algebra (**RA**) is a tuple $\mathcal{A} = (A, 1, 0, -, \vee, Id, ^\smile, ;)$ which satisfies the following axioms, essentially due to Tarski. For all $a, b, c \in A$,

1. $(A, 1, 0, -, \vee)$ is a Boolean algebra (1 is the universal element)
2. $Id; a = a; Id = a$
3. $(a^\smile)^\smile = a$
4. $(a \vee b)^\smile = a^\smile \vee b^\smile$
5. $(a - b)^\smile = a^\smile - b^\smile$. (Throughout this paper, $-$ will be a unary operation, and $a - b$ will be used to abbreviate $a \wedge -b$, where $x \wedge y$ itself abbreviates $-(-x \vee -y)$.)
6. ; is an associative binary operation on A
7. $a; (b \vee c) = a; b \vee a; c$
8. $(a; b)^\smile = b^\smile; a^\smile$
9. $(a; b) \wedge c^\smile = 0 \Leftrightarrow (b; c) \wedge a^\smile = 0$ [triangle axiom].

The last axiom is not an equation, but it is equivalent (in the presence of the other axioms) to the equation

$$a^\smile; -(a; b) \leq -b$$

where $a \leq b$ is an abbreviation for $a \vee b = b$.

RA will also denote the class of all relation algebras.

A *representable relation algebra* (**RRA**) is a relation algebra that is isomorphic to a proper relation algebra. Of course, the isomorphism should preserve the boolean operations and Id, and map the operations $^\smile$ and ';' to converse and composition of relations. **RRA** will also denote the class of all representable relation algebras. It is easy to see that axioms 1–9 defining **RA** are valid over the class **PRA**. Hence, **RRA** is simply the closure of **PRA** under isomorphism.

It is natural to ask whether the axioms defining **RA** are complete over **PRA** — i.e., whether they axiomatise **RRA**. Lyndon [Lyn50] showed that the answer was negative[3]: there are (abstract) relation algebras, i.e. structures satisfying axioms 1 to 9 above, which are not isomorphic to any concrete algebra of binary relations. Lyndon ([Lyn56]; see also [Lyn50]) gives an infinite set of conditions which are sound and complete over representable relation algebras. Later, we will approach this problem using games, which will give us a quite considerable simplification in the definition of the conditions as well as the proof of completeness.

[3] McKenzie [McK70] page 286, provides the simplest example of a relation algebra, with just four atoms, which does not possess a representation. Monk [Mon64] proves that the class **RRA** cannot be finitely axiomatised at all. Jónsson has shown further that the representable relation algebras cannot be characterised by any set of axioms using a fixed number of variables. See also [Mad89, And94] for a strengthening of the basic result.

Definitions

- We defined the boolean ordering '$\leq$' already: it is given by $x \leq y \Leftrightarrow x \vee y = y$.
- An *atom* is a $\leq$-minimal non-zero element of $\mathcal{A}$. $\mathcal{A}$ is *atomic* if every non-zero element is above ($\geq$) an atom. The set of all atoms of an **RA** $\mathcal{A}$ is denoted by $At(\mathcal{A})$.
- A *unit* is an atom below the identity element.
- A relation algebra is *complete* if arbitrary suprema (and infima) exist. Here, a supremum of a set of elements is taken to mean the least upper bound of that set, if it exists, and is undefined otherwise; and similarly for infima.
- A relation algebra which obeys the axiom

$$a;b = 0 \Leftrightarrow a = 0 \text{ or } b = 0$$

  is called *integral*. It is not hard to show that a relation algebra is integral if and only if the identity element is an atom.
- A map $h : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* it it respects the relation algebraic operations, e.g. $h(a;b) = h(a);h(b)$ and similar equations for $0, 1, -, \vee, \smile$ and Id. If a homomorphism is one-one, it is called an isomorphism.
- A relation algebra is *simple* if it has no non-trivial homomorphic images. Equivalently, every non-zero element r satisfies

$$1;r;1 = 1.$$

- An atomic relation algebra $\mathcal{A}$ is a *group relation algebra* if the composition of any two atoms is an atom, and $(At(\mathcal{A}), ;)$ forms a group.

Jónsson and Tarski showed that every relation algebra is isomorphic to a subalgebra of a complete and atomic relation algebra. For the most part the relation algebras that are considered here are atomic. They also showed that every relation algebra can be decomposed into simple relation algebras: see below.

3.2 Representations

Definitions

- A *representation* of a relation algebra $\mathcal{A}$ is a pair (X, D) where D is any non-empty set — the domain of the representation — and X is an isomorphism from $\mathcal{A}$ onto a proper relation algebra on the set D. (Since X, being a map, also has a domain, and one which is quite different from the domain D of the representation, the reader should be careful where confusion may arise.)
- The *cardinality* of a representation (X, D) is defined to be the cardinality $|D|$ of D.
- Such a representation is called a *complete representation* if arbitrary suprema are preserved wherever they exist: i.e., if $a_\lambda \in \mathcal{A}$ for all $\lambda \in \Lambda$, and $\bigvee_{\lambda\in\Lambda} a_\lambda$ exists, then

$$X(\bigvee_{\lambda\in\Lambda} a_\lambda) = \bigcup_{\lambda\in\Lambda}(X(a_\lambda)).$$

- A relation algebra $\mathcal{A}$ is said to be *completely representable* if it has a complete representation.

Evidently, a relation algebra is representable if and only if it has a representation.

Note Where the context is unambiguous the representation (X, D) may be referred to simply as X. Instead of saying that a point d belongs to D, we may just say that the point d is in X, or $d \in X$.

Notation Let X be a representation of a relation algebra $\mathcal{A}$ and let $r \in \mathcal{A}$, $x, y \in X$. We write $X \models a(x, y)$, or, sometimes, $X, (x, y) \models a$, to mean $(x, y) \in X(a)$.

It might be helpful to look at the algebraic properties above from the point of view of a representation. Observe that for any representation (X, D) of $\mathcal{A}$, the universal relation 1 defines an equivalence relation on the domain D. This follows from the equations: $1 \geq Id$; $1 = 1^{\smile}$ and $1 = 1;1$. So D is partitioned into a disjoint union $D_1 \dot{\cup} D_2 \ldots$ and $X(1) = (D_1 \times D_1) \dot{\cup} (D_2 \times D_2) \ldots$.

- For any i the map X_i defined by $X_i(a) = X(a) \cap (D_i \times D_i)$ is a homomorphism from $\mathcal{A}$ into the proper relation algebra $\mathcal{P}(D_i)$ (the full power set algebra on D_i).
- For a simple relation algebra, X_i must be an isomorphism, so each of these sub-domains D_i defines a separate representation of $\mathcal{A}$. Thus, for simple $\mathcal{A}$, a representation can always be chosen so that $X(1) = D \times D$, provided it is representable. If $X(1) = D \times D$, we call the representation *square.*
- A relation algebra $\mathcal{A}$ is a *subdirect product* of the relation algebras $\mathcal{B}_i : i \in I$ if it is a subalgebra of the direct product of the $\mathcal{B}_i$, and for each $j \in I$, the restriction to $\mathcal{A}$ of the canonical projection $\pi_j : \prod_{i \in I} \mathcal{B}_i \to \mathcal{B}_j$ is surjective.
- Any relation algebra $\mathcal{A}$ is isomorphic to a subdirect product of simple relation algebras (called the *simple components* of $\mathcal{A}$) [JT52]. $\mathcal{A}$ is representable if and only if each of its simple components is representable.
- A representation of a general atomic relation algebra partitions the domain into sets S_i ($i \in I$) and each atom of the algebra has domain S_j and range S_k (some j, k). (The domain of a binary relation r is $\{x : \exists y(r(x, y))\}$; the range of r (= the domain of $r^{\smile}$) is defined dually.) $\mathcal{A}$ is integral if and only if for every element $r \in \mathcal{A}$, the domain and the range of r is the whole of the domain of the representation.

Let $\mathcal{A}$ be a relation algebra, and let (X, D) be a representation. (X, D) can be viewed as a directed graph with an edge for each pair of nodes (d, e) in $X(1)$. Each edge (d, e) is labelled by the set $X^{-1}(d, e)$ of all relations that hold on that edge[4]. So $a \in X^{-1}(d, e)$ if and only if $a \in \mathcal{A}$ and $(d, e) \in X(a)$. Now for any edge (d, e), the set $X^{-1}(d, e)$ forms a filter[5] on $\mathcal{A}$, since

$$
\begin{array}{rcl}
1 & \in & X^{-1}(d, e) \\
a \in X^{-1}(d, e) \ \&\ a \leq b & \Rightarrow & b \in X^{-1}(d, e) \\
a, b \in X^{-1}(d, e) & \Rightarrow & a \wedge b \in X^{-1}(d, e)
\end{array}
$$

$X^{-1}(d, e)$ is an ultrafilter, since

$$a \in X^{-1}(d, e) \Leftrightarrow -a \notin X^{-1}(d, e)$$

Under favourable circumstances $X^{-1}(d, e)$ will form a principal ultrafilter and we will be able to define it by a single atomic relation.

[4] There is an abuse of notation here, though if instead of thinking of X as a *function* from $\mathcal{A}$ to $\mathcal{P}(D \times D)$ it is considered as a many : many *relation* between $\mathcal{A}$ and $D \times D$, then X^{-1} coincides with the inverse relation.

[5] Filters and ultrafilters are formally defined in section 8.

Definition An *atomic representation* (X, D) of $\mathcal{A}$ is a representation such that for any pair $(d, e) \in X(1)$, the ultrafilter $X^{-1}(d, e)$ is principal — or, equivalently, there is a unique atom which holds on (d, e).

THEOREM 1 *Let $\mathcal{A}$ be any relation algebra. A representation X of $\mathcal{A}$ is an atomic representation if and only if it is a complete representation.*

PROOF:

First assume that X is a complete representation of $\mathcal{A}$: i.e., arbitrary suprema are preserved. Let $(x, y) \in X(1)$. We must show that $f = X^{-1}(x, y)$ is a principal ultrafilter. Now f has an infimum in $\mathcal{A}$: it is either an atom, if f is principal, or 0, if not. Write $\bigwedge f$ for the infimum of f. Clearly, arbitrary infima are also preserved in the representation. So $X(\bigwedge f) = \bigcap_{a \in f} X(a)$. But $(x, y) \in X(a)$ for all $a \in f$. So $X(\bigwedge f) \neq \varnothing$. Hence $\bigwedge f \neq 0$, so it is an atom, showing that f is principal, as required.

Conversely, let X be any atomic representation of $\mathcal{A}$. It is immediate that $\mathcal{A}$ is an atomic algebra. For if $r \in \mathcal{A}$ is non-zero then $X(r) \neq \varnothing$, and if $(x, y) \in X(r)$ then r lies in the ultrafilter $X^{-1}(x, y)$. As this is a principal filter, it follows that r lies above an atom of $\mathcal{A}$.

For any $a \in \mathcal{A}$, we have

$$X(a) = \bigcup \{X(\alpha) : \alpha \in At(\mathcal{A}), \alpha \leq a\}.$$

So, for any set of elements $a_\lambda \in \mathcal{A}$ such that $\bigvee_\lambda a_\lambda$ exists in $\mathcal{A}$, we have

$$\begin{aligned} X(\bigvee_\lambda a_\lambda) &= \bigcup \{X(\alpha) : \alpha \in At(\mathcal{A}), \alpha \leq \bigvee_\lambda a_\lambda\} \\ &= \bigcup_\lambda \bigcup \{X(\alpha) : \alpha \in At(\mathcal{A}), \alpha \leq a_\lambda\} \\ &= \bigcup_\lambda X(a_\lambda), \end{aligned}$$

and so X is a complete representation. □

Note As we just proved, if a relation algebra $\mathcal{A}$ has an atomic representation it follows that $\mathcal{A}$ is an atomic relation algebra. The converse doesn't follow at all: an atomic (even finite) relation algebra might not have any representations and, in the infinite case, even if it is representable it is not clear that it should have an atomic representation.

Example A proper relation algebra $\mathcal{A}$ on a domain D induces a representation (X, D) of itself, where $X(a) = a$ for all relations $a \in \mathcal{A}$. Thus, we can regard $\mathcal{A}$ as a representation of itself. However, a proper, *atomic* relation algebra need not be a *complete* representation of itself. Consider the proper relation algebra with domain $\mathbb{R}$, the set of real numbers. Let I be an interval with rational endpoints, or an unbounded interval — i.e. the endpoint ∞ is allowed. Intervals may be open, closed or semi-open. Define a basic relation $r(I)$ as $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y - x \in I\}$. The identity relation is $r([0, 0])$, the converse of $r((p, q))$ is $r((-q, -p))$ and compositions is defined by $r((p_1, q_1)); r((p_2, q_2)) = r((p_1 + p_2, q_1 + q_2))$. Let the relation algebra consist of all finite unions of these basic relations. The atomic relations are of the form $r([q, q])$ for some rational number q and because the rationals are dense in the reals the relation algebra is atomic.

But it is not a complete representation since the pair $(0, \pi)$ is not related by any atomic relation.

This relation algebra has another representation, though. Consider instead a proper relation algebra with domain the set of all rational numbers, $\mathbb{Q}$, and all the other definitions being as before. This relation algebra is isomorphic to the first, hence a representation. Furthermore, for every pair of rationals (p, q) it *is* the case that there is an atom $(r([q-p, q-p]))$ which relates (p, q) so the representation is complete.

Further work Roger Maddux constructs an atomic, non-integral, representable relation algebra with *no* complete representation ([Mad69], page 154–173). It is related to an example of Lyndon [Lyn50]. In [HH96], an integral, atomic representable relation algebra with no complete representation is constructed, and used to show that the class of relation algebras that have a complete representation is not elementary.

3.3 First-order theories

There are two types of first order languages corresponding, respectively, to relation algebras in general and to one fixed relation algebra. The first language L_r has constants and operators $(1, 0, -, \vee, Id, ^\smile, ;)$ and has no relation symbols other than equality. Later, when we find axioms and indeed equations that characterise the representable relation algebras, and axioms for other classes of relation algebras, it is in this language that our axiomatisation will be written.

Given a particular relation algebra $\mathcal{A}$, we can define a second type of first-order language, $L(\mathcal{A})$. This purely relational language has one binary predicate symbol for each element of $\mathcal{A}$. We obtain some results about the size of a representation by converting to a first-order theory in the language $L = L(\mathcal{A})$. This theory will be used again later. Define an L-theory, $T(\mathcal{A})$, to consist of all of the following:

$$\begin{array}{rcll}
\sigma_{Id} & = & \forall x, y\,[Id(x,y) \leftrightarrow (x = y)] & : \text{ for each } R, S, T \in \mathcal{A} \text{ with:} \\
\sigma_{\vee}(R,S,T) & = & \forall x, y\,[R(x,y) \leftrightarrow S(x,y) \vee T(x,y)] & : \ R = S \vee T \\
\sigma_{\neg}(R,S) & = & \forall x, y\,[1(x,y) \rightarrow (R(x,y) \leftrightarrow \neg(S)(x,y))] & : \ R = -S \\
\sigma_{\smile}(R,S) & = & \forall x, y\,[R(x,y) \leftrightarrow S(y,x)] & : \ R = S^\smile \\
\sigma_{;}(R,S,T) & = & \forall x, y\,[R(x,y) \leftrightarrow \exists z(S(x,z) \wedge T(z,y))] & : \ R = S;T
\end{array}$$

An L-structure consists of a domain and an interpretation of each predicate symbol in L as a binary relation over the domain. It is clear that an L-structure defines a representation of $\mathcal{A}$ if and only if it is a model of $T(\mathcal{A})$. Note that all the formulas in $T(\mathcal{A})$ are universal except the last which is universal–existential.

The next theorem follows immediately from the Löwenheim–Skolem theorem.

THEOREM 2 *If the cardinality of $\mathcal{A}$ is λ and if $\mathcal{A}$ has an infinite representation then it has representations of size μ for each cardinal $\mu \geq \lambda + \omega$.*

PROBLEM 1 *Let $\mathcal{A}$ be a representable, atomic relation algebra and suppose $|At(\mathcal{A})| = \lambda \geq \omega$. Let L_{ats} be the first-order language with binary predicate symbols for the atoms of $\mathcal{A}$ only, so $|L_{ats}| = \lambda$. Can the representations of $\mathcal{A}$ be axiomatised in the language L_{ats}? That is, is there necessarily an L_{ats}-theory T whose models are precisely those L_{ats}-structures M that expand to a representation of $\mathcal{A}$? Must $\mathcal{A}$ have a representation of size λ? (Observe that if $\mathcal{A}$ has a complete representation then the answer to the last question must be yes.)*

3.4 Networks

Definitions Let $\mathcal{A}$ be a relation algebra.

1. A *network* N over $\mathcal{A}$, or an $\mathcal{A}$-*network*, is a complete, directed graph, possibly empty, with each arc labelled by an element of $\mathcal{A}$, satisfying the two conditions below. We write $n \in N$ if n is a node in the graph, and $N(n,m)$ for the label on the arc from n to m. Thus N stands for the network, the labelling of the edges, and the set of nodes. Occasionally, where there is the possibility of an ambiguity, we will write $dom(N)$ for the set of nodes, though mostly we simply write N.

   The two conditions are:

   (a) Diagonal condition: $N(n,n) \leq Id_{\mathcal{A}}$ for all $n \in N$

   (b) Triangle condition: $N(l,m); N(m,n) \wedge N(l,n) \neq 0$ for all $l, m, n \in N$.

2. A network N is a *subnetwork* of another, M, if N is the restriction of M to a subset of its nodes.

3. For networks N, M, we write $N \subseteq M$ if the nodes of N are a subset of those of M, and $M(l,m) \leq N(l,m)$ for all nodes l, m of N.

4. An *embedding* from a network N to another, M, is a function h mapping nodes of N to nodes of M, and preserving the labelling, i.e.,

$$N(m,n) = M(h(m), h(n)) \text{ for all nodes } m, n \in N.$$

   We stress that h need not be 1–1. However, it does of course respect Id: we have $N(m,n) = Id \Leftrightarrow M(h(m), h(n)) = Id$. So the use of the term 'embedding' is natural.

5. M, N are said to be *isomorphic* if there is a bijective embedding h from M to N.

6. An $\mathcal{A}$-network N is *consistent* (or satisfiable) if $\mathcal{A}$ has a representation (X, D) that *embeds* N — i.e., there is a map $h : N \to D$, *not necessarily 1–1,* such that for all $n, m \in N$,

$$(h(n), h(m)) \in X(N(n,m)).$$

7. A network is *atomic* if each arc is labelled by an atom[6]. An atomic network satisfies $N(m,l) = N(l,m)^{\smile}$ for all $l, m \in N$. It also has the commonly used property known as 'path consistency',

$$N(l,m); N(m,n) \geq N(l,n) \text{ for all } l, m, n \in N,$$

   which, for general networks, is stronger than the triangle condition. (The triangle condition will be helpful in our completeness proof for arbitrary representations of infinite relation algebras.)

8. Recall that a relation algebra $\mathcal{A}$ decomposes as a subdirect product of simple components $\mathcal{A}_\lambda : \lambda \in \Lambda$ and for each component $\mathcal{A}_\lambda$ there is a homomorphism or *projection* from $\mathcal{A}$ onto $\mathcal{A}_\lambda$. For an atomic network N, there is a unique $\lambda \in \Lambda$ such that N projects to an atomic $\mathcal{A}_\lambda$-network and all the other projections map N to a graph labelled everywhere by 0. Thus every atomic $\mathcal{A}$-network is essentially equivalent to an atomic $\mathcal{A}_\lambda$-network for some λ.

Notation For an atomic relation algebra $\mathcal{A}$, let $Net(\mathcal{A})$ be the class of all finite, atomic, $\mathcal{A}$-networks.

Comments

1. In the temporal reasoning literature [All83, VK86, DMP91] a network is considered as an example of a constraint satisfaction problem — given a network, usually with non-atomic relations on the arcs, the problem is to discover whether the network is consistent. Consistency has been taken to mean that the network can be tightened[7] to an atomic network which still obeys the triangle condition. The problem is to choose one atom from the relation on each arc such that the resulting graph obeys the triangle condition and forms an atomic network. We see that an important assumption has been made, namely that an atomic network is consistent. This assumption turns out to be valid only on a certain subclass of the class of all relation algebras — those where $Net(\mathcal{A})$ forms a TAP-class (defined later).

[6] In [Mad82] an atomic network is called an $\mathcal{A}$-labelling of α where α is the size of the matrix or equivalently the number of nodes in the graph.

[7] Network N is a *tightening* of network M if N and M have exactly the same nodes, and for all nodes i and j, $N(i,j) \leq M(i,j)$.

2. Another interpretation of networks will be developed in the following section. A network will be used as a kind of bridge between the purely algebraic properties of the relation algebra and the properties of a representation. It can be seen that any finite substructure of a complete representation defines an atomic network. We shall consider the converse — when does a given atomic network embed in some representation of the relation algebra?

4 Games

An elegant method of determining whether a relation algebra has a representation and, if so, what properties its representations might have, is to try to construct a representation using a game (see [Hod85]). The crucial *conjunction lemma* (below) will allow us to avoid lengthy book-keeping arguments in the proofs.

Let $\mathcal{A}$ be an atomic relation algebra and let $\mathcal{K}$ be a class of finite atomic $\mathcal{A}$-networks. Our games will be played by two players, $\forall$ and $\exists$, respectively male and female, who alternately pick some network from $\mathcal{K}$ which extends the network of the previous player. $\forall$ plays first and the game is of countable length. So a *play* of the game might be

$$N_0 \subseteq N_1 \subseteq \cdots$$

where the networks with even index are chosen by $\forall$ and the odd ones by $\exists$. Here, the networks are atomic, so $N \subseteq N'$ just means that N is a subnetwork of N'. The class of all possible plays of the game is partitioned into two: wins for $\forall$ and wins for $\exists$. There are no draws. The wins for $\exists$ are determined by some property P of the *limit* of the play $\bigcup_{i<\omega} N_i$. We denote by G_P the game with property P. $\exists$ wins this play of the game G_P if and only if the limit satisfies P.

Notation We write $\mathcal{K}^{<\omega}$ for the set of all finite sequences of elements of $\mathcal{K}$.

Definition Let P be a property, and consider the game G_P. Let

$$\sigma : \mathcal{K}^{<\omega} \to \mathcal{K}$$

be such that $\sigma(N_0, N_1, \ldots, N_n)$ is a network extending N_n. $\exists$ can use σ to play the game by responding to the initial play $N_0 \subseteq N_1 \subseteq \ldots N_{2k}$ with $N_{2k+1} = \sigma(N_0, N_1, \ldots, N_{2k})$. σ is a *winning strategy* for $\exists$ if in any play of G_P in which $\exists$ always uses σ, $\exists$ wins the game.

Example If every network in $\mathcal{K}$ has a proper extension in $\mathcal{K}$ then $\exists$ has a strategy to ensure that the limit is infinite: for each turn she must choose a proper extension of $\forall$'s previous move.

LEMMA 3 (Conjunction Lemma) *Let $\{P_i : i < \omega\}$ be a countable set of properties and suppose σ_i is a winning strategy for $\exists$ in G_{P_i}. Then $\exists$ has a winning strategy in the game G_P where $P = \bigwedge_{i<\omega} P_i$, that is, she can achieve all the properties at once.*

PROOF:

See [Hod85] page 30. □

4.1 Using Games to Build Representations

Among finite relation algebras, the ones possessing representations can be characterised by an infinite axiom schema ([Lyn50] page 711). In this section, a representation of a (possibly infinite) relation algebra is further classified as *universal* or *homogeneous.* Relation algebras with such representations are characterised by certain network properties.

Definition Let $\mathcal{K}$ be a class of atomic networks of the atomic relation algebra $\mathcal{A}$.

1. $\mathcal{K}$ is *downward closed* if whenever $M \in \mathcal{K}$ and N is a subnetwork of M then $N \in \mathcal{K}$.
2. $\mathcal{K}$ is *closed under isomorphism* if whenever M, N are isomorphic networks, $M \in \mathcal{K}$ implies $N \in \mathcal{K}$.

Definition Let (X, D) be a representation of the relation algebra $\mathcal{A}$.

1. A *local isomorphism* θ of (X, D) is a finite partial map from D to itself, say $\theta : D_0 \to D$ where D_0 is a finite subset of D, such that $X^{-1}(d, e) = X^{-1}(\theta(d), \theta(e))$, for all $d, e \in D_0$.
2. X is *homogeneous* if every local isomorphism of (X, D) extends to a full automorphism of (X, D) i.e. a permutation α of D preserving the relations between each pair of points: $X^{-1}(d, e) = X^{-1}(\alpha(d), \alpha(e))$, for all $d, e \in D$.

Definition Let $\mathcal{A}$ be an atomic relation algebra, and let $\mathcal{K}$ be any class of atomic networks over $\mathcal{A}$, closed under isomorphism and downward closed. So $\mathcal{K}$ could be $Net(\mathcal{A})$, for example.

1. The *age* of a representation X of $\mathcal{A}$, written $age(X)$, is defined to be the class of all finite atomic networks that embed into X.
2. X is *universal* over $\mathcal{K}$ if every member of $\mathcal{K}$ embeds into X.

   X is a *universal representation* of $\mathcal{A}$ if X is universal over $age(Y)$ for every representation Y of $\mathcal{A}$.
3. $\mathcal{K}$ has the *joint embedding property* (JEP) if any two networks N and M in $\mathcal{K}$ embed jointly into a network L of $\mathcal{K}$.
4. $\mathcal{K}$ has the *Amalgamation Property* (AP) if given any networks L, M_1 and M_2 from $\mathcal{K}$ and embeddings $f_i : L \to M_i$ $(i = 1, 2)$ there is a network $N \in \mathcal{K}$ and embeddings $g_i : M_i \to N$ such that $g_1 f_1 = g_2 f_2$. Less formally, given M_1 and M_2 containing common isomorphic copies of a network L, there is a way of gluing M_1 and M_2 together along L and forming a larger network N, still in $\mathcal{K}$.

   As $\mathcal{K}$ is downward closed, it contains the empty network, L. Hence, the amalgamation property implies the joint embedding property.
5. An *amalgamation class* is a class of finite atomic networks with JEP and AP which is closed under isomorphism and downward closed.
6. $\mathcal{K}$ has the *triangle addition property* (TAP) if (a) every atom of $\mathcal{A}$ appears as the label on some edge of some network in $\mathcal{K}$ and (b) given any atomic network $N \in \mathcal{K}$ and any atomic network T of size three (a triangle) such that there is an edge E (a network of size two) and embeddings $\tau : E \to T$ and $\nu : E \to N$ then there is a network $L \in \mathcal{K}$ and embeddings $\tau' : T \to L$, $\nu' : N \to L$ such that $\tau' \circ \tau = \nu' \circ \nu$. This says that a network and a triangle with an isomorphic edge can be glued together along that edge and extended to a larger network.
7. A *TAP-class* is a class of finite atomic networks with the TAP, which is downward closed and closed under isomorphism.

Notation Write $\tau(\mathcal{A})$ for the union of all the TAP-classes of $\mathcal{A}$; clearly this is still a TAP-class. We will see that X is a universal representation of $\mathcal{A}$ if and only if X is universal over $\tau(\mathcal{A})$.

Definition Let $\mathcal{A}$ be an atomic relation algebra.

- A *congruence relation* $\sim$ on an $\mathcal{A}$-network N is an equivalence relation on the nodes of N such that for all $l, m, n \in N$, if $l \sim m$ then $N(l,n) = N(m,n)$ and $N(n,l) = N(n,m)$.
- Let N be an atomic $\mathcal{A}$-network. Define a relation $\sim$ on the nodes of N by $m \sim n \Leftrightarrow N(m,n) \leq Id$. This can be shown to be a congruence relation. Define an $L(\mathcal{A})$-structure N^* whose points are the $\sim$ equivalence classes $[n]$, and for each pair $([m],[n])$ and each $a \in \mathcal{A}$, we have
$$N^* \models a(([m],[n])) \Leftrightarrow a \geq N(m,n).$$
This is well-defined.
- We can equally view N^* as a map from $\mathcal{A}$ into the full power set algebra $\mathcal{P}(N/\sim)$, given by $N^*(a) = \{([m],[n]) : a \geq N(m,n)\}$ (this is well-defined).

Definition Let $\mathcal{A}, \mathcal{B}$ be relation algebras. A map $\theta : \mathcal{A} \to \mathcal{B}$ is said to be a *complete homomorphism* if it respects all the operators of relation algebra — $\theta(a;b) = \theta(a);\theta(b)$ etc. — and respects arbitrary suprema wherever they are defined. If $\mathcal{B}$ is a proper relation algebra with domain D, and θ is a one–one complete homomorphism $: \mathcal{A} \to \mathcal{B}$, then (θ, D) is a complete representation of $\mathcal{A}$. In this case, we will just say that θ is a complete representation of $\mathcal{A}$.

LEMMA 4 *Let $\mathcal{A}$ be an atomic relation algebra and let N be a non-empty atomic $\mathcal{A}$-network.*

1. *The following are equivalent:*
   (a) *N^* is a complete homomorphism of $\mathcal{A}$;*
   (b) *for every edge (n,m) and every pair of atoms b, c with $b;c \geq N(n,m)$, there is a node $l \in N$ such that $N(n,l) = b$ and $N(l,m) = c$.*[8]
2. *If $\mathcal{A}$ is simple, then (b) above is equivalent to N^* being a complete square representation of $\mathcal{A}$.*

PROOF:

1. $(a \Rightarrow b)$ Suppose that N^* is a complete homomorphism, let b, c be atoms of $\mathcal{A}$, let $n, m \in N$, and suppose that $N(n,m) \leq b;c$. So $([n];[m]) \in N^*(b;c)$. Because N^* respects the composition operator, there must be a point l with $([n],[l]) \in N^*(b)$ and $([l],[m]) \in N^*(c)$, and clearly we must have $N(n,l) = b$, $N(l,m) = c$.
   $(b \Leftarrow a)$ Assume that (b) holds. We first observe that N^* preserves all the operators. It is easy to verify that the Boolean operations and converse are preserved. Property (b) guarantees that composition is preserved as well. Completeness of the homomorphism follows from the proof of theorem 1, because for each pair of points $m, n \in N$ there is an atom a such that $N(m,n) = a$, so $(m,n) \in N^*(a)$.
2. Assume that (b) holds. From (1), N^* is a complete homomorphism. As $\mathcal{A}$ is simple, it has no non-trivial homomorphic images. Clearly, N^* is not the zero map, so it must be an isomorphism.[9] Thus, N^* is a representation of $\mathcal{A}$, and we have already seen that it must be a complete one. It is square by definition.
   Since any complete representation is a complete homomorphism, the converse follows from (1, $a \Rightarrow b$).

$\square$

[8] In [Mad82] a network satisfying this property is called a *complete* $\mathcal{A}$-labelling.

[9] More directly, suppose for contradiction that there is an $a \in \mathcal{A}$ with $a \neq 0$ but $N^*(a) = 0$. Then, since composition is preserved and using the simplicity of $\mathcal{A}$, $N^*(1) = N^*(1;a;1) = N^*(1);N^*(a);N^*(1) = 0$. However N is not empty so this cannot happen.

Notation Let (X, D) be a complete representation of the atomic relation algebra $\mathcal{A}$.

- Let $d, e \in X(1)$. We write $\overleftarrow{X}(d,e)$ for the unique atomic relation which holds between d and e. That is, $\overleftarrow{X}(d,e) = \bigwedge X^{-1}(d,e)$.
- Let $\Delta \subseteq D$. $\overleftarrow{X}(\Delta)$ is defined to be the atomic network with Δ as its set of nodes, where for all $\delta_1, \delta_2 \in \Delta$, the edge from δ_1 to δ_2 is labelled by the atomic relation which holds on (δ_1, δ_2), i.e., $\overleftarrow{X}(\Delta)(\delta_1, \delta_2) = \overleftarrow{X}(\delta_1, \delta_2)$.

The following theorem characterises complete representations of relation algebras by the outcome of games which, in turn, are determined by the existence of appropriate classes of atomic networks. Homogeneous and universal representations are also characterised.

THEOREM 5 *Let $\mathcal{A}$ be an atomic relation algebra with at most countably many atoms.*

1. *$\mathcal{A}$ is a completely representable relation algebra if and only if there is some TAP-class for $\mathcal{A}$ (if and only if $\tau(\mathcal{A})$ is non-empty).*

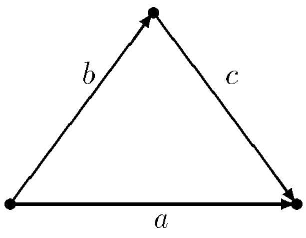
Now assume that $\mathcal{A}$ is simple. Let $\mathcal{K}$ be a class of finite atomic networks that is downward closed and closed under isomorphism.

2. *$\mathcal{A}$ has a complete, square representation X with $age(X) = \mathcal{K}$ if and only if $\mathcal{K}$ has TAP and JEP. $\mathcal{A}$ has a complete, universal, square representation if and only if $\tau(\mathcal{A})$ is non-empty and has JEP.*
3. *$\mathcal{A}$ has a homogeneous, complete, square representation X with $age(X) = \mathcal{K}$ if and only if $\mathcal{K}$ contains every triangle of $\mathcal{A}$ (i.e., every network of size 3) and has JEP and AP. Countable representations of this type will all be base-isomorphic to each other.*
4. *$\mathcal{A}$ has a countable, homogeneous, universal, square representation if and only if $\tau(\mathcal{A})$ is non-empty and has JEP and AP (i.e. $\tau(\mathcal{A})$ forms a non-empty amalgamation class). Such a representation is unique up to base-isomorphism.*

PROOF:

1. $\Rightarrow$

If $\mathcal{A}$ has a complete representation X, then let $\mathcal{K}$ be the class of all finite atomic networks which embed in X. Clearly $\mathcal{K}$ is downward closed and closed under isomorphism, and every atom of $\mathcal{A}$ labels an edge of some network in $\mathcal{K}$. Also, let N be any atomic network and let Ξ embed N into X. If $N(n,m) = a$ and $a \leq b;c$ then it is necessary to show that the triangle T



can be joined onto the edge (n,m) of N and form an atomic network that still belongs to $\mathcal{K}$. But since X is a representation there must be some point $x \in X$ say with $\overleftarrow{X}(\Xi(n), x) = b$ and $\overleftarrow{X}(x, \Xi(m)) = c$. The points $\Xi(N) \cup \{x\} \subseteq X$ define the required extension. Therefore $\mathcal{K}$ has TAP.

$\Leftarrow$
First suppose that $\mathcal{A}$ is simple.
Assume that some TAP-class $\mathcal{K}$ exists. Let $\exists$ win the game $G_R(\mathcal{K})$ if and only if the limit is a network L satisfying the conditions of lemma 4 part 2, or equivalently, L^* is a complete, square representation of $\mathcal{A}$. (Formally, we define property R of the limit to be just this.)
Claim: $\exists$ has a winning strategy for $G_R(\mathcal{K})$ and therefore $\mathcal{A}$ has a complete representation. Moreover, each network $N \in \mathcal{K}$ embeds into some complete representation[10] (if $\forall$'s first move is N then the limit is a complete representation embedding N).
Proof of claim: $\exists$ needs to ensure that the conditions of lemma 4 are met. She first needs to ensure that the limit is non-empty. This is easy: it is clear that any TAP-class contains non-empty networks, so if $\forall$ should begin with the empty network, $\exists$ can replace his move by a non-empty one, without loss.

Next, she needs to ensure that for each edge (n, m) which appears in the play (labelled by the atom a, say), and each pair of atoms b, c such that $b; c \geq a$, there is a node l such that (n, l) is labelled by b and (l, m) by c. Now since $\mathcal{K}$ has the triangle addition property, for any *one* edge (n, m) and any *single* pair of atoms b, c, $\exists$ can ensure that the required node is added. After the edge has been included in the play she simply chooses from $\mathcal{K}$ some network with the relevant triangle added on. Since there are only countably many edges which appear, and at most countably many pairs of suitable atoms, we can use the conjunction lemma to conclude that $\exists$ can arrange that for *every* edge, and *every* suitable pair of atoms, the extra node exists in the limit. This proves the claim.

If $\mathcal{A}$ is not simple, it can be shown that the TAP-class $\mathcal{K}$ decomposes into disjoint classes, each one being a TAP-class over a simple component of $\mathcal{A}$. Therefore each simple component of $\mathcal{A}$ has a complete representation, so the disjoint union of complete representations of each of the simple components will form a complete representation of $\mathcal{A}$.

2. (a) $\Rightarrow$
   Suppose $age(X) = \mathcal{K}$. As earlier, it follows that $\mathcal{K}$ has TAP. Also, $\mathcal{K}$ has JEP. For if N, M are in $\mathcal{K}$, they must embed in X, say as N_x and M_x. Let $L = \overleftarrow{X}(N_x \cup M_x)$ be the atomic network formed by the nodes of $N_x \cup M_x$. $L \in \mathcal{K}$, so JEP holds.
   $\Leftarrow$
   Let $\mathcal{K}$ have TAP and JEP. Define a win for $\exists$ in the game $G_U(\mathcal{K})$ by requiring that the limit is a complete, square representation with age $\mathcal{K}$. It is required to prove that $\exists$ has a winning strategy for this game. Since $\mathcal{K}$ has TAP, $\exists$ can certainly ensure that the limit is a complete, square representation. Also, given any one network $N \in \mathcal{K}$, she can ensure that N embeds in the limit. She can do this in one move: if $\forall$'s last turn was M, then JEP implies that she can choose some network $L \in \mathcal{K}$ extending M and embedding N. Now, up to isomorphism, $\mathcal{K}$ is countable (because $At(\mathcal{A})$ is countable). So by the conjunction lemma, she can enforce that the limit is a complete representation and that *each* network $N \in \mathcal{K}$ embeds in the limit.

   (b) Let $\mathcal{K} = \tau(\mathcal{A})$.
3. $\Rightarrow$
   Suppose $\mathcal{A}$ has a homogeneous, complete, square representation X with $age(X) = \mathcal{K}$. Clearly, $\mathcal{K}$ contains every triangle of $\mathcal{A}$. Let $L, M_1, M_2 \in \mathcal{K}$, and let $f_i : L \to M_i$ $(i = 1, 2)$ be embeddings. Let $= \theta_1(M_1)$ and $\theta_2(M_2)$ be respective appearances of M_1 and M_2 in X (θ_i are embeddings from M_i into X). There

[10] But note that the networks in $\mathcal{K}$ do not necessarily all embed into the same square representation.

is a local isomorphism from $\theta_1(f_1(L))$ to $\theta_2(f_2(L))$, namely, $\theta_2 f_2 {f_1}^{-1} {\theta_1}^{-1}$. By homogeneity, this extends to a full automorphism α of X. To find a network in $\mathcal{K}$ embedding M_1 and M_2, let $N = \overleftarrow{X}(\alpha(\theta_1(M_1)) \cup \theta_2(M_2))$. There are embeddings from M_1 and M_2 to N making a commutative diagram with the embeddings from L. This defines N, which embeds in X and therefore belongs to $\mathcal{K}$. The case where L is empty covers the JEP, and the general case covers AP.

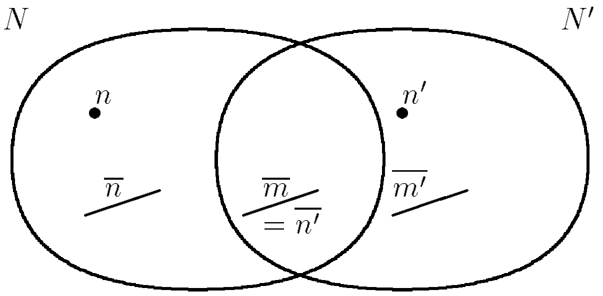
$\Leftarrow$

Suppose $\mathcal{K}$ has JEP and AP. Let $\exists$ win $G_A(\mathcal{K})$ if the limit is a homogeneous representation of $\mathcal{A}$ of age $\mathcal{K}$ (the limit is automatically square, as every edge of each network has a label on it). Since $\mathcal{K}$ contains every triangle of $\mathcal{A}$, the triangle addition property follows from AP. Thus $\exists$ can certainly ensure that the limit is a complete representation with age $\mathcal{K}$. She must also make sure that that the limit is homogenous, i.e., that any local isomorphism of the limit extends to a full automorphism. As the limit is countable, using a back-and-forth argument it is enough if:

($*$) for every local isomorphism θ, every node n of the limit is in the domain and range of some local isomorphism θ' that extends θ.

Since the limit is countable, there are only countably many local isomorphisms of the limit, and countably many nodes n to extend a local isomorphism to. So by the conjunction lemma, it is enough if $\exists$ can ensure that ($*$) holds for a single arbitrary (θ, n).

She can achieve this in one move. Let (θ, n) be given. At some stage during the game, where the current network is N, say, θ will be a local isomorphism of N, and n will be a node of N. To define $\theta(n)$, $\exists$ takes a copy N' of N and identifies the tuple $\bar{n}'$ of N' with $\bar{m}$ of N (this can be done since θ preserves all edge relations from $\bar{n}$ to $\bar{m}$). The amalgamation property allows her to extend the result to a structure in $\mathcal{K}$ containing N and N'. Let n' be the element of N' that corresponds to n. She sets $\theta(n) = n'$.



Similarly we can extend θ and N so that $\theta^{-1}(n)$ is defined.

Thus, $\exists$ can ensure that the limit is a complete, homogeneous representation of $\mathcal{A}$.

Let X and Y be any two countable, homogeneous, complete, square representations of $\mathcal{A}$ with age $\mathcal{K}$. We get a base-isomorphism by a back and forth construction. At each stage we have two finite sets M and N in X and Y, respectively, defining isomorphic networks (initially empty). We extend one of them (say M) by adding a new point, say x in X. Let M^+ be the atomic network formed by the points $M \cup \{x\}$. To find the corresponding point in Y use the fact that $age(X) = age(Y)$ to show that M^+ embeds in Y, and then use homogeneity to show that such a network extends N to $N \cup \{y\}$ for some $y \in Y$. Provided we ensure that each point of X and Y gets chosen eventually, the limit of this process will be a base-isomorphism between X and Y.

4. Set $\mathcal{K} = Net(\mathcal{A})$ in the previous part to show that $\mathcal{A}$ has a complete, homogeneous, universal, square representation if and only if $Net(\mathcal{A})$ has JEP and AP.

$\square$

Definition The countable, universal, homogeneous, square representation constructed in the last part of the preceding theorem is called the Fraïssé limit of the amalgamation class. See [Fra54].

5 First-order characterisation of the finite, representable relation algebras

Recall that the language L_r of relation algebras has constants $1, 0, Id$ and function symbols $\vee, -, \smile, ;$, and, as predicates, only the logical $=$. In order to derive first-order L_r-axioms for the class of finite, representable relation algebras, we use a second type of game. In this section, we consider only finite relation algebras, though some of our definitions and results possess generalisations to infinite ones. In later sections we will tackle the general case.

So let $\mathcal{A}$ be any finite relation algebra.

Definition Let N be any finite atomic $\mathcal{A}$-network. A play of the game $G^{at}_n(N, \mathcal{A})$ is a sequence of finite, atomic $\mathcal{A}$-networks

$$N = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n.$$

Each network extends the previous by a single node. In the kth round $(0 \leq k < n)$ $\forall$ picks any edge (l, m) from N_k and any pair of atoms b, c with $b; c \geq N_k(l, m)$. $\exists$ must choose a one point extension N_{k+1} of N_k with an extra point n_{k+2} such that $N_{k+1}(l, n_{k+2}) = b$ and $N_{k+1}(n_{k+2}, m) = c$. (We need to add 2 to the suffices because n_0, n_1 are constructed in the initial round, defined next.) That completes the round and if $\exists$ can continue to make a legitimate move until N_n is constructed she has won the play.

$G^{at}_n(\varnothing, \mathcal{A})$ is a special case where there is an additional, initial round. In this initial round $\forall$ chooses any atom $a \in \mathcal{A}$ and $\exists$ responds with the atomic network $N_0(a)$ with two nodes n_0 and n_1 such that $N_0(a)(n_0, n_1) = a$. $N_0(a)$ is determined up to isomorphism by a — it is not hard to show that the other labels must be $N_0(a)(n_0, n_0) = Id \wedge a; a^{\smile}$, $N_0(a)(n_1, n_1) = Id \wedge a^{\smile}; a$, $N_0(a)(n_1, n_0) = a^{\smile}$ (see [Mad82] lemma 5.12(1)). The game then continues as in $G^{at}_n(N_0(a), \mathcal{A})$. (Not wishing to confuse the reader, it might be helpful to think of $G^{at}_n(\varnothing, \mathcal{A})$ as two separate games played in succession. In the first game — the initial round — $\forall$ is pretty well in charge, and the point of this game is to set up a network so that the second game can get under way. $\forall$ is restricted in that he can only set up a two-point network of the form $N_0(a)$ for some a. Then the second game $G^{at}_n(N_0(a), \mathcal{A})$ starts, as described above.)

The game $G^{at}_\omega(N, \mathcal{A})$ is similar but in order to win $\exists$ must survive a countable sequence of rounds.

LEMMA 6 *Let $\mathcal{A}$ be a finite relation algebra. $\exists$ has a winning strategy for $G^{at}_\omega(\varnothing, \mathcal{A})$ if and only if $\mathcal{A}$ is representable.*

PROOF:

$\Leftarrow$: Let X be a representation of $\mathcal{A}$. Throughout a play of $G^{at}_\omega(\varnothing, \mathcal{A})$, $\exists$ maintains an embedding $' : N_k \to X$ to help her win the play. In the initial round of the game let $\forall$ choose $a \in At(\mathcal{A})$. $\exists$ responds with $N_0(a)$ and picks two points x_0, x_1 in X with $X \models a(x_0, x_1)$; she can find such points because X is a representation. She lets $n'_0 = x_0, n'_1 = x_1$. In round k if $\forall$ plays $m, n \in N_k$ and $r, s \in At(\mathcal{A})$ then $\exists$ finds a point $y \in X$ such that $X \models r(m', y) \wedge s(y, n')$. Such a point must exist in X because $r; s \geq N_k(m, n)$ and $X \models [N_k(m, n)](m', n')$. $\exists$ plays the atomic network determined by $N'_k \cup \{y\}$. These points do determine an atomic network N_{k+1} because $\mathcal{A}$ is finite, and hence every pair of points in X defines a unique atom. (For infinite relation algebras this does not always hold — see theorem 1.)

$\Rightarrow$: Assume $\exists$ has a winning strategy for $G^{at}_\omega(\varnothing, \mathcal{A})$ and suppose, for now, that $\mathcal{A}$ is simple. Let $\forall$ cooperate with $\exists$ in a play of the game by eventually picking each edge e, constructed in the ith round say, and each pair of atoms b, c with $b; c \geq N_i(e)$. Here, again, the finiteness (or countability) of the set of atoms of $\mathcal{A}$ is used. By part 2 of lemma 4 the limit will define a representation.

Now in an arbitrary (not necessarily simple) finite relation algebra $\mathcal{A}$, let $a \in At(\mathcal{A})$ be $\forall$'s initial move and suppose, as before, that he cooperates by picking each edge constructed and all suitable pairs of atoms. By part 1 of lemma 4 the limit will define a homomorphism X_a into a proper relation algebra on some domain D_a, say, such that $X_a(a) \neq \varnothing$. Construct X_a in this way for all $a \in At(\mathcal{A})$. We may assume without loss of generality that the D_a $(a \in At(\mathcal{A}))$ are pairwise disjoint. Now define a representation X of $\mathcal{A}$ by

$$X(r) = \bigcup_{a \in At(\mathcal{A})} X_a(r)$$

where r is an arbitrary element of $\mathcal{A}$. X is still a homomorphism, but now constructed to be one-one, and thus a representation. $\square$

LEMMA 7 *Let $\mathcal{A}$ be a finite relation algebra. If, for every $n < \omega$, $\exists$ has a winning strategy for $G^{at}_n(N, \mathcal{A})$, then she has a winning strategy for $G^{at}_\omega(N, \mathcal{A})$.*

PROOF:

In a play of $G^{at}_\omega(N, \mathcal{A})$, at each round $\exists$ has to maintain the property that for each $n < \omega$ she has a winning strategy for $G^{at}_n(M, \mathcal{A})$, where M is the current position. This is true initially by assumption. Provided she can maintain this each round, she can survive forever and win the play.

Suppose M is a network such that for all $n < \omega$ $\exists$ has a winning strategy in $G^{at}_n(M, \mathcal{A})$. Let $\forall$ pick an edge e from M and atoms b, c with $b; c \geq M(e)$. Because $\mathcal{A}$ is finite, there are only finitely many possible moves (up to isomorphism) that $\exists$ can respond with; and because we are assuming she has a winning strategy, there is a finite, non-empty set $\Xi_n(M, e, b, c)$ of winning moves she can make. If $n \geq m$, a winning strategy for $G^{at}_n(M, \mathcal{A})$ will clearly be a winning strategy for $G^{at}_m(M, \mathcal{A})$, too. Therefore $\Xi_n(M, e, b, c) \subseteq \Xi_m(M, e, b, c)$. Thus there is a nested sequence of finite sets

$$\Xi_1(M, e, b, c) \supseteq \Xi_2(M, e, b, c) \supseteq \ldots \Xi_n(M, e, b, c) \supseteq \ldots$$

Because the sets are finite and non-empty, the intersection $\bigcap_{n<\omega} \Xi_n(M, e, b, c)$ is non-empty. If $\exists$ chooses a network from this set, it will be part of a winning strategy for the game $G^{at}_n(M, \mathcal{A})$ for each $n < \omega$. Thus the property is maintained for another round.

The special case where $M = \varnothing$ is dealt with similarly. $\square$

LEMMA 8 *Let $\mathcal{A}$ be any finite relation algebra. There is an L_r-sentence ϕ_n that holds in $\mathcal{A}$ if and only if $\exists$ has a winning strategy for the game $G^{at}_n(\varnothing, \mathcal{A})$.*

PROOF:

Notation Let Δ be some non-empty, finite index set, and for each $i, j \in \Delta$, let x_{ij} be a variable that ranges over the atoms of a relation algebra. The L_r-formula $ANet(x_{ij} : i, j \in \Delta)$ is defined to be

$$\bigwedge_{i \in \Delta}(x_{ii} \leq Id) \wedge \bigwedge_{i,j,k \in \Delta}(x_{ij}; x_{jk} \geq x_{ik}).$$

Roughly, this formula means that the graph formed by the variables x_{ij} is an atomic network. More precisely, let h be any assignment from the variables x_{ij} to atoms of a relation algebra $\mathcal{A}$. Let $h(x_{ij} : i, j \in \Delta)$ be the graph with nodes Δ and each edge (i, j) labelled by the atom $h(x_{ij})$. Then $\mathcal{A}, h \models ANet(x_{ij} : i, j \in \Delta)$ if and only if $h(x_{ij} : i, j \in \Delta)$ is an atomic $\mathcal{A}$-network.

Next we define formulas $\psi_n(x_{ij} : i, j \in \Delta)$. A straightforward induction will show that for any assignment h from the variables into $At(\mathcal{A})$ we have $\mathcal{A}, h \models \psi_n(x_{ij} : i, j \in \Delta)$ if and only if $\exists$ has a winning strategy for the game $G_n^{at}(h(x_{ij} : i, j \in \Delta), \mathcal{A})$.

The Lyndon Conditions The formulas are defined inductively:

$$\begin{aligned}
\psi_0(x_{ij} : i, j \in \Delta) &= ANet(x_{ij} : i, j \in \Delta) \\
\psi_{n+1}(x_{ij} : i, j \in \Delta) &= \bigwedge_{i,j \in \Delta} \forall y_{id}, y_{dj}[(y_{id}; y_{dj} \geq x_{ij}) \rightarrow (\exists_{k \in \Delta} x_{kd}, x_{dk}, x_{dd}) \\
&\quad (x_{id} = y_{id}) \wedge (x_{dj} = y_{dj}) \wedge \psi_n(x_{ij} : i, j \in \Delta \cup \{d\})],
\end{aligned}$$

where the variables $x_{ij}, y_{ij} : i, j \in \Delta \cup \{d\}$ range over the *atoms* of a relation algebra, and $d \notin \Delta$ is a new index. As a special case, corresponding to a game starting from the empty network, we define

$$\phi_n =_{\text{def.}} (\forall x_{ij} : i, j < 2) ANet(x_{ij} : i, j < 2) \rightarrow \psi_n(x_{ij} : i, j < 2)$$

The formulas ϕ_n, $n < \omega$ are equivalent to the Lyndon conditions which appeared in [Lyn50]. □

The following theorem was proved first in [Lyn50] and follows immediately from the previous three lemmas.[11]

THEOREM 9 *A finite relation algebra satisfies ϕ_n for all $n < \omega$ if and only if it is representable.*

REMARK 10 *One can ask whether, for infinite atomic relation algebras, the Lyndon conditions imply the existence of a complete representation. It turns out that the answer is no, as is shown in [HH96]. However, it is also shown there that an arbitrary atomic relation algebra satisfies the Lyndon conditions if and only if it is elementarily equivalent to a relation algebra that has a complete representation.*

5.1 Axiomatising Relation Algebras with Homogeneous Representations

We can use the same method to define a set of axioms that hold in a finite relation algebra $\mathcal{A}$ if and only if it has a homogeneous representation. Again, some of the definitions and results have generalisations to certain kinds of infinite relation algebra. But we postpone the treatment of the infinite case until section 10.

First, we examine what happens to homogeneity when we pass from a relation algebra to its simple components. Given a relation algebra $\mathcal{A}$, let us say that *ample simple components of $\mathcal{A}$ have a homogeneous representation* if for every non-zero $a \in \mathcal{A}$, there is a simple component A_λ of $\mathcal{A}$

[11] The paper contains an important error concerning infinite relation algebras. The correction can be found in [Lyn56].

such that (i) $\mathcal{A}_\lambda$ has a homogeneous representation, and (ii) if ν is the projection homomorphism from $\mathcal{A}$ onto $\mathcal{A}_\lambda$, then $\nu(a) \neq 0$. (That is, if $\mathcal{S}$ is the class of simple relation algebras with homogeneous representations, then $\mathcal{A}$ is residually $\mathcal{S}$.)

LEMMA 11 *Let $\mathcal{A}$ be any relation algebra. Then $\mathcal{A}$ has a homogeneous representation if and only if ample simple components of $\mathcal{A}$ have a homogeneous representation.*

PROOF:

$\Rightarrow$

Let X be any representation of $\mathcal{A}$. Then $X(1_\mathcal{A})$ is an equivalence relation on the domain of X. For any $X(1_\mathcal{A})$-equivalence class E, the map $X\lceil_E$ defined by $X\lceil_E(a) = X(a) \cap (E \times E)$, for any $a \in \mathcal{A}$, is a relation algebra homomorphism from $\mathcal{A}$ onto a simple, proper relation algebra $\mathcal{A}_E$, say, with domain E. Then $\mathcal{A}_E$ is simple, since if $a \in \mathcal{A}$ is such that $X\lceil_E(a) \neq \varnothing$, we have

$$1_{\mathcal{A}_E} ; X\lceil_E(a) ; 1_{\mathcal{A}_E} = (E \times E) ; X\lceil_E(a) ; (E \times E) = E \times E = 1_{\mathcal{A}_E}.$$

Thus, $\mathcal{A}_E$ is isomorphic to a simple component of $\mathcal{A}$.

Assume further that X is a homogeneous representation of $\mathcal{A}$. Now (I, E) is a representation of $\mathcal{A}_E$, where $I(a) = a$ for $a \in \mathcal{A}_E$. We claim that it is a homogeneous representation. Let θ be a local isomorphism of (I, E). We will extend θ to an automorphism of (I, E). We may assume that $\theta \neq \varnothing$; otherwise, the identity extends it. Now θ preserves the relations of $\mathcal{A}_E$. Since these are simply the intersection with $E \times E$ of the relations $X(a)$ $(a \in \mathcal{A})$, and $dom(\theta) \subseteq E$, it follows that θ is a local isomorphism of the representation X of $\mathcal{A}$. Hence it extends to an automorphism α of X. Clearly, α fixes E setwise, so its restriction $\alpha\lceil_E$ to E is a permutation of E. We now see that $\alpha\lceil_E$ is an automorphism of the representation (I, E) of $\mathcal{A}_E$ extending θ. This proves the claim.

Since for any non-zero $a \in \mathcal{A}$ there is an $X(1_\mathcal{A})$-equivalence class E with $X(a) \cap (E \times E) \neq \varnothing$, we are done.

$\Leftarrow$

Conversely, assume that ample simple components $\mathcal{A}_\lambda$ of $\mathcal{A}$ have homogeneous representations X_λ. For each λ, let π_λ be the projection from $\mathcal{A}$ onto $\mathcal{A}_\lambda$. By deleting some of the $\mathcal{A}_\lambda$ if need be, we can assume that the kernels $\{a \in \mathcal{A} : \pi_\lambda(a) = 0\}$ of the π_λ are pairwise distinct. By simplicity, they are pairwise $\subseteq$-incomparable.

A homogeneous representation X of $\mathcal{A}$ can now be constructed as follows. The domain of X is taken to be the disjoint union of the domains of the X_λ, and for any $a \in \mathcal{A}$,

$$X(a) = \bigcup_\lambda X_\lambda(\pi_\lambda(a)).$$

X is evidently a relation algebra homomorphism of $\mathcal{A}$ into a proper relation algebra, and as the $\mathcal{A}_\lambda$ are 'ample', X is 1–1. Hence it is a representation of $\mathcal{A}$. To see that it is a homogeneous one, observe that any local isomorphism f of X maps points in the domain of any given X_λ back into X_λ. (For let $x \in X_\lambda$, $f : x \mapsto y$, $y \in X_\mu$. If $\lambda \neq \mu$, then by the assumption on kernels there is $a \in \mathcal{A}$ with $\pi_\lambda(a) \neq 0$, $\pi_\mu(a) = 0$. Let $b = 1; a; 1$. Then $\pi_\lambda(b) = 1$, $\pi_\mu(b) = 0$. So $(x, x) \in X(b)$, but $(y, y) \notin X(b)$, a contradiction.) Thus f is a union of local isomorphisms f_λ of X_λ (some of the f_λ may be empty). By homogeneity, each f_λ extends to an automorphism α_λ of X_λ. The required automorphism α of X is then the union of the α_λ; as in the first part, this preserves the relations of $\mathcal{A}$. □

PROBLEM 2 *Is it true that if a relation algebra has a homogeneous representation then all its simple components do? (This is true for finite relation algebras.)*

Games

- For non-empty N, let a play of the game $G_n^{at*}(N, \mathcal{A})$ be a nested sequence of atomic networks $N = N_0 \subseteq N_1 \subseteq \ldots N_n$. In the kth round ($0 \leq k < n$), if N_k has just been defined, $\forall$ does one of two things. Either he plays as before by picking any edge $e \in N_k$ and a pair of atoms b, c with $b; c \geq N_k(e)$. $\exists$ responds by extending N_k to N_{k+1} so that there are edges $f, g \in N_{k+1}$ with $e = f \circ g$ and $N_{k+1}(f) = b$, $N_{k+1}(g) = c$ ($\circ$ is just composition of edges). Alternatively $\forall$ picks any local isomorphism $\Theta : L \cong L'$ where L, L' are isomorphic subnetworks of N_k. In this case $\exists$ must respond by defining a network N_{k+1} with two embeddings $\sigma, \tau : N_k \to N_{k+1}$ such that $\sigma \circ \Theta \subseteq \tau$. In other words she has to glue two copies of N_k together along L and L'. $\exists$ wins the play if she survives till the nth round.
- $G_\omega^{at*}(N, \mathcal{A})$ is similar but a play is of countable length.
- As before we can define games $G_n^{at*}(\emptyset, \mathcal{A}), G_\omega^{at*}(\emptyset, \mathcal{A})$ where there is an initial round starting from the empty network. The initial round of these games is the same as in $G_n^{at}(\emptyset, \mathcal{A})$ and $G_\omega^{at}(\emptyset, \mathcal{A})$ respectively.

LEMMA 12 *Let $\mathcal{A}$ be a finite relation algebra.*

1. *$\exists$ has a winning strategy for $G_\omega^{at*}(\emptyset, \mathcal{A})$ if and only if $\mathcal{A}$ has a homogeneous representation.*
2. *If $\exists$ has a winning strategy for $G_n^{at*}(N, \mathcal{A})$ (all $n < \omega$) then she has a winning strategy for $G_\omega^{at*}(N, \mathcal{A})$.*

PROOF:

(1): If $\mathcal{A}$ has a homogeneous representation then, as in lemma 6, she can use it to win the game.

Conversely, suppose she has a winning strategy for $G_\omega^{at*}(\emptyset, \mathcal{A})$. For each atom a of $\mathcal{A}$, there is a (unique) simple component $\mathcal{A}_a$ such that the projection $\pi_a : \mathcal{A} \to \mathcal{A}_a$ does not map a to zero. By lemma 11, it suffices to show that for each atom a there is a homogeneous representation X_a of $\mathcal{A}_a$ — then, a homogeneous representation of $\mathcal{A}$ can be obtained by taking the disjoint union of the X_as. But if $\forall$ plays a in the initial round and then cooperates to make sure every edge and suitable pair of atoms gets picked eventually and also every local isomorphism that occurs in the play, the outcome defines a homogeneous representation X_a of $\mathcal{A}_a$.

(2): This is similar to the proof of lemma 7. $\Box$

LEMMA 13 *There is a formula η_n that holds in $\mathcal{A}$ if and only if $\exists$ has a winning strategy for $G_n^{at*}(\emptyset, \mathcal{A})$.*

PROOF:

As before we deal with the game starting from a non-empty network first. We define formulas θ_n ($n < \omega$) to do this. Let Δ, x_{ij}, and the formula $Anet$ be as before. We let

$$\theta_0(x_{ij} : i, j \in \Delta) = ANet(x_{ij} : i, j \in \Delta).$$

In the definition of θ_{n+1}, below, the first formula is just the same as in the definition of the Lyndon conditions, and corresponds to the case where $\forall$ chooses an edge and a pair of atoms for his move. The index d is not to be in Δ. In the second formula, Δ' is a disjoint copy of Δ, and for $i \in \Delta$, i' is the element of Δ' that corresponds to i. The notation $\sigma : \Delta \rightsquigarrow \Delta$ means that σ is a partial, one-one map from Δ to itself. This formula says that if σ defines a partial isomorphism then there is a copy network (indexed by Δ') which can be glued to the original network along the isomorphic part to form a bigger network on which θ_n holds. In full, $\theta_{n+1}(x_{ij} : i, j \in \Delta)$ is the conjunction of the following two formulas:

$$\begin{aligned}&\bigwedge_{i,j\in\Delta}(\forall y_{id}, y_{dj})[(y_{id}; y_{dj} \geq x_{ij})\\ &\quad \to (\exists_{k\in\Delta} x_{kd}, x_{dk}, x_{dd})[(x_{id} = y_{id}) \wedge (x_{dj} = y_{dj}) \wedge \theta_n(x_{ij} : i, j \in \Delta \cup \{d\})],\end{aligned}$$

and

$$\begin{aligned}&\bigwedge_{\sigma:\Delta\rightsquigarrow\Delta,\, i,j\in dom(\sigma)} \Big((x_{ij} = x_{\sigma(i),\sigma(j)})\\ &\quad \to [(\exists_{i,j\in\Delta} x_{i',j'}, x_{i,j'}, x_{i',j}) \bigwedge_{i,j\in\Delta}(x_{i',j'} = x_{ij})\\ &\qquad \wedge \bigwedge_{i\in dom(\sigma)}(x_{i',\sigma(i)} \leq Id) \wedge \theta_n(x_{ab} : a, b \in \Delta \cup \Delta')]\Big).\end{aligned}$$

Now let $\eta_n = (\forall x_{ij} : i, j < 2)\ ANet(x_{ij} : i, j < 2) \to \theta_n(x_{ij} : i, j < 2)$. □

Putting these results together we have

THEOREM 14 *A finite relation algebra satisfies $\{\eta_n : n < \omega\}$ if and only if it has a homogeneous representation.*

PROBLEM 3 *Is it possible to find a set of axioms that characterise the finite relation algebras with universal representations?*

6 Some examples of Relation Algebras

The Allen interval algebra [All83] is a finite relation algebra with 13 atoms, representing the 13 possible relations that two non-empty intervals of a linear order can bear to each other. It is not hard to show [Hir94] that the atomic networks of the Allen interval algebra form an amalgamation class. The last part of theorem 5 now tells us that it has a countable, universal, homogeneous, square representation, and that any two such representations are base-isomorphic to each other. It is tempting to draw the conclusion that a relation algebra is 'ω-categorical' (that all its countable representations are base-isomorphic — a theorem first proved for the Allen interval algebra in [LM87][12]) if its atomic networks form an amalgamation class. However, this conclusion would be far too hasty, and to illustrate this we define a relation algebra whose atomic networks form an amalgamation class and which therefore has only one countable, universal, homogeneous, square representation (up to base-isomorphism), but which has non-base-isomorphic countable square representations which are either not universal or inhomogeneous (or neither).

Let $\mathcal{B}$ have atoms $\{Id, e, d\}$ where Id is the identity, $e^\smile = e$ and $d^\smile = d$ and composition is defined by

$$\begin{aligned} e; e &= Id \vee e \\ e; d = d; e &= d \\ d; d &= 1 \end{aligned}$$

So the relation $Id \vee e$ is an equivalence relation which partitions the domain of a representation into at least three clusters. The countable, universal, homogeneous, square representation consists of a countable collection of countable 'clusters'. Within one cluster two distinct elements are related by e. Two elements from different clusters are related by d.

However, the representation defined similarly but with always three elements in a cluster is homogeneous but not universal. A representation with clusters of size n for all $n > 2$ is universal but not homogeneous. If there are infinitely many clusters of size 7 and 11, but no other clusters, then the representation is neither universal nor homogeneous. Thus $\mathcal{B}$ has many non-base-isomorphic, countable, square representations.

[12] See [Hir94] for a generalisation of Ladkin and Maddux's result to all interval algebras.

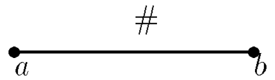
6.1 Examples[13]

- The *point algebra* for linear order has three atoms $\{Id, <, >\}$, and is given by
  - $<^{\smile} = >$ and $>^{\smile} = <$
  - $(<;<) = <$ and $(>;>) = >$
  - $(<;>) = (>;<) = 1$.

  The networks of the point algebra, and those for the Allen interval algebra, both form amalgamation classes. In fact, the universal, homogeneous representation constructed by theorem 5 (the Fraïssé limit) is in each case the only countable representation of each algebra — respectively the rationals $\mathbb{Q}$ and intervals of rationals.

- The relation algebra $\mathcal{B}$ (above). $Net(\mathcal{B})$ forms an amalgamation class so $\mathcal{B}$ has a unique countable, homogeneous, universal, square representation. However $\mathcal{B}$ has uncountably many other non-base-isomorphic countable, square representations.

- Let $\mathcal{C}$ be the relation algebra with atoms Id and $\#$ (both self converse), composition being defined by $\#;\# = Id$. $\mathcal{C}$ has only one square representation, which has exactly two elements, and looks like this.



- Let $\mathcal{D}$ have atoms Id and $\#$ (both self converse) and composition defined by $\#;\# = Id \vee \#$. $Net(\mathcal{D})$ forms an amalgamation class and the Fraïssé limit consists of a countable set of points and each pair of distinct points is related by $\#$. However, for each $n \geq 3$, it also has finite representations consisting of a set of size n with each pair of distinct points related by $\#$. Still, $\mathcal{D}$ has only one countable, square representation (up to base-isomorphism) and thus is 'weakly ω-categorical' (see section 7).

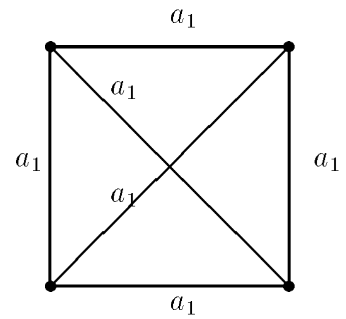
- **Lyndon Algebras** [Lyn61] Let $\lambda \geq 4$. The λth *Lyndon Algebra* is an atomic relation algebra $\mathcal{A}_\lambda$ with atoms $Id, a_1, \ldots, a_{\lambda-1}$. Each atom is self-converse, and composition of atoms is given by $a_i;a_i = Id \vee a_i$ $(0 < i < \lambda)$ and $a_i;a_j = \bigvee_{k \neq i,j} a_k$ $(i, j < \lambda, i \neq j)$. Composition and converse are then determined on the non-atomic elements by the distributive laws. These algebras are integral and symmetric. They are very useful as counter-examples to a number of conjectures, as we will see later. Note that $|At(\mathcal{A}_\lambda)| = \lambda$.

  Consider the $(n+2)$th Lyndon algebra, $\mathcal{A}_{n+2}$, with atoms $Id, a_1 \ldots, a_{n+1}$, and suppose there exists a projective plane G of order n (so $n+1$ points on a line).

  A square representation (X, U) of $\mathcal{A}_{n+2}$ can (in fact *must*) be constructed as follows. Form an affine geometry by picking one line l in G (the "line at infinity") and removing it, leaving $U = G \setminus l$. Enumerate the $n+1$ points of l in some way: $l = \{l_1, \ldots l_{n+1}\}$. We use U as the domain for the representation; in fact there are n^2 points in U. For any pair of distinct points $x, y \in U$ there is a unique projective line $\overline{xy}$ of G through them, and this line meets l at a unique point. Define $X(a_i)$ to be the set of pairs (x, y) such that $x \neq y$ and $\overline{xy}$ meets l at l_i (for each i with $1 \leq i \leq n+1$). Of course $X(Id) = \{(x, x) : x \in U\}$. X can be extended to non-atomic elements by taking the union of the representations of the atoms contained in the element. It is not hard to see that this is a representation of $\mathcal{A}_{n+2}$.

  Consider $\mathcal{A}_{3+2}$. Since 3 is prime and so there is a projective plane of order 3, such a geometry and such an affine representation exist. But the representation is not universal over $Net(\mathcal{A})$, as the following network does not embed in U.

[13] A large collection of finite, symmetric relation algebras is defined in [Mad82] and [Mad85].



This is because there are only three distinct points in U on a line through l_1.

It can be shown (see [Lyn61] theorem 1) that all square representations of $\mathcal{A}_{n+2}$ are base-isomorphic to affine representations, hence the quadrangle above does not embed in any representation of $\mathcal{A}_5$.

6.2 A Relation Algebra with no Homogeneous Representation

Lyndon algebras can be used to construct a relation algebra with no homogeneous representation.

Lyndon gives a very general definition of a geometry G, which we quote [Lyn61]. It consists of a set of *points*, together with certain subsets, called *lines*. The geometry must satisfy three axioms:

- there exists at least one line, and each line contains at least four points
- each pair of distinct points p and q lies on a unique line, $\overline{pq}$
- if p, q and r are distinct points, and a line meets $\overline{pq}$ and $\overline{pr}$ in distinct points, then it meets $\overline{qr}$.

In such a geometry it follows from these axioms that every line contains the same number $n+1$ of points and the *order* of the geometry is defined to be n. To define the *dimension* of a geometry, let $\bar{l} = l_1, \ldots, l_d$ be a sequence of lines. Say that $\bar{l}$ is *independent* if for any sequence of points $p_0, \ldots, p_d$ with $p_i \neq p_{i+1}$, if $\overline{p_i p_{i+1}} = l_i$ $(i = 1, \ldots, d)$ then $p_0 \neq p_d$. The *dimension* of G is defined to be the size of a maximal, independent sequence of lines.

Given a geometry G, Lyndon defines a relation algebra $\mathcal{A}(G)$. The elements of $\mathcal{A}(G)$ are all subsets of $G \cup \{Id\}$ where Id is a new element, disjoint from G. The boolean operations $\vee, -$ are interpreted as $\cup, \setminus$, as usual. The converse of any set is itself. Id is the identity element, and composition is defined on the other atoms by

$$p;p = \{Id, p\}$$
$$p;q = \overline{pq} \setminus \{p, q\} \text{ for } p \neq q,$$

where we identify a point p with the singleton $\{p\}$, and $\overline{pq}$ is the unique line of G containing p and q. $\mathcal{A}(G)$ forms a Lyndon algebra (see above), and so a relation algebra.

If G can be embedded as a hyperplane in a geometry H of one higher dimension then we can define the *affine* space $D = H \setminus G$. An *affine representation* Δ of $\mathcal{A}(G)$ is defined on the atoms by letting $\Delta(Id)$ be the identity relation on D and for a point $p \in G$

$$\Delta(p) = \{(x, y) : x \neq y \text{ and } p \in \overline{xy}\}.$$

This induces in a unique way a complete representation of $\mathcal{A}(G)$. We now quote a theorem from [Lyn61] page 24.

THEOREM 15 *Each affine representation of $\mathcal{A}(G)$ is a representation, and each complete[ly additive] representation of $\mathcal{A}(G)$ is equivalent [base-isomorphic] to some affine representation.*

For finite relation algebras, there is no difference between complete representations and general representations. So if we can prove the lemma that follows, then by taking a finite G such that the algebra $\mathcal{A}(G)$ is representable, we will have shown that there are representable relation algebras with no homogeneous representation.

LEMMA 16 *An affine representation of dimension greater than one and order greater than three is not homogeneous.*

PROOF:

Let (D, Δ) be an affine representation of $\mathcal{A}(G)$ as defined above. First we show that any automorphism α of D that fixes two distinct points d and e must be the identity. Let $f \in D$ be any point and suppose first that f is not on the line $\overline{de}$. Let $\overline{df}$ meet G at p i.e. $(d, f) \in \Delta(p)$. Since α is an automorphism fixing d, $\overline{\alpha(d)\alpha(f)} = \overline{d\alpha(f)} \in \Delta(p)$. So $\alpha(f)$ is on the line $\overline{df}$. Similarly it is on the line $\overline{ef}$, and as these two lines meet only once, it follows that f is fixed by α. The case where f is on the line $\overline{de}$ can be covered by two iterations of this argument: once to fix some other point off the line and a second time to fix f.

Now in an affine space the order of the geometry is the same as the number of points on each line. By assumption the number of points on a line is at least four. Let l be any line and let a_1, a_2, a_3, a_4 be any distinct points on the line. If l meets G at p then each pair $(a_i, a_j) \in \Delta(p)$ $(i \neq j)$. So the mapping

$$(a_1, a_2, a_3, a_4) \to (a_2, a_1, a_3, a_4)$$

preserves the relations between the four points and is therefore a local isomorphism. It cannot extend to a full automorphism since it fixes two points and it is not the identity. Consequently this representation is not homogeneous. □

So an example of a representable relation algebra with no homogeneous representation can be formed by taking a field of order four and embedding a two-dimensional vector space in a three-dimensional one over that field. These can be turned into a one-dimensional projective space G embedded in a two-dimensional projective space H. Let D be the affine space $H \setminus G$. G has five points so the relation algebra $\mathcal{A}(G)$ has six atoms: $Id, g_1, \ldots, g_5$. Id is the identity element and compositions of the other atoms are defined by

$$g_i ; g_i = Id \vee g_i \qquad \text{(for } i = 1 \ldots 5)$$

$$g_i ; g_j = \bigvee_{k \neq i,j} g_k \qquad (i \neq j)$$

All of its representations must be equivalent to an affine representation into D and therefore none of them are homogeneous.

COROLLARY 17 *The class of relation algebras possessing homogeneous representations is not axiomatisable by universal sentences in the language of relation algebra.*

PROOF:

Every representable relation algebra $\mathcal{A}$ can be extended to a relation algebra with a homogeneous representation. To see this, let $\mathcal{P}$ be a proper relation algebra, with top element $1_{\mathcal{P}}$, isomorphic to $\mathcal{A}$. Extend $\mathcal{P}$ to a proper relation algebra $\mathcal{P}^+$ on the same domain by taking the power set of the set of pairs in $1_{\mathcal{P}}$. This is now *rigid* in that there are no non-trivial local isomorphisms of $\mathcal{P}^+$. Thus $\mathcal{P}^+$ provides a homogeneous representation of itself.

So a representable relation algebra with no homogeneous representations (such as $A(G)$ above) extends to one which does have a homogeneous representation. Thus the class of relation algebras with homogeneous representations is not closed under subalgebras and therefore cannot be axiomatised by universal sentences. □

6.3 Lyndon algebras can have non-base-isomorphic square representations

Suppose there is a projective plane G of order n (for example this holds if n is prime). We have seen that for the Lyndon algebra $\mathcal{A}_{n+2}$ with atoms $\{Id, a_1 \ldots, a_{n+1}\}$ there is an affine representation U formed by removing a single line l from G and enumerating the points of l as $\{l_1, \ldots, l_{n+1}\}$. The representation X is defined by letting $X(a_i) = \{(x,y) : \overline{xy} \cap l = \{l_i\}\}$ $(i = 1, \ldots n+1)$. In general, the base-isomorphism type of the representation obtained above does depend on the enumeration, for if any two enumerations give base-isomorphic representations, it means that any permutation of l is induced by a collineation (line-preserving permutation) of G. But this is not the case in general.

PROPOSITION 18 *The Lyndon algebra $\mathcal{A}_{31}$ has non-base-isomorphic, square representations.*

PROOF:

Let G be a projective plane of order $n = 29$, and let l be a line of G. Take a permutation σ of the points of l, consisting of five disjoint cycles of orders $2, 3, 5, 7$, and 11, respectively. This is possible, as $2+3+5+7+11 = 28$, while l has 30 elements. Then σ has order $2.3.5.7.11 = 2310$. But σ can't be induced by any collineation, as we can show (cf. theorem 13.1 of [HP73]) that no collineation of G has order greater than the number of points of G, viz., $|G| = n^2 + n + 1 = 871$.

To see this, let Γ be the group generated by any collineation of G (or, more generally, any abelian collineation group of G). We must prove that $|\Gamma| \leq |G|$.

The group Γ acts both on the points and on the lines of G. Let X be an orbit of Γ of maximal size. We can assume that $|X| > 1$ — the result is trivial otherwise. By duality of points and lines, we can assume that X is a set of points. If $x \in X$, write Γ_x for $\{\gamma \in \Gamma \mid \gamma(x) = x\}$, the stabiliser of x. Bear in mind that because Γ is abelian, for any $y \in X$ we have

$$\Gamma_y = \gamma \Gamma_x \gamma^{-1} = \Gamma_x,$$

where γ is any element of Γ such that $\gamma(x) = y$. Thus Γ_x fixes X pointwise.

We show that each Γ_x is the identity 1, so that $\gamma \mapsto \gamma(x)$ is a bijection from Γ to X, and hence $|\Gamma| = |X| \leq |G|$.

If not, then it can be checked that there is $x \in X$ such that some line through x is moved by Γ_x. So some Γ_x-orbit Y_x of lines through x has size at least 2. For each $y \in X$, let $Y_y = \gamma Y_x \gamma^{-1}$ for some $\gamma \in \Gamma$ such that $\gamma(x) = y$; this is well-defined as Y_x is an orbit of Γ_x. So Y_y is a set of at least two lines through y, and is a Γ_y-orbit. Now since Γ_x fixes the whole of X, any line through x and moved by Γ_x intersects X only in x. Hence, if $y, z \in X$ and $y \neq z$ then Y_y and Y_z are disjoint. We now get a Γ-orbit (on lines) of at least twice the size of X, by taking $\bigcup_{y \in X} Y_y$. This contradicts the maximality of X, and completes the proof. □

6.4 A Hierarchy of Representations?

Call a representation *n-transitive* if any partial isomorphism of size at most n extends to a full automorphism of the representation. A representation is n-transitive for all $n < \omega$ if and only if it is homogeneous. A representation is called *permutational* if it is one-transitive. We can call the relation algebra n-transitive (permutational) if it has *some* n-transitive (respectively permutational) representation. In [McK70] it is shown that the subalgebras of group relation algebras are not finitely axiomatisable over the permutational relation algebras. Andréka *et. al.* [ADN92] go on to show that the permutational relation algebras are not finitely axiomatisable over the integral relation algebras. They do this by constructing an infinite set of integral non-permutational relation algebras but show that they have a permutational ultraproduct.

PROBLEM 4 *Use this method to show that the relation algebras with homogeneous representations are not finitely axiomatisable over the representable relation algebras.*

PROBLEM 5 *Prove that the relation algebras with homogeneous representations are not finitely axiomatisable over the permutational relation algebras.*

PROBLEM 6 *Show that the $n+1$-transitive relation algebras are not finitely axiomatisable over the n-transitive relation algebras. This might be shown by finding an infinite collection of n-transitive relation algebras none of which are $n+1$-transitive but with an $n+1$-transitive ultraproduct.*

Alternatively, at the other extreme, it might be the case that for some values of n, the n-transitive relation algebras are all $n+1$-transitive.

Can the n-transitive relation algebras be axiomatised at all?

7 ω-categorical relation algebras

Definitions

- A simple relation algebra $\mathcal{A}$ is *weakly ω-categorical* if it has a countably infinite, square representation and all such representations are base-isomorphic.
- $\mathcal{A}$ (simple) is *(strongly) ω-categorical* if it has exactly one square representation (up to base-isomorphism) of size ω or less.
- For non-simple $\mathcal{A}$, we say $\mathcal{A}$ is *weakly ω-categorical* if it has a countably infinite representation X such that X embeds in any infinite representation of $\mathcal{A}$. $\mathcal{A}$ is *strongly ω-categorical* if it has a representation X of size at most ω such that X embeds in any representation of $\mathcal{A}$.
- Let $\mathcal{A}$ be a simple relation algebra and $\mathcal{K}$ a TAP-class of atomic $\mathcal{A}$-networks. Let $N^+ \in \mathcal{K}$ and let $N \in \mathcal{K}$ be obtained from N^+ by the deletion of a single point n (alternatively N^+ is a *one-point extension* of N). The *Type* of n over N, written $Type(n, N)$, is defined to be the pair (N^+, n). We may also write this type as $Type(N^+, N)$ without any ambiguity. Intuitively, the type of a one point extension to N tells you exactly how the new point relates to all the old ones.

Equivalence Games Strongly ω-categorical, simple relation algebras can be characterised by properties of their networks. To do this, we will use games again. Let $\mathcal{A}$ be a finite, simple relation algebra and let τ be a TAP-class. Again there are two players, $\forall$ and $\exists$, who build a nested sequence of networks from τ of finite or countable length.

$$N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$$

Suppose N_i has been built ($i < \omega$). First, $\exists$ chooses $n, m \in N_i$ and atoms $a, b \in \mathcal{A}$ such that $N_i(n, m) \leq a; b$. Then, $\forall$ chooses from τ an extension N_{i+1} of N_i with at most one extra point so that there is some l in N_{i+1} with $N_{i+1}(n, l) = a$ and $N_{i+1}(l, m) = b$. $\forall$ can always do this since τ has the triangle addition property. This ends the round and the game continues from N_{i+1}.

Now let $N^+ \supseteq N$ belong to τ and suppose $|N^+ \setminus N| = 1$. The game $G(N^+, N, \mathcal{A})$ is played as above, starting with N. Let the play be $N = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$. $\exists$ wins the play if there is some $i < \omega$ and a point $n \in N_i$ such that $N_i \downarrow_{N_0 \cup \{n\}} \cong N^+$ and the isomorphism fixes N_0 pointwise. Otherwise, if there is no such i and n, $\forall$ wins the play.

THEOREM 19 *Let $\mathcal{A}$ be any finite, simple relation algebra. The following are equivalent.*

1. *$\mathcal{A}$ has a countable, homogeneous representation and $\mathcal{A}$ is strongly ω-categorical*
2. *$\mathcal{A}$ has a unique TAP-class τ and for all one point extensions $N \subset N^+ \in \tau$ $\exists$ has a winning strategy for $G(N^+, N, \mathcal{A})$.*

PROOF:

$2 \Rightarrow 1$.

Assume the hypothesis of 2. By theorem 5, $\mathcal{A}$ is representable. As it is simple, by the Löwenheim–Skolem theorem we see that it has a countable square representation. Let X, Y be countable, square representations of $\mathcal{A}$. We show that they are base-isomorphic. By assumption, X and Y have age τ. It suffices (by back and forth) to show that if $\theta : X \to Y$ is a finite partial isomorphism and $x \in X$, then θ extends to x. (The 'back' case from Y to X is similar).

Let $\overleftarrow{X}(dom(\theta)) = N$ and $\overleftarrow{X}(dom(\theta) \cup \{x\}) = N^+$. We can assume that $N^+ \supset N$. Starting with $\overleftarrow{Y}(Range(\theta))$, $\forall, \exists$ play a game on Y. In the ith round, if the play so far has been
$\overleftarrow{Y}(Range(\theta)) = M_0 \subset M_1 \subset \ldots \subset M_i \, (\subseteq Y)$ (where each M_{k+1} is a one point extension of M_k), $\exists$ chooses $n, m \in M_i$ and atoms $a, b \in \mathcal{A}$ such that $(n, m) \in Y(a; b)$. $\forall$ then chooses $y \in Y$ with $(n, y) \in Y(a)$ and $(y, m) \in Y(b)$. This is always possible for $\forall$ as Y is a representation of $\mathcal{A}$. Letting $M_{i+1} = \overleftarrow{Y}(M_i \cup \{y\})$ the game continues.

This gives a play of the game $G(N^+, N, \mathcal{A})$ so $\exists$ can use her winning strategy. If she does so, then for some $i < \omega$ we find a point $z \in M_i$ such that $N^+ \cong \overleftarrow{Y}(M_0 \cup \{z\})$ and the isomorphism extends θ. Thus θ extends to a partial map $\theta^+ : X \to Y$ where $\theta^+ = \theta \cup \{(x, z)\}$. The same argument can be used to show that the countable representation is homogeneous.

$1 \Rightarrow 2$.

Clearly if $\mathcal{A}$ is strongly ω-categorical, there is a unique TAP-class, τ say. Let $\exists$ play so that the outcome is a representation, Y, say. By homogeneity and ω-categoricity, Y is isomorphic to X by an isomorphism that fixes $N = N_0$ pointwise. But then $N^+ \subseteq Y$, so $N^+ \subseteq N_i$ for some i. Hence $\exists$ has a winning strategy as required. $\square$

PROBLEM 7 *Find a (finite, simple) relation algebra $\mathcal{A}$ with a unique TAP-class but which is not strongly ω-categorical. Such an algebra must have at least one inhomogeneous, countable, square representation.*

PROBLEM 8 *Find a recursive set of axioms that characterises the finite, strongly ω-categorical relation algebras.*

Infinite relation algebras Theorem 19 applies only to finite relation algebras. When extended to countably infinite relation algebras we get the following, weaker result.

THEOREM 20 *Let $\mathcal{A}$ be a simple, countably infinite relation algebra. The following are equivalent.*

1. *$\mathcal{A}$ has a countable, homogeneous, complete, square representation and all its countable, complete, square representations are base-isomorphic*

2. *$\mathcal{A}$ has a unique TAP-class τ and for all one point extensions $N \subset N^+ \in \tau$ $\exists$ has a winning strategy for $G(N^+, N, \mathcal{A})$.*

But what about incomplete representations? This theorem falls short of characterising the ω-categorical relation algebras.

Note An infinite relation algebra cannot be (strongly or weakly) ω-categorical. This is a corollary of the Ryll-Nardzewski–Engeler–Svenonius theorem on model-theoretic ω-categoricity ([CK90], theorem 2.3.13).

PROBLEM 9 *Can an infinite, atomic relation algebra be categorical in any uncountable cardinal (under the obvious definition)?*

PROBLEM 10 *Find an infinite axiomatisation, in the language of relation algebras, that characterises the subalgebras of group relation algebras. McKenzie [McK70] showed that the subalgebras of group relation algebras cannot be finitely axiomatised relative to the integral, representable ones. It may well be that the class of relation algebras whose networks form an amalgamation class is, similarly, not finitely axiomatisable relative to the class of integral, representable relation algebras (see problem 4). Question: can an integral relation algebra whose networks form an amalgamation class be extended to a group relation algebra? Does the converse hold?*

It is fairly easy to use these results to show that some of the best known relation algebras are ω-categorical; see [Hir94] for more details. The point algebra and the Allen interval algebra can both be shown to be ω-categorical (first proved in [VK86] and [vBC90]). The metric linear order of [DMP91] has only one countable, square representation: the rationals.

8 Ultrafilter extensions

Up to now, we have mainly dealt with complete representations. The corresponding networks have been labelled by atoms. When we generalise the work, below, it will be necessary to use ultrafilters instead of atoms as the labels on the edges of the networks. In later sections we will handle ultrafilters by convergent sequences of elements of a relation algebra. In this section, we introduce some of the relevant notation and prove a few basic results concerning ultrafilters in relation algebras.

Definitions Let $\mathcal{A}$ be any relation algebra.

- An *$\mathcal{A}$-filter* (or just filter) is a non-empty set of elements of $\mathcal{A}$, closed upwards under the boolean ordering '$\geq$', and closed under finite intersections ('$\wedge$'). The *principal* filter generated by $a \in \mathcal{A}$ is $F(a) =_{\text{def}} \{b \in \mathcal{A} : a \leq b\}$. A *proper* filter is a filter not containing 0.
- An *ultrafilter* z is a filter with the property that for each $a \in \mathcal{A}$ exactly one of $a, -a$ belongs to z. Let $Uf(\mathcal{A})$ be the set of all ultrafilters over $\mathcal{A}$.
- The *product* and *converse* of ultrafilters w, z over $\mathcal{A}$ are defined by

$$\begin{aligned} z^{\smile} &= \{a^{\smile} : a \in z\} \in Uf(\mathcal{A}) \\ w;z &= \{x \in Uf(\mathcal{A}) : \forall a \in w, b \in z(a;b \in x)\} \subseteq Uf(\mathcal{A}) \end{aligned}$$

- If X is a representation of $\mathcal{A}$, $x, y \in X$ and F is an ultrafilter over $\mathcal{A}$, we write $X,(x,y) \models F$ if $\forall f \in F\ (X,(x,y) \models f)$. If S is a set of ultrafilters over $\mathcal{A}$, we write $X,(x,y) \models S$ for $\exists F \in S\ (X,(x,y) \models F)$.
- The *ultrafilter extension* $\mathcal{A}^+$ of $\mathcal{A}$ is the atomic relation algebra consisting of all sets of ultrafilters over $\mathcal{A}$, where the operations $\vee, -$ are interpreted as $\cup, \setminus$. Converse and composition are defined above and extend, by an infinite distribution rule, to arbitrary sets of ultrafilters. We identify a singleton $\{F\} \subseteq \mathcal{A}^+$ with the ultrafilter F. Under this identification, $\mathcal{A}^+$ forms a complete, atomic extension of $\mathcal{A}$; but note that if $\mathcal{A}$ is infinite, the natural embedding of $\mathcal{A}$ in $\mathcal{A}^+$ given by $a \mapsto \{z \in Uf(\mathcal{A}) : a \in z\}$ is not a complete embedding. (For example, if f is a non-principal ultrafilter of $\mathcal{A}$ then the infimum of f in $\mathcal{A}$ is 0, whilst the imfimum of its image in $\mathcal{A}^+$ is $\{f\} \neq \varnothing$. So the embedding does not preserve infima.)
- If $a \in \mathcal{A}$ let

$$< a >= \{z \in Uf(\mathcal{A}) : a \in z\}.$$

  Then $\{< a >: a \in \mathcal{A}\}$ is a base of clopen sets for the *Stone space topology* on $Uf(\mathcal{A})$. This topology is compact, Hausdorff and completely metrizable [CK90] (theorems 3.3.5 and 3.3.6).

- A *U-network* N (over $\mathcal{A}$) is a complete directed graph, possibly empty, with each edge labelled by an ultrafilter such that for all nodes n, m, l of N,
  - $Id \in N(n,n)$ and
  - $N(l,n) \in N(l,m); N(m,n)$,

  where $N(n,m)$ is the ultrafilter labelling the edge (n,m). Note that it follows that $N(m,n) = N(n,m)^{\smile}$, for all nodes n, m.

Definition Let N be a U-network over $\mathcal{A}$. Define an equivalence relation $\sim$ on the nodes of N by $n \sim m \Leftrightarrow Id \in N(m,n)$. Note that $\sim$ is a congruence: $m \sim m'$, $n \sim n' \Rightarrow N(m,n) = N(m',n')$. Let the equivalence class of the node $n \in N$ be denoted $[n]$. Let $N^* : \mathcal{A} \to \mathcal{P}(N/{\sim})$ be defined by

$$N^*(a) = \{([m],[n]) \in (N/\sim) \times (N/\sim) : a \in N(m,n)\}$$

Observe that for any pair of nodes $[m],[n] \in N/\sim$ we have $([m],[n]) \in N^*(Id)$ if and only if $[m] = [n]$.

LEMMA 21 *cf. Lemma 4.*

1. *Let N be a non-empty U-network over the relation algebra $\mathcal{A}$. The mapping N^* is a relation algebra homomorphism if and only if for every pair of nodes $m, n \in N$ for every pair $b, c \in \mathcal{A}$ with $b;c \in N(m,n)$ there is some node $l \in N$ with $b \in N(m,l)$ and $c \in N(l,n)$.*
2. *Let $\mathcal{A}$ be a simple relation algebra and let N be a non-empty U-network. The mapping N^* is a representation of $\mathcal{A}$ if and only if for every pair of nodes $m, n \in N$ for every pair $b, c \in \mathcal{A}$ with $b;c \in N(m,n)$ there is some node $l \in N$ with $b \in N(m,l)$ and $c \in N(l,n)$.*

PROOF:

1. Suppose we have some U-network N that satisfies the condition. In order to show that N^* is a homomorphism it is necessary to show that it respects all the operations of the relation algebra. Since there are ultrafilters on edges it follows that complement and finite disjunctions are preserved, though arbitrary suprema may not be. N^* has been constructed so that the identity relation is preserved. The preservation of converse follows easily from the definition of a U-network. We show that composition is preserved. Let $m, n \in N$ and $b, c \in \mathcal{A}$. Then:

$$\begin{aligned}
&([m],[n]) \in N^*(b;c) \\
&\Leftrightarrow \quad b;c \in N(m,n) \\
&\Leftrightarrow \quad \exists l \in N[b \in N(m,l) \wedge c \in N(l,n)] \\
&\qquad (\Leftarrow \text{ by transitive closure of } N, \Rightarrow \text{ by hypothesis of lemma}) \\
&\Leftrightarrow \quad \exists l \in N[([m],[l]) \in N^*(b) \wedge ([l],[n]) \in N^*(c)].
\end{aligned}$$

   It follows that $N^*(b;c) = N^*(b);N^*(c)$ as required.
   Conversely if N^* is a homomorphism then the edge condition for N follows from the preservation of composition.
2. If $\mathcal{A}$ is simple then any non-zero homomorphism can only be an isomorphism. Thus N^* is a homomorphism if and only if it is a representation.

$\square$

The following theorem is due to J. D. Monk. Here we give an alternative, simpler proof than the original one (reported in [McK66], theorem 2.12).

THEOREM 22 *(J. D. Monk) A relation algebra $\mathcal{A}$ is representable if and only if its ultrafilter extension $\mathcal{A}^+$ has a complete representation.*[14]

PROOF:

$\Leftarrow$ Since $\mathcal{A}$ can be identified with a subalgebra of its ultrafilter extension, any representation of the extension automatically induces a representation of $\mathcal{A}$.

$\Rightarrow$ Recall from section 3.3 that $T(\mathcal{A})$ is a theory of the first-order language $L(\mathcal{A})$ that exactly characterises the representations of $\mathcal{A}$. For each ultrafilter U of $\mathcal{A}$ add two new constant symbols c_U, d_U to the language. Consider the first-order theory

$$\Sigma = T(\mathcal{A}) \cup \{r(c_U, d_U) : U \in Uf(\mathcal{A}), r \in U\}.$$

We use compactness to show that Σ is consistent. For any ultrafilter U, a finite subset of $\{r(c_U, d_U) : r \in U\}$ is equivalent, modulo $T(\mathcal{A})$, to the single formula $R(c_U, d_U)$ where R is the conjunction of the rs appearing in the finite set. So it is enough to show that any set of formulas of the form

$$T(\mathcal{A}) \cup \{R_1(c_1, d_1), R_2(c_2, d_2) \ldots R_n(c_n, d_n)\}$$

is consistent. Let X be any representation of $\mathcal{A}$. Then this set of formulas holds in the model X, given appropriate interpretations of the constant symbols. So Σ is consistent and has a model, Y, say. Y can be treated as an $L(\mathcal{A}^+)$-structure as follows. Let S be any set of ultrafilters. Let $Y(S) = \{(x, y) \in Y : Y \models S\}$. Every non-zero element S of $\mathcal{A}^+$ is interpreted as a non-empty binary relation in this structure — as $Y \models \Sigma$, for every ultrafilter U there is a pair of points in Y realising U.

To be a representation of the ultrafilter extension we need more: Y is a representation of $\mathcal{A}^+$ if for all ultrafilters V, W of $\mathcal{A}$ and for all $y_1, y_2 \in Y$,

$$(y_1, y_2) \models (V; W) \Rightarrow (\exists z \in Y)\ (y_1, z) \models V \wedge (z, y_2) \models W. \tag{1}$$

(Note that the ultrafilters of $\mathcal{A}$ are the atoms of $\mathcal{A}^+$.)

Our strategy is to build a representation Y_ω embedding Y and satisfying (1) step by step. The idea is that we first embed the representation Y in a larger representation Y_1 such that for all (V, W, y_1, y_2) where property (1) fails in Y there is a new point $x \in Y_1$ making that property true. This extension Y_1 may contain new pairs of points where the property fails, but these get repaired in the next step. After ω iterations we take the limit Y_ω and then show that it is a representation of $\mathcal{A}^+$.

Now, in more detail, let $L(Y)$ be the first-order language $L(\mathcal{A})$ augmented with a new constant symbol c_y for each element $y \in Y$ and let $\triangle(Y)$ be the *diagram* of Y consisting of all quantifier-free $L(Y)$-sentences that hold in Y. Clearly Y is a model of $\triangle(Y)$ and Y embeds in any model of $\triangle(Y)$.

For each tuple (V, W, c_1, c_2) such that (1) fails in Y on the points named by c_1, c_2 add a new constant c to $L(Y)$ and define a new theory Γ to include $\Sigma \cup \triangle(Y)$ and all sentences $v(c_1, c)$ and $w(c, c_2)$ for all $v \in V, w \in W$. This should be done for all tuples (V, W, c_1, c_2). As before, we can show that for any finite subset of Γ, Y is already a model; hence Γ is consistent. Let Y_1 be a model of Γ. Since $Y_1 \models \Sigma$ it follows that Y_1 is a representation of $\mathcal{A}$; and $Y_1 \models \triangle(Y)$ implies that Y embeds in Y_1.

[14] Another way of proving the theorem is to take an ω-saturated model M of the $L(\mathcal{A})$-theory $T(\mathcal{A})$ of section 3.3 [CK90]. Then check that M, regarded as an $L(\mathcal{A}^+)$-structure in the natural way, is a complete representation of $\mathcal{A}^+$.

By iterating this procedure we get a chain of representations

$$Y = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$$

(Since Y_n embeds isomorphically in Y_{n+1} we can assume without loss of generality that Y_{n+1} is actually an extension of Y_n.) Notice that Σ is an $\forall\exists$ theory and each Y_i is a model of Σ. Therefore the limit of the chain $Y_\omega = \bigcup_{i<\omega} Y_i$ is also a model of Σ and hence a representation of $\mathcal{A}$. Also, for all possible tuples (V, W, y_1, y_2) $(y_1, y_2 \in Y_\omega)$ equation (1) is satisfied in Y_ω so Y_ω is a representation of $\mathcal{A}^+$.
Now we show that the representation Y_ω of $\mathcal{A}^+$ is a complete representation. This holds because Y_ω is a representation of $\mathcal{A}$, so any pair of points (y_1, y_2) from $Y_\omega(1)$ are related by an ultrafilter of $\mathcal{A}$, that is, an atom of $\mathcal{A}^+$. By theorem 1 Y_ω must be a complete representation of $\mathcal{A}^+$.

□

9 Axiomatising RRA

In this section we find axioms for **RRA**. The standard reference work, Cylindric Algebras, by Henkin, Monk & Tarski (vol. 1, page 461) identifies one of the two outstanding problems of the representation theory as "…the problem of providing a simple intrinsic characterization for all representable cylindric algebras …" (A characterisation does exist — for example, see page 112 of vol. 2 — but the axioms are certainly not simple.) Our methods here are very simple, and are easily modified (in section 12) to handle cylindric algebras. Thus, we may have solved this problem.

In theorem 9 we derived the Lyndon conditions which characterise the finite, representable relation algebras. These conditions can be falsified on certain infinite representable relation algebras.[15] In the following, we derive conditions that characterise *all* relation algebras with arbitrary, not necessarily complete, representations (**RRA**). The corresponding set of equations in [Lyn56] is rather involved and the equations themselves are very hard to read. However, the meaning of the formulas derived here is clear, because of their correspondence to the moves in a two player game. They can also be transformed into equations quite easily.

Recall that given a relation algebra $\mathcal{A}$, a (not necessarily atomic) $\mathcal{A}$-network N is a map : $N^2 \to \mathcal{A}$ such that (writing the map as N, as usual) we have $N(i,i) \leq Id$ and $N(i,j);N(j,k) \wedge N(i,k) \neq 0$ for all $i, j, k \in N$.

9.1 The game G_n

We define a game, $G_n(N, \mathcal{A})$ (where N is a finite $\mathcal{A}$-network), of length $n \leq \omega$, between players $\forall$ and $\exists$, to build a chain $N = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$ of finite networks if n is finite, and an infinite chain $N = N_0 \subseteq N_1 \subseteq \cdots$ if $n = \omega$. Here, $N \subseteq N'$ means that every node of N is a node of N' and for all $l, m \in N$, $N(l,m) \geq N'(l,m)$. In the ith round $(0 \leq i < n)$, if N_i has been constructed,

- $\forall$ chooses $l, m \in N_i$, and elements $r, s \in \mathcal{A}$;
- $\exists$ responds, if possible, with a finite network $N_{i+1} \supseteq N_i$ (with exactly one more node $d \notin N_i$), such that one of the following holds:

[15] For example the relation algebra M' in [Lyn50] falsifies the Lyndon conditions but, contrary to the conclusion of that paper, it *does* have a representation, though not a complete one. See [Lyn56], and problem 10 of the current paper.

(reject) $N_{i+1}\lceil_{dom(N_i)}$ is the same as N_i, except that
$N_{i+1}(l,m) = N_i(l,m) - (r;s)$,
$N_{i+1}(d,d) = Id$,
and for any other edge involving the new node d,
$N_{i+1}(k,d) = N_{i+1}(d,k) = 1$ (any $k \in N_i$)

(accept) $N_{i+1}(l,d) = r$ and $N_{i+1}(d,m) = s$
$N_{i+1}(d,d) = Id$
$N_{i+1}\lceil_{dom(N_i)}$ is the same as N_i, except that
$N_{i+1}(l,m) = N_i(l,m) \wedge (r;s)$,
and all other node pairs $(k,d), (d,k)$ in N_{i+1} are labelled by 1.

- Let us define the two directed, labelled graphs resulting from these two responses as $Out(N_i,l,m,r,s)$ and $In(N_i,l,m,r,s)$ respectively. Each of these two graphs may fail to be a network (it may have an edge labelled with 0) and, as we will see, this would correspond to a losing move by $\exists$.

There is a special case in the 0th round, if $N = \varnothing$. In $G_n(\varnothing,\mathcal{A})$ there is an additional, *initial round,* so that a play of this game is a sequence $\varnothing \subset N_0 \subseteq \ldots \subseteq N_n$. In the initial round, $\forall$ chooses any element $a \neq 0$ and $\exists$ responds with the 2-point network $N_0(a)$ with set of nodes $\{0,1\}$ and with $N_0(a)(0,1) = a, N_0(a)(1,0) = a^{\smile}, N_0(a)(0,0) = N_0(a)(1,1) = Id$. This is the only round where $\exists$ has no choice in her response. The game then goes on for another n rounds as in $G_n(N_0(a),\mathcal{A})$.

$\exists$ wins if she never gets stuck.

So in each round, apart from the special case just mentioned, $\forall$ challenges $\exists$ to add a certain triangle to an edge, and $\exists$ must either do so, or claim instead that the label on the edge is disjoint from the label on the matching side of the triangle.

Now we can characterise the countable representable relation algebras. The assumption of countability will be removed later.

PROPOSITION 23 *Let $\mathcal{A}$ be any countable relation algebra. Then $\mathcal{A} \in$ **RRA** if and only if $\exists$ has a winning strategy in $G_n(\varnothing,\mathcal{A})$ for all $n < \omega$.*

PROOF:

$\Rightarrow$: If X is a representation of $\mathcal{A}$, $\exists$ can use X to help her decide whether to accept or reject in each round. In more detail, she preserves the condition that for each i there's a map $' : N_i \to X$ with

$$X, (l', m') \models N_i(l,m) \text{ for all nodes } l, m \text{ of } N_i.$$

Clearly, if there is such a map then N_i is a network, so it is enough to show that $\exists$ can keep this condition in each round.

In the initial round, if $\forall$ plays $a \neq 0$, she defines $0', 1'$ to be any points of X such that $X \models a(0',1')$. Such points exist because X is a representation of $\mathcal{A}$. Let $\forall$ play l,m,r,s in round i:

1. If $X, (l',m') \models r;s$, she accepts, and extends the map $'$ by mapping the d of the game definition above to any $d' \in X$ with $X, (l',d') \models r$ and $X, (d',m') \models s$. Again, such a point exists because X is a representation.
2. If $X, (l',m') \not\models r;s$, she rejects and lets $d' \in X$ be arbitrary within the square component of X containing N_i', say $d' = 0'$ for example. Since $X, (l',m') \models 1$ and X is a representation, we have $X, (l',m') \models -(r;s)$, so that the condition on the map "$'$" is kept.

Thus the condition on the map "$'$" is kept in either case, and $\exists$ can continue into the next round.

$\Leftarrow$: Suppose that $\exists$ has a winning strategy in $G_n(\varnothing, \mathcal{A})$ for infinitely many $n < \omega$. We claim that she can also win $G_\omega(\varnothing, \mathcal{A})$. This is because we can show, inductively, that in any round, k, she can apply her winning strategies in the games $G_n(N_k, \mathcal{A})$, for infinitely many $n \geq 1$, to the current position N_k. These strategies say whether to accept or reject $\forall$'s triangle. If infinitely many of them tell her to accept, then she accepts. If not, then infinitely many will advise rejection, and she rejects. In either case, she arrives at the next round in a position N_{k+1} where infinitely many winning strategies are still running. So she can continue, and win $G_\omega(\varnothing, \mathcal{A})$.

Now assume that $\exists$ has a winning strategy in $G_n(\varnothing, \mathcal{A})$ for all $n < \omega$. By the above, she also has a winning strategy in $G_\omega(\varnothing, \mathcal{A})$. To show that $\mathcal{A}$ is representable, consider a play of $G_\omega(\varnothing, \mathcal{A})$ in which $\exists$ uses her winning strategy, but also persuades $\forall$ to play at some stage l, m, r, s, for each pair l, m of nodes that arise during the game, and for every $r, s \in \mathcal{A}$. This is possible, as both $\mathcal{A}$ and the networks N_i in the game are countable. Let

$$\varnothing \subseteq N_0 \subseteq N_1 \subseteq \ldots$$

be the resulting play, and define the $\mathcal{A}$-structure M to be the *limit* of the play. So, the domain of M is the union of the domains of the N_i $(i < \omega)$, and for any two points $m, n \in M$, we let

$$M, (m,n) \models a \Leftrightarrow \text{ for some } i < \omega, m, n \in N_i \text{ and } N_i(m,n) \leq a.$$

For each pair $m, n \in M$ and every $a \in \mathcal{A}$, we know that m, n, Id, a is played by $\forall$ at some stage. It follows that either $M, (m,n) \models a$ (if $\exists$ accepts) or $M, (m,n) \models -a$ (if she rejects), and so M defines a labelled U-graph. It is easily seen to be a U-network.

Claim. Assume that $b, c \in \mathcal{A}$ and $M, (x,y) \models b;c$. Then there is $z \in M$ with $M, (x,z) \models b$ and $M, (z,y) \models c$.

Proof of Claim. Assume that $M, (x,y) \models b;c$, so that $N_i(x,y) \leq b;c$ for some $i < \omega$. Suppose that $\forall$ played x, y, b, c in round j. If $\exists$ rejected, then $N_j(x,y) \leq \neg(b;c)$, so $N_k(x,y) \leq N_i(x,y) \wedge N_j(x,y) \leq b;c \wedge \neg(b;c) = 0$ for any $k \geq i, j$. This contradicts the fact that N_k is a network. So she must have accepted. This means that there is $z \in N_j$ with $N_j(x,z) = b$ and $N_j(z,y) = c$. Hence $M, (x,z) \models b$ and $M, (z,y) \models c$, as claimed.

Defining $m \sim n$ if and only if $M, (m,n) \models Id$, we deduce from lemma 21 that the mapping $M^* : \mathcal{A} \to \mathcal{P}(M/{\sim})$ is a relation algebra homomorphism of $\mathcal{A}$.

Yet for non-simple $\mathcal{A}$, M^* may not be one–one: in other words, there may be an element $a \in \mathcal{A}$ with $M^*(a) = \varnothing$. However, given any non-zero $a \in \mathcal{A}$, $\forall$ can use the initial round to force that a labels an edge of some N_i (in fact N_0) and therefore if the $L(\mathcal{A})$-structure obtained in the limit is called M_a^* we have $(0,1) \in M_a^*(a)$ so $M_a^*(a) \neq \varnothing$. For a representation of $\mathcal{A}$, we may assume that the domains of the structures $M_a^* : a \in \mathcal{A}$ are pairwise disjoint. Then let M^* be defined by

$$M^*(b) = \bigcup_{a \in \mathcal{A} \setminus \{0\}} M_a^*(b)$$

for any $b \in \mathcal{A}$. M^* is still a homomorphism and has now been constructed to be one–one. Thus, it is a representation of $\mathcal{A}$. $\Box$

9.2 $\mathcal{A}$-graphs and L_r-graphs

Recall that L_r is the first-order language with similarity type $(1, 0, -, \vee, Id, ^\smile, ;)$. In order to use proposition 23 to obtain an axiomatisation of **RRA** in the language L_r, it will be helpful to define an L_r-graph, which is similar to an $\mathcal{A}$-graph but with edges labelled by *terms* of L_r.

Definitions

- An L_r*-graph* N is a set of nodes and a binary labelling function (both denoted by the same letter N) such that for any pair $m, n \in N$, $N(m, n)$ is a term of L_r. For example, we could have $N(m, n) = x; y^{\smile} \wedge (Id - x)$ where x and y are variables, and $\wedge, -$ are the usual abbreviations.

- If N is an L_r-graph and h is an assignment of the variables into some relation algebra $\mathcal{A}$, then $h(N)$ is the $\mathcal{A}$-labelled graph obtained from N by replacing each variable x by $h(x)$.

- Let $m \geq 0$, let N be an L_r-graph with nodes $0, 1, \ldots, m-1$, let $i, j < m$ and let x, y be variables. Analagously with the definitions of In and Out for $\mathcal{A}$-graphs, we can define two L_r-graphs $In = In(N, i, j, x, y)$ and $Out = Out(N, i, j, x, y)$ (we'll omit all the arguments to In and Out in the following for brevity) with nodes $0, \ldots, m-1, m$ as follows.

$$\begin{aligned} In(i, m) &= x \\ In(m, j) &= y \\ In(i, j) &= N(i, j) \wedge x; y, \end{aligned}$$

  and for any other pairs of nodes $i', j' < m$ not covered by the above,

$$\begin{aligned} In(i', j') &= N(i', j') \\ In(i', m) &= 1 \\ In(m, i') &= 1 \\ In(m, m) &= Id. \end{aligned}$$

  Similarly,

$$\begin{aligned} Out(i, j) &= N(i, j) - x; y \\ Out(i', j') &= N(i', j') \ \text{ if } (i', j') \in N^2 \setminus \{(i, j)\} \\ Out(i', m) &= 1 \\ Out(m, i') &= 1 \\ Out(m, m) &= Id. \end{aligned}$$

Returning to the axiomatisation in L_r of the countable, representable relation algebras, by proposition 23, it remains to find an L_r-sentence ϕ_n which holds in a relation algebra if and only if $\exists$ has a winning strategy for $G_n(\varnothing, \mathcal{A})$. First we define formulas $\psi_n^m(x_{ij} : i, j < m)$ for $m > 0$, with free variables x_{ij} $(i, j < m)$. The intention here is that if N is an L_r-graph with nodes $0, \ldots, m-1$ and h an assignment to the variables of the terms labelling the edges of N, then $\exists$ has a winning strategy in the game $G_n(h(N), \mathcal{A})$ if and only if $\mathcal{A}, h \models \psi_n^m(N(i, j) : i, j < m)$. We define these formulas by recursion on n.

Definition Let $m > 0$. Define the formula $Net^m(x_{ij} : i, j < m)$ (cf. the definition of $ANet$ in lemma 8) to be

$$\bigwedge_{i<m} (x_{ii} \leq Id) \wedge \bigwedge_{i,j,k<m} (x_{ij}; x_{jk} \wedge x_{ik} \neq 0).$$

$Net^m(x_{ij} : i, j < m)$ is a quantifier-free L_r-formula. For the base case, $n = 0$, we know that $\exists$ has a winning strategy in $G_0(h(N), \mathcal{A})$ if and only if $h(N)$ forms an $\mathcal{A}$-network. So we'll let $\psi_0^m(N(i, j) : i, j < m) = Net^m(N(i, j) : i, j < m)$.

The idea for the inductive step is, roughly, that $\exists$ has a winning strategy for $G_{n+1}(N, \mathcal{A})$ if and only if for all possible $\forall$-moves (i, j, x, y), $\exists$ has a winning strategy in either $G_n(In(N, i, j, x, y), \mathcal{A})$ or $G_n(Out(N, i, j, x, y), \mathcal{A})$. So we define $\psi_{n+1}^m(N(i, j) : i, j < m)$ to be the following formula:

$$\begin{array}{l}\forall x, y \bigwedge_{i,j<m} [\psi_n^{m+1}(In(N,i,j,x,y)(i',j') : i',j' < m+1) \\ \qquad \vee\ \psi_n^{m+1}(Out(N,i,j,x,y)(i',j') : i',j' < m+1)],\end{array}$$

where x, y are new variables.

THEOREM 24 *Let $\mathcal{A}$ be any relation algebra, and N any L_r-graph.*

1. *For any assignment h of the variables occurring in the terms of N to elements of $\mathcal{A}$, we have*

$$\mathcal{A}, h \models \psi_n^m(N(i,j) : i,j < m)$$

if and only if $\exists$ has a winning strategy for the game $G_n(h(N), \mathcal{A})$.

2. *Let*

$$\phi_n = (\forall x_0 \neq 0)\ \psi_n^2(N_0(x_0)(i,j) : i,j < 2),$$

where $N_0(x_0)$ is the L_r-graph with nodes $0,1$ and $N_0(x_0)(0,0) = N_0(x_0)(1,1) = Id, N_0(x_0)(0,1) = x_0, N_0(x_0)(1,0) = x_0^{\smile}$. Then $\mathcal{A} \models \phi_n$ if and only if $\exists$ has a winning strategy in $G_n(\varnothing, \mathcal{A})$.

PROOF:

1. The proof goes by induction on n. The case $n = 0$ is straightforward ($\exists$ wins a game of length 0 if and only if the start position forms a network). Now consider ψ_{n+1}^m. We have

$$\mathcal{A}, h \models \psi_{n+1}^m(N(i,j) : i,j < m)$$

if and only if

$$\begin{array}{rl}\mathcal{A}, h \models \forall x, y \bigwedge_{i,j<m} & [\psi_n^{m+1}(In(N,i,j,x,y)(i',j') : i',j' < m+1) \\ \vee & \psi_n^{m+1}(Out(N,i,j,x,y)(i',j') : i',j' < m+1))].\end{array}$$

Letting h^* be an arbitrary assignment that agrees with h except perhaps on x and y, we see that this is equivalent to saying, for any $\forall$-move $(i, j, h^*(x), h^*(y))$, that

$$\begin{array}{rl}\mathcal{A}, h^* \models & \psi_n^{m+1}(In(N,i,j,x,y)(i',j') : i',j' < m+1) \\ \vee & \psi_n^{m+1}(Out(N,i,j,x,y)(i',j') : i',j' < m+1))\end{array}$$

which, by the induction hypothesis, holds if and only if, for all such $\forall$-moves, $\exists$ has a winning strategy in $G_n(h^*(In), \mathcal{A})$ or $G_n(h^*(Out), \mathcal{A})$. This is equivalent to the existence of a winning strategy for $\exists$ in the game $G_{n+1}(h(N), \mathcal{A})$, as required.

2. $\mathcal{A} \models \phi_n$ if and only if, for all assignments h with $h(x_0) \neq 0$,

$$\mathcal{A}, h \models \psi_n^2(N_0(x_0)(i,j) : i,j < 2).$$

By the previous part this holds if and only if $\exists$ has a winning strategy in all games $G_n(h(N_0(x_0)), \mathcal{A})$ which is true if and only if for any initial $\forall$-move $h(x_0)$, $\exists$ has a winning strategy for the remainder of the game $G_n(\varnothing, \mathcal{A})$, or equivalently she had a winning strategy to start with.

$\square$

Hence a countable relation algebra is representable if and only if it satisfies ϕ_n, for all $n < \omega$.

The two subformulas ψ_n^{m+1} in the definition of ψ_{n+1}^m both sit below a disjunction and contain universal quantifiers. Inductively, all the universal quantifiers occur positively and therefore ϕ_n can be put in prenex normal form as a universal formula (i.e. only universal quantification and all quantifiers at the front of the formula). We'll show how this is done in more detail next.

9.3 Explicit universal axioms

Consider a play $\varnothing \subset N_0 \subset \ldots \subset N_n$ of the game $G_n(\varnothing, \mathcal{A})$. N_0 has nodes $\{0,1\}$ and each subsequent round an extra node gets added, so, without loss of generality, we assume that the node $i+2$ gets added in the ith round $(0 \le i < n)$, so the nodes of N_i are $\{0,1,\ldots,i+1\}$. At the end of a play there will be a graph N_n with nodes $\{0,\ldots,n+1\}$. In the initial round, if $\forall$ plays $a \neq 0$ then $\exists$ has to let $N_0 = N_0(a)$, the network with nodes $0,1$ with $N_0(0,1) = a$, defined in theorem 24. Each round after the first, $\exists$ may respond in either of the two ways defined in the rules of the game. Let $\bar{q} = (q_0, \ldots, q_{n-1}) \in \{In, Out\}^n$ be an n-tuple that defines one way for her to play the game: so in the ith round $(0 \le i < n)$, if the last network constructed was N_i and $\forall$ plays $a, b \in \mathcal{A}$ and the edge $(j,k) \in N_i^2$, then $\exists$ responds with $N_{i+1} = In(N_i, j, k, a, b)$ if $q_i = In$ and with $N_{i+1} = Out(N_i, j, k, a, b)$ if $q_i = Out$. Let $\forall$'s initial move be $a \neq 0$ and let the n following moves of $\forall$ be defined by $(a_i, b_i, j_i, k_i : 0 \le i < n)$: so in the ith round $(0 \le i < n)$ he plays (j_i, k_i, a_i, b_i) where $a_i, b_i \in \mathcal{A}$ and $j_i, k_i \in \{0, \ldots, i+1\}$. Let $\Gamma(a, \bar{q}, (j_i, k_i, a_i, b_i : 0 \le i < n))$ be the graph resulting at the end of the play and let (j,k) be any edge of the graph $(0 \le j, k \le n+1)$. Unfolding the definitions of In and Out over the n rounds of the play, we get:

$$
\begin{aligned}
&\Gamma(a, \bar{q}, (j_i, k_i, a_i, b_i : 0 \le i < n))(j,k) = \\
&\qquad \bigwedge\{-(a_i ; b_i) : 0 \le i < n, (j,k) = (j_i, k_i), q_i = Out\} \wedge \\
&\qquad \bigwedge\{-(a_i ; b_i)^{\smile} : 0 \le i < n, (j,k) = (k_i, j_i), q_i = Out\} \wedge \bigwedge_{0 \le l < n} t_l \wedge s
\end{aligned}
$$

where

$$
t_l = \begin{cases} a_l & \text{if } q_l = In, (j,k) = (j_l, l+2) \\ b_l & \text{if } q_l = In, (j,k) = (l+2, k_l) \\ a_l ; b_l & \text{if } q_l = In, (j,k) = (j_l, k_l) \\ 1 & \text{otherwise} \end{cases}
$$

and

$$
s = \begin{cases} Id & \text{if } j = k \\ a & \text{if } (j,k) = (0,1) \\ a^{\smile} & \text{if } (j,k) = (1,0) \\ 1 & \text{otherwise} \end{cases}
$$

The sentence ϕ_n corresponds to a winning strategy for $\exists$ in the game $G_n(\varnothing, \mathcal{A})$. In order to express ϕ_n as a universal formula more explicitly, let us consider what these winning strategies are. First, since G_n is of finite length it is a *determined game,* meaning that if $\exists$ does not have a winning strategy then $\forall$ does [GS53]. So what is a strategy for $\forall$? It is a set of instructions telling him how to play each round of the game and the instructions for the ith round of the game $(0 \le i < n)$ must depend only on *player $\exists$'s previous moves,* i.e., on a sequence $\bar{p} = (p_0, \ldots, p_{i-1}) \in \{In, Out\}^i$. Let us write $\{In, Out\}^{<n}$ for the set of all sequences from $\{In, Out\}$ of length less than n. If $\bar{p}$ is such a sequence, it defines one way for $\exists$ to play up to the ith round. We call $\bar{p}$ an initial $\exists$-play.

A strategy σ for $\forall$ consists of an initial move $a \neq 0$, and then, for any initial $\exists$-play $\bar{p} \in \{In, Out\}^i$, a response for $\forall$. Let

$$
\sigma(\bar{p}) = (j(\bar{p}), k(\bar{p}), a^{\bar{p}}, b^{\bar{p}}) \qquad (*)
$$

where $a^{\bar{p}}, b^{\bar{p}} \in \mathcal{A}$ and j and k are functions : $\{In, Out\}^{<n} \to \{0, \ldots, , n+1\}$ telling $\forall$ which pair of points to choose given an initial $\exists$-play $\bar{p}$. In order that j and k give a legal strategy it is necessary that $j(\bar{p}), k(\bar{p}) \in N_{|\bar{p}|} = \{0, \ldots, |\bar{p}|+1\}$. To tidy up the notation we'll let F be the set of all legal functions of this type:

$$
F = \{f : \{In, Out\}^{<n} \to \mathbb{N} \quad | \bigwedge_{\bar{p} \in \{In, Out\}^{<n}} f(\bar{p}) \le |\bar{p}| + 1\}.
$$

Let $\bar{q} = (q_0, \ldots, q_{n-1}) \in \{In, Out\}^n$, and for each $i < n$ let us write $\bar{q}\lceil_i$ for the initial $\exists$-play $(q_0, \ldots, q_{i-1})$. If $\exists$ plays according to $\bar{q}$ and $\forall$ uses the strategy σ, then the resulting graph is $\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))$, in the notation of $(*)$.

A strategy σ is a *winning $\forall$-strategy* if in any play of the game $G_n(\varnothing, \mathcal{A})$, no matter what sequence of moves $\bar{q} \in \{In, Out\}^n$ $\exists$ makes, the resulting $\mathcal{A}$-graph is not a network. So, the statement '$\forall$ has a winning strategy' can be written

$$\exists a \neq 0, \exists_{(\bar{p} \in \{In, Out\}^{<n})} a^{\bar{p}}, b^{\bar{p}} \bigvee_{j,k \in F} \bigwedge_{\bar{q} \in \{In, Out\}^n} \neg Net(\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n)).$$

It is worth noting that the definitions of $N_0(a)$, $In(N, j, k, a, b)$, $Out(N, j, k, a, b)$ and $\Gamma(a, \bar{q}, (j_i, k_i, a_i, b_i : 0 \leq i < n))$ guarantee that the first condition of Net (viz. $\Gamma(u, u) \leq Id$) is automatically satisfied. Also, by the triangle axiom, if Γ is an $\mathcal{A}$-graph and $u, v, w \in \Gamma$ then $\Gamma(u, v); \Gamma(v, w) \wedge \Gamma(u, w) \neq 0 \Rightarrow \Gamma(v, w); \Gamma(w, u) \wedge \Gamma(v, u) \neq 0$ etc. Thus, $\forall$ has a winning strategy if and only if $\mathcal{A}$ satisfies

$$\exists a \neq 0, \exists_{(\bar{p} \in \{In, Out\}^{<n})}\ a^{\bar{p}}, b^{\bar{p}} \bigvee_{j,k \in F} \quad \bigwedge_{\bar{q} \in \{In, Out\}^n} \quad \bigvee_{0 \leq u < v < w \leq n+1}$$

$$\begin{aligned} &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(u, v) \ ; \\ &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(v, w) \wedge \\ &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(u, w) = 0. \end{aligned}$$

Taking the negation:

THEOREM 25 *The formula ϕ_n is equivalent in relation algebras to $\phi'_n =$*

$$\forall a \neq 0, \forall_{(\bar{p} \in \{In, Out\}^{<n})}\ a^{\bar{p}}, b^{\bar{p}} \bigwedge_{j,k \in F} \quad \bigvee_{\bar{q} \in \{In, Out\}^n} \quad \bigwedge_{0 \leq u < v < w \leq n+1}$$

$$\begin{aligned} &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(u, v) \ ; \\ &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(v, w) \wedge \\ &\Gamma(a, \bar{q}, (j(\bar{q}\lceil_i), k(\bar{q}\lceil_i), a^{\bar{q}\lceil_i}, b^{\bar{q}\lceil_i} : 0 \leq i < n))(u, w) \neq 0. \end{aligned}$$

9.4 Summary

Let us summarise these results. Proposition 23 says that for countable relation algebras, $\exists$ has a winning strategy in each game $G_n(\varnothing, \mathcal{A})$ $(n < \omega)$ just when $\mathcal{A}$ is representable. Theorems 24 and 25 imply that a relation algebra $\mathcal{A}$ satisfies ϕ'_n if and only if $\exists$ has a winning strategy in $G_n(\varnothing, \mathcal{A})$. Thus, for countable relation algebras, the universal sentences ϕ'_n exactly characterise the representable relation algebras.

By an argument of Henkin this result carries over from countable to arbitrary relation algebras. A theorem of Henkin ([Hen53] theorem 1, essentially compactness) implies that if we can show that every finitely generated relation algebra satisfies $\{\phi'_n : n < \omega\}$ if and only if it is representable then the result holds for arbitrary relation algebras too. Since a finitely generated relation algebra is necessarily countable, we obtain this result by the preceding paragraphs. Thus

THEOREM 26 *$\{\phi'_n : n < \omega\}$, together with the basic Tarski axioms of section 3.1, axiomatises the representable relation algebras.*

As an addendum to this section, we show how to turn these universal axioms into equations. We can do this because relation algebras form a *discriminator class,* and therefore these universal sentences ϕ'_n are equivalent *over simple relation algebras* to equations eq_n.

LEMMA 27 *Let L_r be the language of relation algebras. For every universal L_r-sentence $\forall\bar{x}\psi(\bar{x})$ there is an L_r-equation $\theta(\bar{x})$, of the form $t(\bar{x}) = 0$ for some L_r-term t, such that in any simple relation algebra $\forall\bar{x}(\phi(\bar{x}) \leftrightarrow \theta(\bar{x}))$ holds. Hence, $\forall\bar{x}\psi(\bar{x})$ is equivalent in simple relation algebras to the equation $\forall\bar{x}(t(\bar{x}) = 0)$.*

PROOF:

By induction on ψ. For the atomic case, the equation $t = u$ is equivalent in any relation algebra to $(t - u) \vee (u - t) = 0$.

Assume inductively that ψ is equivalent in simple relation algebras to $t = 0$, and χ to $u = 0$. Then:

- $\neg\psi$ is equivalent to $\neg(t = 0)$ and so (in simple relation algebras) to $1; t; 1 = 1$, and so to $(-(1; t; 1)) = 0$.
- $\psi \wedge \chi$ is clearly equivalent in any relation algebra to $(t \vee u) = 0$.

□

Thus, for each sentence ϕ'_n above, there is an equation eq_n equivalent to ϕ'_n over simple relation algebras. We'll show that the equations $\{eq_n : n < \omega\}$ exactly characterise the class of *all* representable relation algebras, using the facts that **RRA** is a variety [JT48], and so closed under subdirect products and homomorphic images; that every relation algebra $\mathcal{A}$ is a subdirect product of simple relation algebras $\mathcal{A}_\lambda$ $(\lambda \in \Lambda)$, each of which is a homomorphic image of $\mathcal{A}$ (the $\mathcal{A}_\lambda$ are the simple components of $\mathcal{A}$); and hence that a relation algebra is representable if and only if all its simple components are. We have

$$\begin{array}{llll} \mathcal{A} \in \mathbf{RRA} & \Leftrightarrow & \mathcal{A}_\lambda \in \mathbf{RRA} & (\text{all } \lambda \in \Lambda) \\ & \Leftrightarrow & \mathcal{A}_\lambda \models eq_n & (\text{all } n, \lambda, \text{ since the } \mathcal{A}_\lambda \text{ are simple}) \\ & \Leftrightarrow & \mathcal{A} \models eq_n & (\text{all } n, \text{since equations are preserved under} \\ & & & \ \text{homomorphisms, products and subalgebras.}) \end{array}$$

Thus

THEOREM 28 *A relation algebra satisfies $\{eq_n : n < \omega\}$ if and only if it is representable.*

10 Axiomatising the Relation Algebras with Homogeneous Representations

By the same technique we can define an axiom schema that characterises the countable, relation algebras with homogeneous representations. The games and results are more complicated than was the case for arbitrary representations because here the existential player cannot just say 'in' or 'out' when it is her move, but has to actually choose elements of the relation algebra. This means that the formulas obtained contain existential quantifiers and do not reduce to universal formulas (see corollary 17). Also, we do not expect the axiomatisation to characterise uncountable relation algebras with homogeneous representations. The outline of the argument is presented below. First we explain how a graph labelled by ultrafilters can arise as the *limit* of a sequence of networks. This allows us to derive axioms in the language of relation algebras, without refering to ultrafilters.

Definition

- A *U-graph* M is a complete graph with nodes D, say, each edge being labelled by an ultrafilter: i.e., $M : D^2 \to Uf(\mathcal{A})$.
- A finite U-graph M obeying (for all $x, y, z \in D$)
  - $Id \in M(x, x)$ and

– $M(x,z) \in M(x,y); M(y,z)$

is then a U-network.

- For $i < \omega$ let N_i be $\mathcal{A}$-graphs on the same domain D, and M a U-graph, also with domain D. Say M is the *limit* of the N_is, and write $(N_i) \to M$, if for all edges $e \in D^2$ and for all $a \in M(e)$, we have $N_i(e) \leq a$ for cofinitely many $i < \omega$.
- As a special case we consider the empty U-graph to be the limit of a sequence of empty networks.

LEMMA 29 *If N_i $(i < \omega)$ are $\mathcal{A}$-networks and $(N_i)_{i<\omega} \to M$ then M is a U-network.*

PROOF:

It is necessary to show that M obeys the conditions for U-networks (above). This is not hard and here we only check the last one — the triangle condition. Let $x, y, z \in M$. Let $a \in M(x,y), b \in M(y,z)$. We must show that $a; b \in M(x,z)$. If not, then $-(a;b) \in M(x,z)$. Choose $i < \omega$ such that

$$N_i(x,y) \leq a, N_i(y,z) \leq b, N_i(x,z) \leq -(a;b).$$

As N_i is a network, $N_i(x,y); N_i(y,z) \wedge N_i(x,z) \neq 0$ implying $a;b \wedge -(a;b) \neq 0$, a contradiction. □

Next, we define the games $G^*_n(N, \mathcal{A})$ $(n < \omega)$ and $F^*_\omega(M, \mathcal{A})$ as follows.

A play of $G^*_n(N, \mathcal{A})$ is a sequence of finite $\mathcal{A}$-networks $N = N_0 \subseteq \ldots \subseteq N_n$. In the ith round, $(0 \leq i < n)$ after N_i has been constructed, $\forall$ may do one of two things. As before, he may pick an edge $e \in N_i^2$ and a pair of elements $a, b \in \mathcal{A}$. $\exists$ then responds by accepting or rejecting, exactly as in the definition of $G_n(N, \mathcal{A})$ in section 9.1. Alternatively, $\forall$ may pick any partial 1–1 map $\sigma : N_i \to N_i$. In this latter case $\exists$ responds in one of two ways. She may *tighten* N_i to N_{i+1} in such a way that for some edge $e \in dom(\sigma)^2$, $N_{i+1}(e) \wedge N_{i+1}(\sigma(e)) = 0$ (so σ is definitely not a partial isomorphism). Otherwise, she must make embeddings $\theta, \phi : N_i \to N_{i+1}$, for some new network N_{i+1}, such that $\forall e \in N_i^2$, $N_i(e) \geq N_{i+1}(\theta(e))$, $N_{i+1}(\phi(e))$, and $\theta \circ \sigma \subseteq \phi$ (amalgamate the two isomorphic networks). The initial round of $G^*_n(\varnothing, \mathcal{A})$ is the same as in $G_n(\varnothing, \mathcal{A})$ — $\forall$ picks any $a \neq 0$ and $\exists$ responds with the two-point network $N_0(a)$ labelled by a.

If M is a finite ultrafilter network, the game $F^*_\omega(M, \mathcal{A})$ is played to build ultrafilter networks, as follows. In the ith round $\forall$ can pick an edge (m,n) from M_i^2 and a pair of elements $a, b \in \mathcal{A}$ with $a; b \in M_i(m,n)$, in which case $\exists$ must respond by extending M_i to M_{i+1} so that there is a node $l \in M_{i+1}$ with $a \in M_{i+1}(m,l)$, $b \in M_{i+1}(l,n)$. Alternatively, if he wishes, $\forall$ can pick a partial 1–1 map $\sigma : M_i \to M_i$ such that for all edges e in the domain of σ, $M_i(e) = M_i(\sigma(e))$ (in other words, σ is a partial isomorphism). In this case, $\exists$ must respond by making embeddings $\theta, \phi : M_i \to M_{i+1}$ such that $\theta \circ \sigma \subseteq \phi$. In the initial round of $F^*_\omega(\varnothing, \mathcal{A})$, $\forall$ picks an element $a \neq 0$, and $\exists$ responds with any two-point ultrafilter network M_0 with nodes $0, 1$, say, and $a \in M_0(0,1)$.

LEMMA 30 *Let $\mathcal{A}$ be a countable relation algebra. $\mathcal{A}$ has a homogeneous representation if and only if $\exists$ has a winning strategy for $F^*_\omega(\varnothing, \mathcal{A})$*

PROOF:

Similar to the proofs of lemmas 6, 12 and proposition 23. Note that we need lemma 11 to cover the case where $\mathcal{A}$ is not simple. □

LEMMA 31 *If $\mathcal{A}$ has a homogeneous representation X then $\exists$ has a winning strategy for $G^*_n(\varnothing, \mathcal{A})$ (all $n < \omega$).*

PROOF:

As in the first part of the proof of proposition 23, $\exists$ maintains an embedding $'$ from the current network N_i into X to help her win the game. If $\forall$ plays a 1–1 map $\bar{m} \to \bar{n}$ of N_i such that the corresponding map $\bar{m}' \to \bar{n}'$ is a local isomorphism of X then she amalgamates; if not she can tighten the network to show that it is not a local isomorphism. In other respects the proof is the same as that of proposition 23. $\Box$

LEMMA 32 *Let $\mathcal{A}$ be a countable relation algebra and let $(N_i) \to M$. If $\exists$ has a winning strategy for $G^*_n(N_n, \mathcal{A})$ for each $n < \omega$ then $\exists$ has a winning strategy for $F^*_\omega(M, \mathcal{A})$.*

PROOF:

Assume the hypothesis of the lemma for non-empty finite M. It is sufficient to show that for any first move by $\forall$ in a play of $F^*_\omega(M, \mathcal{A})$, $\exists$ can respond with a U-network $M^* \supseteq M$ such that there is a sequence of networks $(N^*_n) \to M^*$ and $\exists$ has a winning strategy for $G^*_n(N^*_n, \mathcal{A})$ for all $n < \omega$. Clearly if she can do this then she can survive forever and win the play.

There are two types of moves that $\forall$ can make and we consider them separately. In $F^*_\omega(M, \mathcal{A})$ let $\forall$ make the first type of move, so he chooses $(x, y) \in M^2$ and $a, b \in \mathcal{A}$ with $a; b \in M(x, y)$. To calculate her response, $\exists$ first deletes the N_n for small n, so that $N_n(x, y) \le a; b$ for all n (not just cofinitely many n). After renumbering the N_n as $0, 1, 2, \ldots$, it is still the case that $\exists$ has a winning strategy for $G^*_n(N_n, \mathcal{A})$ for all $n < \omega$. Indeed it is possible to delete some of the N_n so that there are infinitely many left and, after renumbering, $\exists$ has a winning strategy for $G^*_{l(n)}(N_n, \mathcal{A})$ for all n, where $l(n) = n + 1 + (n + 1)^3$.

Now, in the first round of this game $G^*_l(N_n, \mathcal{A})$ let $\forall$ choose the same edge (x, y) and elements $a, b \in \mathcal{A}$ that he chose in the main game $F^*_\omega(M, \mathcal{A})$. Let $\exists$ respond using her winning strategy, with N'_n having an extra node k such that $N'_n(x, k) \le a$, $N'_n(k, y) \le b$. Her strategy is winning, so the 'rejection' option $N'_n(x, y) \wedge a; b = 0$ is impossible, because $N'_n(x, y) \le N_n(x, y) \le a; b$. So $\exists$ has no choice but to accept and add a triangle. Then $\forall$ plays $(n + 1)^3$ further rounds, choosing successively all edges f of N'_n and all pairs of elements $a_i, Id \in \mathcal{A}$ (for all $i \le n$), where a_i is the ith element in a fixed enumeration of $\mathcal{A}$. $\exists$ responds with her winning strategy on each occasion, without adding any new nodes. (Strictly she adds $(n + 1)^3$ new nodes, but for each new node there is an old node such that the edge between them is labelled by Id. So we can identify these new nodes with the nodes of N'_n.) The outcome is a network $N^*_n \supseteq N_n$, extending N_n with a single extra node, k, such that

1. $\exists$ has a winning strategy for $G^*_n(N^*_n, \mathcal{A})$
2. $N^*_n(f) \le a_i$ or $N^*_n(f) \wedge a_i = 0$ for all $f \in (N^*_n)^2, i \le n$
3. $N^*_n(x, k) \le a$, $N^*_n(k, y) \le b$
4. $N^*_n(f) \le N_n(f)$ for all $f \in (N_n)^2$.

Thus $\exists$ can define N^*_n for each n.

For each N^*_n, choose a U-graph M^+_n on the same domain with $N^*_n(f) \in M^+(f)$ for all edges $f \in (N^*_n)^2$. Then M^+_n is essentially an element of the space $Uf(\mathcal{A})^{(n+1)^2}$. This is a compact topological space in the product topology induced from the compact Stone space topology on $Uf(\mathcal{A})$ (we use Tychonoff's theorem here). So the sequence $(M^+_n)_{n<\omega}$ has an infinite convergent subsequence with a unique limit M^* say. M^* is a U-graph but not necessarily, at least at first sight, a U-network.

Delete all networks N^*_n such that M^+_n is not in the chosen convergent subsequence and renumber the remaining networks (again, $\exists$ still has a winning strategy for $G^*_n(N^*_n, \mathcal{A})$ for each n).

Now $(N^*_n)_{n<\omega} \to M^*$. To see this, let f be an edge of M^* and let $a \in M^*(f)$. It is required to show that $N^*_n(f) \le a$ for cofinitely many n. Since $(M^+_n)_{n<\omega}$ converges to M^*, we must have $a \in (M_n^+)(f)$ for cofinitely many n. Also, as $N_n^*(f) \in M_n^*(f)$ for all n, we have $a \wedge N_n^*(f) \neq 0$ for cofinitely many n. Let a be the jth element a_j of the enumeration of $\mathcal{A}$. By condition (2) above, if $n \geq j$, $a \wedge N_n^*(f) \neq 0$ implies that $N_n^*(f) \leq a$ as required.

By lemma 29 it follows that M^* is a U-network. Evidently $M^* \lceil dom(M) = M$, so M^* extends M. Hence $\exists$ has a response M^* to the first type of $\forall$-move in $F_\omega^*(M, \mathcal{A})$ and she still has a winning strategy for $G_n^*(N_n^*, \mathcal{A})$ for all $n < \omega$.

The second case we have to consider in a play of $F_\omega^*(M, \mathcal{A})$ is when $\forall$ chooses a partial isomorphism $\sigma : M \to M$. Then in each game $G_n^*(N_n, \mathcal{A})$, $\exists$ can either tighten N_n, so that σ is not a partial isomorphism, or she must amalgamate by finding suitable embeddings θ, ϕ. Crucially, as $(N_n) \to M$, she must amalgamate for cofinitely many games $G_n^*(N_n, \mathcal{A})$. Delete all the others and then follow the same argument that applied to a $\forall$-move of the first type (above). □

LEMMA 33 *Let N be any L_r-graph. There is a formula $\theta_n(N(i,j) : i,j \in N)$ such that for any assignment h of the variables occurring in the labels of N to elements of $\mathcal{A}$, $\mathcal{A}, h \models \theta_n(x_{ij} : i,j \in N)$ if and only if $\exists$ has a winning strategy in $G_n^*(h(N), \mathcal{A})$.*

PROOF:

Defining the formula is just a matter of translating the requirement that $\exists$ survives n rounds into first-order logic. The definition is not included here but the proof of lemma 13 shows the method. □

Let $\theta_n' = (\forall x \neq 0)\theta_n(N_0(x)(i,j) : i,j < 2)$. Combining lemmas 30, 32, 31 and 33:

THEOREM 34 *A countable relation algebra satisfies θ_n' for all $n < \omega$ if and only if it has a homogeneous representation.*

PROBLEM 11 *Investigate the characterisation of those relation algebras of arbitrary size which possess homogeneous representations. Note first that there are different definitions of homogeneity to distinguish, for uncountable relation algebras — say a representation is κ-homogeneous if a local isomorphism of size less than κ extends to a full automorphism. So far we have only considered ω-homogeneity. Also, we cannot use Henkin's device to carry the result from the countable to the uncountable case (as we did with representable relation algebras) because we do not know in advance that the class of all relation algebras with homogeneous representations is elementary.*

11 Weakly Associative Algebras and Relativized Relation Algebras

Definitions

- A *weakly associative algebra* (**WA**) obeys all the axioms for relation algebras except the associativity axiom, but satisfies instead[16]

$$(Id \wedge x); (1; 1) = ((Id \wedge x); 1); 1$$

  for all x belonging to the algebra.

- Let W be a set of edges, *reflexive over its domain* and *symmetric* i.e. $(x,y) \in W \Rightarrow (x,x), (y,y), (y,x) \in W$. A *relativized relation algebra* is similar to a proper relation algebra but all the operations are *relativized* to W: so for example if $(x,y) \in W$ then

$$W, (x,y) \models R; S \Leftrightarrow \exists z[(x,z), (z,y) \in W \text{ and } W, (x,z) \models R \text{ and } W, (z,y) \models S].$$

[16] Of course $1;1 = 1$ but the equation is left in this form to indicate the connection with associativity.

These definitions, the following two lemmas and the first part of theorem 37 appear in [Mad82], theorems 5.13, 5.19.

LEMMA 35 *Let $\mathcal{A}$ be a weakly associative algebra. Let $x \in At(\mathcal{A})$. There are unique units x_{dom}, x_{range} satisfying*

$$x_{dom}; x = x = x; x_{range}$$

PROOF:

The required elements are $x_{\rm dom} = (x; x^{\smile}) \wedge Id$ and $x_{\rm range} = (x^{\smile}; x) \wedge Id$. The lemma can be proved directly — or see theorem 3.4 in [Mad82]. □

LEMMA 36 *Let x, y be atoms. Then $x; y \neq 0$ if and only if $x_{range} = y_{dom}$.*

PROOF:

Simple. □

Definition A *partial, atomic network* (PAN) is a set of nodes N and a partial labelling function $l : N \times N \to At(\mathcal{A})$ such that

- l is defined on a reflexive, symmetric set of edges (that is, $(x, y) \in dom(l) \Rightarrow (x, x), (y, x) \in dom(l)$)
- l obeys the diagonal and transitivity conditions defined for ordinary atomic networks, but relativized to the edges where l is defined. That is,
  - if l is defined on (i, j), then $i = j$ if and only if $N(i, j) \leq Id$ (For simplicity, we insist here that Id is contained in equality.)
  - $l(i, j); l(j, k) \geq l(i, k)$, for all $i, j, k \in N$ with l defined on $(i, j), (j, k)$ and (i, k).

THEOREM 37

- *The axioms for weakly associative algebras are sound and complete over the class of relativized relation algebras.*
- *Every countable, weakly associative algebra has a complete, universal, homogeneous 'relativized' representation.*

PROOF:

Soundness is not hard to check; we concentrate on showing that any **WA** is isomorphic to a relativized relation algebra. As with relation algebras, a **WA** can be embedded in a complete, atomic **WA**. So without loss of generality, we assume that $\mathcal{A}$ is a complete, atomic weakly associative algebra.

Let $\kappa = |At(\mathcal{A})| + \omega$. As before, $\forall$ and $\exists$ play a game. This time, they build a continuous chain of partial atomic networks N_i $(i < \kappa)$, each one extending the last, and starting with a PAN $N_0 = N_a$ with at most two nodes, containing an edge labelled by an arbitrary non-zero element $a \in \mathcal{A}$. At limit stages, the union of the networks is taken. $\exists$ wins the game if the limit (N, l) of the game defines a relativized relation algebra. This will happen provided she ensures for all $x, y \in N$ with l defined on (x, y) and for all atoms $r, s \in \mathcal{A}$

$$l(x, y) \leq r; s \Leftrightarrow \exists z[l(x, z) = r \wedge l(z, y) = s].$$

Suppose this condition fails on a pair (x, y) at some stage in the game. That must be because '$\Rightarrow$' fails. She can repair the fault permanently in one turn, by adding to the current network a single extra node z, with the labellings $l(x, z) =$

r, $l(z,x) = r^{\smile}$, $l(z,y) = s$, $l(y,z) = s^{\smile}$ and $l(z,z) = r_{\mathrm{range}} = s_{\mathrm{dom}}$. (Use the triangle axioms to show that $t \leq r;s$ implies $s \leq t^{\smile};r$ etc. in a **WA**). *No other new edges are labelled by l* (this is where the construction is much easier than for relation algebras). Because there are at most κ pairs of atoms and at most κ edges that arise during the game, $\exists$ can do this for all cases of this type, and so she can win. The limit of the play is a PAN M_a, and it is clear that the disjoint union $\bigcup_{a\in\mathcal{A}\setminus\{0\}} M_a$ will define a relativized relation algebra isomorphic to $\mathcal{A}$.

To prove the second part, observe that the class of all PANs has the joint embedding and amalgamation properties, so the proof of theorem 5 gives the result. □

12 Cylindric Algebras

Cylindric algebras generalize relation algebras to relations of arity larger than two. One of the fundamental problems of cylindric algebra theory is to characterise representable cylindric algebras. A characterisation does exist, for example in [HMT85] page 112, but the axioms are certainly not simple. We approach this problem in the same way that we tackled relation algebras, by playing a game with finite structures and seeing if we can ensure that the limit is a representation. The correspondence with games ensures that the axioms have a clear intuitive meaning. They turn out to be universal axioms which are equivalent, over simple cylindric algebras, to equations. For finite dimensional cylindric algebras, since every cylindric algebra is a subdirect product of simple cylindric algebras, we can obtain an equational axiomatisation of the representable cylindric algebras. We can generalise to cylindric algebras of any dimension using algebraic techniques.

Definitions

- If U is a non-empty set and α an ordinal, ${}^{\alpha}U$ denotes the set of functions from α to U. A subset of ${}^{\alpha}U$ is called an α-ary relation on U. $D_{\kappa\lambda}$ denotes the set of all elements y of ${}^{\alpha}U$ such that $y(\kappa) = y(\lambda)$. Given an α-ary relation X on U, define the cylindrification $C_{\kappa}X$ to be the set of all elements of ${}^{\alpha}U$ that agree with some element of X except, perhaps, on the κth co-ordinate.

- A *cylindric set algebra* of dimension α is a structure consisting of a set S of α-ary relations on some domain U, equipped with the operations $0, 1 = {}^{\alpha}U, \cup$, and $\setminus$ (complement), the diagonal elements $D_{\kappa\lambda}$ ($\kappa, \lambda < \alpha$), and the cylindrifications C_{κ} ($\kappa < \alpha$). S is of course closed under all these operations.

  The class of all cylindric set algebras of dimension α is denoted Cs_{α}.

- A *cylindric algebra* of dimension α is defined to be a structure

  $$\mathcal{C} = (C, \vee, -, 0, 1, c_{\kappa}, d_{\kappa\lambda})_{\kappa,\lambda<\alpha}$$

  obeying the following axioms [HMT71] for every $x, y \in C$, $\kappa, \lambda, \mu < \alpha$:

  1. $(C, \vee, -, 0, 1)$ is a boolean algebra
  2. $c_{\kappa}0 = 0$
  3. $x \leq c_{\kappa}x$
  4. $c_{\kappa}(x \wedge c_{\kappa}y) = c_{\kappa}x \wedge c_{\kappa}y$
  5. $c_{\kappa}c_{\lambda}x = c_{\lambda}c_{\kappa}x$
  6. $d_{\kappa\kappa} = 1$
  7. if $\kappa \neq \lambda, \mu$, then $d_{\lambda\mu} = c_{\kappa}(d_{\lambda\kappa} \wedge d_{\kappa\mu})$
  8. if $\kappa \neq \lambda$, then $c_{\kappa}(d_{\kappa\lambda} \wedge x) \wedge c_{\kappa}(d_{\kappa\lambda} \wedge -x) = 0$.

These axioms are valid over cylindric set algebras.

We write $\mathbf{CA}_\alpha$ for the class of all cylindric algebras of dimension α.

- A cylindric algebra is said to be *representable* if it is isomorphic to a subdirect product of cylindric set algebras. $\mathbf{RCA}_\alpha$ denotes the class of all representable cylindric algebras of dimension α.
- A *component* of $\mathcal{C}$ is a non-zero element x of $\mathcal{C}$ such that for all $\kappa < \alpha$, $c_\kappa x = x$. Let α be finite. Then $\mathcal{C}$ is simple if and only if the only component is 1, and a representable cylindric algebra is simple if and only if it is isomorphic to a cylindric set algebra. (We note that an algebra is called simple if it has no non-trivial congruences.) If α is finite, then any cylindric algebra is a subdirect product of simple cylindric algebras.
- The operations on a cylindric algebra $\mathcal{C}$ can be extended to the Stone space of sets of ultrafilters over $\mathcal{C}$. For example if u, v are ultrafilters and c_i is a cylindric operator, we say $u \leq c_i v$ if for all $b \in v$, $c_i b \in u$.
- L_c^α is the first-order language of α-dimensional cylindric algebras with similarity type $(1, 0, -, \vee, d_{ij}, c_i : i, j < \alpha)$ and with only the equality predicate.
- If $\mathcal{C}$ is an n-dimensional cylindric algebra then $L(\mathcal{C})$ is the first-order relational language with one n-ary predicate symbol for each element of $\mathcal{C}$. As with relation algebras, we can easily define an $L(\mathcal{C})$-theory $T(\mathcal{C})$ whose models are exactly the representations of $\mathcal{C}$.

12.1 Representations and games

Throughout this and the next section let $\mathcal{C}$ be a cylindric algebra of finite dimension $n \geq 3$. For relation algebra the representation-approximations we used were called networks, and we use the same term here.

Notation Let $\bar{\mu} = (\mu_0, \ldots, \mu_{n-1})$ be an n-tuple and let $i < n$. The n-tuple $\bar{\mu}[i \to \nu]$ is identical to $\bar{\mu}$, except that the ith element is replaced by ν. Thus, $(a, b, c)[0 \to x] = (x, b, c)$, for example.

Definitions

- A *$\mathcal{C}$-graph* G is an n-dimensional hypergraph with each n-tuple of nodes labelled by an element of $\mathcal{C}$, i.e., a set of nodes D and a map : $D^n \to \mathcal{C}$. As before, we use the same symbol G to denote the set of nodes, the mapping, and the graph itself.
- A *$\mathcal{C}$-network* (or simply a network) is a $\mathcal{C}$-graph N satisfying:
  - for each $i, j < n$ and any n-tuple from $dom(N)$ (written '$\bar{\mu} \in N^n$'), if $\mu_i = \mu_j$ then $N(\bar{\mu}) \leq d_{ij}$
  - for any $\bar{\mu} \in N^n$, any $\nu \in N$, and any distinct $i, j < n$, if $N(\bar{\mu}[j \to \nu]) \leq d_{ij}$ then $N(\bar{\mu}) \wedge N(\bar{\mu}[i \to \nu]) \neq 0$.
  - for any $\bar{\mu} \in N^n$, any $i < n$ and any $\nu \in N$ we have $N(\bar{\mu}[i \to \nu]) \wedge c_i N(\bar{\mu}) \neq 0$
- As in the relation algebra case, for networks N, N', we write $N \subseteq N'$ if the nodes of N' include those of N, and $N'(\bar{\mu}) \leq N(\bar{\mu})$ for all $\bar{\mu} \in N^n$.
- As in section 9.2, we define an L_c^n-graph to be an n-dimensional graph with each n-tuple of nodes labelled by a term of L_c^n.

12.2 Games and representations

We can find equations that characterise the representable cylindric algebras in the same way that we did it for relation algebras. The presentation is more concise here; for more details, look at the axiomatisation of **RRA** in section 9 above. Let us first define an appropriate game G^c_n for cylindric algebras.

Definitions Let N be a non-empty network (over the fixed cylindric algebra $\mathcal{C}$), and let $t \leq \omega$. The game $G^c_t(N, \mathcal{C})$ is of length t. As before, $N_0 = N$, a play of $G^c_t(N, \mathcal{C})$ is a sequence of networks $N_0 \subseteq \cdots \subseteq N_t$ if t if finite, and a play of $G^c_\omega(N, \mathcal{C})$ is a sequence $N_0 \subseteq N_1 \subseteq \cdots$.

In the sth round ($s < t$), let the last network played be N_s. $\forall$ may pick $i < n, \bar{\mu} \in N^n_s$ and $a \in \mathcal{C}$. $\exists$ must respond to this move (if possible) with a network $N_{s+1} \supseteq N_s$ such that either $N_{s+1}(\bar{\mu}) \wedge c_i a = 0$, or else $N_{s+1}(\bar{\mu}[i \to \nu]) \leq a$ for some node $\nu \in N_{s+1}$.

Alternatively, $\forall$ may pick $i = n$, $\bar{\mu} \in N^n_s$, and any element $a \in \mathcal{C}$. $\exists$ responds to this, if she can, with a network $N_{s+1} \supseteq N_s$ such that either $N_{s+1}(\bar{\mu}) \leq a$ or $N_{s+1}(\bar{\mu}) \leq -a$. The '$i$' plays no role here: it is just an indicator of the type of move $\forall$ makes.

In the game $G^c_t(\varnothing, \mathcal{C})$ there is an extra, initial move and a play of this game is a sequence $\varnothing \subset N_0 \subseteq N_1 \subseteq \cdots$. Here, $\forall$ picks any element $c \neq 0$ of $\mathcal{C}$. $\exists$ responds with the network $N_0(c)$ with n nodes, $0, \ldots, n-1$, where

- $N_0(c)(0, \ldots, n-1) = c$
- for any other n-tuple $\bar{\mu} \in \{0, \ldots, n-1\}^n$, we have $N_0(c)(\bar{\mu}) = \bigwedge_{\mu_i = \mu_j} d_{ij}$.

The play continues for another t rounds as in $G^c_t(N_0(c), \mathcal{C})$.

If $\exists$ can move legally for t rounds she has won the play of $G^c_t(N, \mathcal{C})$.

THEOREM 38 *Let $\mathcal{C}$ be a countable, n-dimensional cylindric algebra. $\mathcal{C}$ is representable if and only if $\exists$ has a winning strategy in each game $G^c_t(\varnothing, \mathcal{C})$ $(t < \omega)$.*

PROOF:

If $\mathcal{C}$ is representable then, as with proposition 23, $\exists$ can use a representation to give her a winning strategy in each of the games. Conversely, if she can win each game $G^c_t(\varnothing, \mathcal{C})$ for $t < \omega$, we can show that she also has a winning strategy for $G^c_\omega(\varnothing, \mathcal{C})$. Again, the argument is the same as in the proof of proposition 23.

Next we want to use $\exists$'s winning strategy in $G^c_\omega(\varnothing, \mathcal{C})$ to construct a representation of $\mathcal{C}$. As before, $\exists$ will use her winning strategy, and $\forall$ will be persuaded to pick every n-tuple ever constructed, every $i \leq n$, and every $a \in \mathcal{C}$ eventually during the game.

Write M for the set of all nodes introduced during the game. For an n-tuple $\bar{\mu} \in M^n$, and $a \in \mathcal{C}$, write $M \models a(\bar{\mu})$ if for some $s < \omega$ we have $\bar{\mu} \in N^n_s$ and $N_s(\bar{\mu}) \leq a$. $\forall$-moves of the second kind ('$i = n$') guarantee that for any n-tuple $\bar{\mu}$ and any $a \in \mathcal{C}$, for sufficiently big s we have either $N_s(\bar{\mu}) \leq a$ or $N_s(\bar{\mu}) \wedge a = 0$. This suffices to prove that $M \models a(\bar{\mu})$ or $M \models -a(\bar{\mu})$ for all $a, \bar{\mu}$. So M is essentially an labelled ultrafilter-graph.

Define a relation $\sim$ on the nodes of M, by

$$x \sim y \quad \Leftrightarrow \quad \exists \bar{\mu} \in M^n\, \exists i, j < n\, (\mu_i = x \wedge \mu_j = y \wedge M \models d_{ij}(\bar{\mu})).$$

Using the axioms defining n-dimensional cylindric algebras, it can be shown that $x \sim y$ if and only if

$$\forall \bar{\mu} \in M^n\, \forall i, j < n\, (\mu_i = x \wedge \mu_j = y \Rightarrow M \models d_{ij}(\bar{\mu})) \qquad (*)$$

(Here, we require the dimension n of $\mathcal{C}$ to be at least three.) It follows easily that $\sim$ is transitive. It is clearly reflexive and symmetric. So it is an equivalence relation on M.

Suppose that $\bar{\mu}, \bar{\nu} \in M^n$ agree except possibly at index i, and $\mu_i \sim \nu_i$. Let $j \neq i$. Then by $(*)$, $M \models d_{ij}(\bar{\mu}[j \to \nu_i])$. So for large enough $s < \omega$, we have $N_s(\bar{\mu}[j \to \nu_i]) \leq d_{ij}$. As N_s is a network, we have $N_s(\bar{\mu}) \wedge N_s(\bar{\mu}[i \to \nu_i]) \neq 0$. That is, $N_s(\bar{\mu}) \wedge N_s(\bar{\nu}) \neq 0$. As this holds for all large enough s, it follows that $M \models a(\bar{\mu}) \leftrightarrow a(\bar{\nu})$ for all $\in \mathcal{C}$. Hence, $\sim$ is a congruence on M.

Now we construct M^* via $M/\!\sim$ as in proposition 23. It is easy to show that M^* defines a representation of $\mathcal{C}$, at least when $\mathcal{C}$ is simple.

If $\mathcal{C}$ is not simple then, as in the proof of proposition 23, $\forall$ can force in his initial move that an arbitrary element $c \in \mathcal{C}$ has non-zero interpretation in the limit of the game. Thus for each $c \in \mathcal{C}$ there is a homomorphism from $\mathcal{C}$ to an $L(\mathcal{C})$-structure X_c such that c does not map to zero. The disjoint union of all the X_c is then a representation of $\mathcal{C}$. $\square$

12.3 First-order axioms

As before, we can define formulas ψ_t saying that $\exists$ has a winning strategy in the game $G^c_t(N, \mathcal{C})$.

Definition As with relation algebras, the winning positions in G^c_t are 'closed upwards'. So, given an L^n_c-graph N, an index $i \leq n$, an n-tuple $\bar{\mu} \in N^n$, and an L^n_c-term τ, we define two L^n_c-graphs, $Out = Out(N, i, \bar{\mu}, \tau)$ and $In = In(N, i, \bar{\mu}, \tau)$ $(i \leq n)$, corresponding to the two ways that $\exists$ can respond and 'as big (carrying as little information) as possible'. In each case, we have

- $dom(Out) = dom(In) = dom(N) \cup \{\lambda\}$ for some new node λ
- For all $\bar{\nu} \in N^n \setminus \{\bar{\mu}\}$, $In(\bar{\nu}) = Out(\bar{\nu}) = N(\bar{\nu})$

For $i < n$, we define In and Out as follows.

- $Out(\bar{\mu}) = N(\bar{\mu}) - c_i\tau$
- For all $\bar{\nu} \in Out^n$ involving the new node λ, $Out(\bar{\nu}) = 1$ if all co-ordinates are distinct; more generally, for n-tuples involving λ we have $Out(\bar{\nu}) = \bigwedge_{\nu_j = \nu_k} d_{jk}$ (that completes the definition of Out)
- $In(\bar{\mu}) = N(\bar{\mu})$
- $In(\bar{\mu}[i \to \lambda]) = \tau \wedge \bigwedge_{j,k \neq i,\, \mu_j = \mu_k} d_{jk}$
- for all other n-tuples $\bar{\nu}$ involving λ, $In(\bar{\nu}) = \bigwedge_{\nu_j = \nu_k} d_{jk}$ (and that defines In).

For $i = n$, we let

- $In(\bar{\mu})$ $(= In(N, n, \bar{\mu}, \tau)(\bar{\mu})) = N(\bar{\mu}) \wedge \tau$
- $Out(\bar{\mu}) = N(\bar{\mu}) - \tau$
- $In(\bar{\nu}) = Out(\bar{\nu}) = \bigwedge_{\nu_j = \nu_k} d_{jk}$ for all n-tuples $\bar{\nu}$ involving the new node λ

Definition Let N be a non-empty L^n_c-graph. We define the formula $CNet^n(N)$ to be the conjunction of the following three formulas:

$$\bigwedge_{\bar{\mu} \in N^n,\, i,j < n,\, \mu_i = \mu_j} N(\bar{\mu}) \leq d_{ij}$$

$$\bigwedge_{i,j < n,\, i \neq j,\, \bar{\mu} \in N^n,\, \nu \in N} (N(\bar{\mu}[j \to \nu]) \leq d_{ij}) \to (N(\bar{\mu}[i \to \nu]) \wedge N(\bar{\mu}) \neq 0).$$

$$\bigwedge_{i<n,\bar\mu\in N^n,\nu\in N} N(\bar\mu[i\to\nu])\wedge c_i N(\bar\mu)\neq 0$$

Clearly, if h is an assignment of the variables occurring in the terms labelling the hyperedges of N into an n-dimensional cylindric algebra $\mathcal{C}$, then

$$\mathcal{C}, h \models CNet^n(N) \;\Leftrightarrow\; h(N) \text{ is a } \mathcal{C}\text{-network}.$$

Now we define

$$\begin{array}{rcl} \psi_0^n(N) &=& CNet^n(N) \\ \psi_{t+1}^n(N) &=& \forall x \bigwedge_{i\leq n,\bar\mu\in N^n} \quad \psi_t^n(In(N,i,\bar\mu,x)) \\ && \qquad\qquad\qquad \vee \quad \psi_t^n(Out(N,i,\bar\mu,x)), \end{array}$$

where x is a new variable not occurring in any of the terms of N. One can show by induction on t that for all $t<\omega$ and all assignments h of the variables in the terms of N into $\mathcal{C}$, we have $\mathcal{C}, h\models\psi_t^n(N)$ if and only if $\exists$ has a winning strategy for $G_t^c(h(N),\mathcal{C})$.

As a special case corresponding to the empty network, for $t\geq 0$, define sentences

$$\phi_t^n = (\forall x\neq 0)\;\psi_t^n(N_{n,x}),$$

where $N_{n,x}$ is a L_c^n-graph with nodes $0,\ldots,n-1$, and with $N_{n,x}(0,\ldots,n-1)=x$ and $N_{n,x}(\bar\nu)=\bigwedge_{\nu_i=\nu_j} d_{ij}$ for all other $\bar\nu\in N_{n,x}^n$.

12.4 Universal and equational axioms

For those who prefer it, there is a universal, prenex equivalent $\phi_t'^n$ to ϕ_t^n. We can obtain it in the same way as for relation algebras. In any play of $G_t^c(\varnothing,\mathcal{C})$, we can suppose that the nodes constructed are $0,\ldots,t+n-1$, where $s+n$ gets added in the sth round ($0\leq s<t$). An arbitrary sequence $\bar p=(p_0,\ldots,p_{t-1})\in\{In,Out\}^t$ defines a possible way for $\exists$ to choose her moves, after the initial round, in a play of $G_t^c(\varnothing,\mathcal{C})$.

We can represent a strategy for $\forall$ in the game $G_t^c(\varnothing,\mathcal{C})$ as a function from initial plays $\bar q$ by his opponent $\exists$ into the set of legal choices he can make at that point of the game. So it is given by a (for his first move), and then the items $i_{\bar q},\bar\mu_{\bar q},a_{\bar q}$ for each initial play $\bar q\in\{In,Out\}^{<t}$ of $\exists$. Here, we take a and the $a_{\bar q}$ to be distinct variables of L_c^n; their values in $\mathcal{C}$ are what really determine $\forall$'s strategy. For each $\bar q$, the index $i_{\bar q}$ must be at most n, and the n-tuple of nodes $\bar\mu_{\bar q}$ must be in $\{0,1,\ldots,|\bar q|+n-1\}^n$ — i.e., in $(|\bar q|+n)^n$, recalling that the ordinal $s+n$ is the set of all smaller ordinals. Let us write $\mathcal{F}$ for the finite set

$$\begin{array}{ll} \{(\iota,\eta)\mid & \iota:\{In,Out\}^{<t}\to n+1,\quad \eta:\{In,Out\}^{<t}\to(t+n)^n, \\ & \qquad\qquad\qquad \forall\bar q\in\{In,Out\}^{<t}\;(\eta(\bar q)\in(|\bar q|+n)^n)\}. \end{array}$$

Thus, a strategy for $\forall$ is essentially a choice of elements of $\mathcal{C}$ to assign to the variables $a, a_{\bar q}$, and a pair in $\mathcal{F}$.

Given that $\exists$ plays $\bar p$ and that $\forall$ uses a strategy involving $(\iota,\eta)\in\mathcal{F}$, we can represent the outcome of the game by an L_c^n-graph $\Gamma=\Gamma(\bar p,\iota,\eta)$, together with an assignment of the variables $a, a_{\bar q}$ to the elements of $\mathcal{C}$ actually chosen by $\forall$ during the game. The edges of Γ will be labelled by L_c^n-terms involving the variables a and $a_{\bar p\lceil s}$ for $s<t$. We can explicitly define Γ as follows. The set of nodes of Γ is of course $\{0,\ldots,t+n-1\}=t+n$. Let $\bar\nu$ be any n-tuple of nodes of Γ. Then the label $\Gamma(\bar\nu)$ is the conjunction of the following L_c^n-terms:

d_{rs}	for each $r<s<n$ such that $\nu_r=\nu_s$
a	if $\bar\nu=(0,\ldots,n-1)$
$-c_{\iota(\bar p\lceil s)}a_{\bar p\lceil s}$	for each $s<t$ such that $p_s=Out,\bar\nu=\eta(\bar p\lceil s),\iota(\bar p\lceil s)<n$
$-a_{\bar p\lceil s}$	for each $s<t$ such that $p_s=Out,\bar\nu=\eta(\bar p\lceil s),\iota(\bar p\lceil s)=n$
$a_{\bar p\lceil s}$	for each $s<t$ such that $p_s=In,\bar\nu=\eta(\bar p\lceil s)[\iota(\bar p\lceil s)\to s+n],\iota(\bar p\lceil s)<n$
$a_{\bar p\lceil s}$	for each $s<t$ such that $p_s=In,\bar\nu=\eta(\bar p\lceil s),\iota(\bar p\lceil s)=n$

Now, just as with theorem 25, we obtain an expanded, universal equivalent $\phi_t'^n$ of ϕ_t^n for each $t < \omega$, saying that whatever strategy $\forall$ adopts, $\exists$ can always play so that the result Γ is a network:

$$\forall a \neq 0\, \forall_{(\bar{q} \in \{In, Out\}^{<t})} a_{\bar{q}} \bigwedge_{(\iota,\eta)\in\mathcal{F}} \bigvee_{\bar{p}\in\{In,Out\}^t} CNet^n(\Gamma(\bar{p}, \iota, \eta)).$$

This is true in $\mathcal{C}$ if and only if $\forall$ has no winning strategy in $G_t^c(\varnothing, \mathcal{C})$, which, as the game is determined, is if and only if $\exists$ has a winning strategy. Thus, we have

THEOREM 39 *$\exists$ has a winning strategy for $G_t^c(\varnothing, \mathcal{C})$ if and only if $\mathcal{C}$ satisfies $\phi_t'^n$.*

Theorems 38 and 39 imply that $\{\phi_t'^n : t < \omega\}$ axiomatises the countable, representable n-dimensional cylindric algebras. As before, by Henkin's argument, the result goes over from the countable to arbitrary cylindric algebras. Combining these results gives

THEOREM 40 *An n-dimensional cylindric algebra satisfies $\{\phi_t'^n : t < \omega\}$ if and only if it is representable.*

These universal formulas can be replaced by equations ε_t^n as in lemma 27. (The use of the property $x \neq 0 \Rightarrow 1; x; 1 = 1$ of simple relation algebras is replaced by $x \neq 0 \Rightarrow c_0 c_1 \ldots c_{n-1} x = 1$.) We obtain

THEOREM 41 *An n-dimensional cylindric algebra satisfies the equations $\{\varepsilon_t^n : t < \omega\}$ if and only if it is representable.*

12.5 Axiomatising $\mathbf{RCA}_\alpha$

The axiomatisation carries over to the class $\mathbf{RCA}_\alpha$ of α-dimensional representable cylindric algebras, for any $\alpha \geq \omega$. This can be easily read off from the following known algebraic results.

Definition Recall that we write L_c^α for the signature of α-dimensional cylindric algebras. For an ordinal α, let Ax_α consist of the axioms CA_α defining cylindric algebras of dimension α, together with all L_c^α-sentences of the following form:

$$\forall v_1, \ldots, v_n (\exists v_0 \varphi(v_0, c_i v_1, \ldots, c_i v_n) \to \exists v_0 \varphi(c_i v_0, c_i v_1, \ldots, c_i v_n)),$$

where $\varphi(v_0, \ldots, v_n)$ is arbitrary, and $i < \alpha$ is such that c_i, d_{ij}, d_{ji} do not occur in φ for any $j < \alpha$.

Fact Let $\alpha \geq 3$ be an ordinal, and ε an equation of L_c^α. Then ε is valid in $\mathbf{RCA}_\alpha$ if and only if $Ax_{\max(\alpha,\omega)} \vdash \varepsilon$. See [HMT85], corollaries 4.1.15, 4.1.16.

To exploit this, we need some notation for renaming symbols of a signature. Let α, β be ordinals, and let $\mu : \alpha \to \beta$ be any 1–1 partial map.[17] For an L_c^α-formula φ, we define φ^μ to be the L_c^β-formula obtained by replacing every c_i in φ by $c_{\mu(i)}$, and replacing every d_{ij} by $d_{\mu(i),\mu(j)}$. So φ^μ is only defined if μ is defined on i, j, k for every c_i, d_{jk} occurring in φ.

For sets Φ of formulas on which μ is defined, we define $\Phi^\mu = \{\varphi^\mu : \varphi \in \Phi\}$.

LEMMA 42 *Let α, β be ordinals, and $\mu : \alpha \to \beta$ be a partial 1–1 map.*

1. *Let Φ be any set of L_c^α-sentences, and suppose that μ is defined on them and also on some other L_c^α-sentence σ. Then $\Phi \vdash \sigma$ if and only if $\Phi^\mu \vdash \sigma^\mu$.*

2. *If μ is a total map, then $(CA_\alpha)^\mu \subseteq CA_\beta$, and $(Ax_\alpha)^\mu \subseteq Ax_\beta$.*

PROOF:

[17] Recall that an ordinal is the set of all previous ordinals.

One can prove (1) by applying μ or its inverse to every sentence in a proof of σ from Φ, for example. (2) follows from the definitions. $\square$

We use this lemma implicitly in the following.

THEOREM 43 *Let $\alpha \geq \omega$ be an ordinal. Then $\mathbf{RCA}_\alpha$ is axiomatised by the set*

$$\Sigma = CA_\alpha \cup \{(\varepsilon_m^n)^\mu : m, n < \omega, \ \ \mu : n \to \alpha \textit{ is a 1–1 map}\}.$$

Here, the ε_m^n are as in theorem 41.

PROOF:

We begin with a claim.

Claim. For any L_c^α-equation ε, we have $Ax_\alpha \vdash \varepsilon$ if and only if $\Sigma \vdash \varepsilon$.

Proof of claim. For '$\Rightarrow$', fix an L_c^α-equation ε such that $Ax_\alpha \vdash \varepsilon$. There is a finite set $\Phi \subseteq Ax_\alpha$ such that $\Phi \vdash \varepsilon$. Choose a partial, surjective 1–1 map $\mu : \alpha \to \omega$ such that $\varepsilon^\mu, \varphi^\mu$ $(\varphi \in \Phi)$ are all defined. Then $\Phi^\mu \vdash \varepsilon^\mu$. Examining the definition of Ax, we see that $\Phi^\mu \subseteq Ax_\omega$. Thus, $Ax_\omega \vdash \varepsilon^\mu$. By the fact quoted above, ε^μ is valid in $\mathbf{RCA}_n$ for all large enough $n < \omega$. Fix such an n. By theorem 41, $CA_n \cup \{\varepsilon_m^n : m < \omega\} \vdash \varepsilon^\mu$. So $(CA_n)^{\mu^{-1}} \cup \{(\varepsilon_m^n)^{\mu^{-1}} : m < \omega\} \vdash \varepsilon$, whence $\Sigma \vdash \varepsilon$, as required.

The converse is easier. It is enough to show that any $(\varepsilon_m^n)^\mu \in \Sigma$ is a consequence of Ax_α. Well, by theorem 41, ε_m^n is an equation valid in $\mathbf{RCA}_n$, so, by the 'fact' again, $Ax_\omega \vdash \varepsilon_m^n$. Let $\nu : \omega \to \alpha$ be any 1–1 extension of μ to ω. Then $(Ax_\omega)^\nu \vdash (\varepsilon_m^n)^\mu$. By definition of Ax, we have $(Ax_\omega)^\nu \subseteq Ax_\alpha$. So $Ax_\alpha \vdash (\varepsilon_m^n)^\mu$, as required. This proves the claim.

As $\mathbf{RCA}_\alpha$ is a variety, any L_c^α-structure $\mathcal{C}$ is in $\mathbf{RCA}_\alpha$ if and only if it satisfies all equations of L_c^α that are valid in $\mathbf{RCA}_\alpha$. By the 'fact' once more, an L_c^α-equation is valid in $\mathbf{RCA}_\alpha$ if and only if it is a logical consequence of Ax_α. By the claim, this is if and only if it is a logical consequence of the set Σ given in the corollary. So $\mathcal{C}$ is in $\mathbf{RCA}_\alpha$ if and only if it satisfies all equational consequences of Σ. Since Σ itself consists of equations, this is if and only if $\mathcal{C} \models \Sigma$, as required. $\square$

12.6 Other Results

The work on amalgamation carries through to cylindric algebra unchanged. As with relation algebras we can define a *homogeneous* representation to be one where every local isomorphism extends to a full automorphism. A *universal* representation X is one where every network that embeds in some representation embeds in X.

THEOREM 44

1. *A countable representable cylindric algebra $\mathcal{C}$ has a homogeneous representation if and only there is a class of networks with the amalgamation property.*

2. *$\mathcal{C}$ has a universal representation if and only if the union of all* ages *of representations has the joint embedding property.*

13 Conclusion

The game theoretic approach is appropriate for a wide class of step by step completeness constructions. We can also use this framework to characterise the representable relation algebras (and cylindric algebras) by playing a game with networks. This gives an easier alternative to other approaches to this problem. Hopefully this work opens an avenue from model theory to algebraic logic which can broaden and enrich the subject. Interesting problems in model theory can be looked at anew in the framework of algebraic logic, for example the analysis of homogeneous structures. Here we were able to characterise universal and homogeneous representations using games and we were able to characterise the countable relation algebras with homogeneous representations by first order axioms, though the uncountable case remains unsolved. Other concepts and results from model theory would also, surely, be worth investigating in algebraic logic.

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