Completely Representable Relation Algebras

Robin Hirsch*

October 27, 1995

Abstract

A boolean algebra is shown to be completely representable if and only if it is atomic whereas it is shown that the class of completely representable relation algebras is not elementary.

1 Introduction

There are two types of representation in algebraic logic: ordinary and complete representations. Ordinary representations, or just representations, have been studied extensively [Jónsson and Tarski1948, Lyndon1950, Lyndon1956, McKenzie1970, Maddux1978, Maddux1982, Henkin *et al.*1985, Andréka *et al.*1991, Venema1992, Monk1993] and are isomorphisms from a boolean algebra with operators to a more concrete structure where the boolean operators $\vee$ and $-$ are replaced by $\cup$ and $\setminus$, and the other operators have certain set-theoretically definable interpretations. For example, in relation algebra the binary operation ; gets interpreted as composition of binary relations.

Complete representations have the additional property that they preserve arbitrary unions (hence intersections too), wherever these unions are defined. Historically, there has been a certain confusion here with complete and arbitrary representations at times being mistaken for each other (e.g. [Lyndon1950]).

Representation classes - i.e. a class of all structures of the appropriate type possessing a representation - form *varieties* in almost every case in algebraic logic. This means that they can be characterised by a set of equational axioms, though in most cases it has been shown that infinitely many axioms are required.

Consider instead the class of *completely representable* structures of some type of algebraic logic. In most cases we discover quickly that they are not closed under the taking of substructures and therefore cannot be characterised by universal axioms and cannot form a variety. In this paper we investigate whether the classes of completely representable boolean algebras, and relation algebras form *elementary classes*, that is, in each case we ask whether there is *any* set of first-order formulas that characterises the class. Now this might sound rather unlikely, after all, the idea of a complete representation is an essentially second-order one, so why should the completely representable structures be first-order definable? Corollary 3 is a warning against drawing this conclusion too rapidly: the class of completely representable boolean algebras is finitely axiomatisable, in fact a boolean algebra has a complete representation if and only if it is atomic.

For relation algebras we do get a negative answer - the class of completely representable relation algebras is not elementary. The proof uses an ultrapower construction. The results are presented in a game-theoretic framework and this is for a number of reasons. Firstly, the game-theoretic method has been used effectively to simplify and clarify quite difficult problems in the field [Hirsch and HodkinsonSubmitted]. This is largely because the goal of the second player is to build a representation or, in effect, to prove the theorem. This makes the proof natural. Secondly, we believe that techniques in game-theory should become standard scientific methods, just like proof by induction. Having said that we should re-assure the reader that absolutely no prior

*Supported by SERC grant reference:GR/H46343. Thanks to Roger Maddux and Mark Reynolds for valuable remarks on the text. Thanks particularly to Ian Hodkinson his collaboration.

knowledge of game-theory is required here. The definitions and results concerning games used here are very simple and explained in the text. Throughout, **BA** will denote the class of all boolean algebras and **RA** denotes the class of all relation algebras.

2 Boolean Algebra

Definition A *representation* of a **BA** $\mathcal{B}$ is an injection

$$h : \mathcal{B} \to \mathcal{P}(X)$$

(some set X) such that

$$\begin{aligned} h(0) &= \emptyset \\ h(1) &= X \\ h(a \vee b) &= h(a) \cup h(b) \\ h(-a) &= X \setminus h(a) \end{aligned}$$

for all $a, b \in \mathcal{B}$.

Notation If $h : \mathcal{B} \to \mathcal{P}(X)$ is a representation and $x \in X$, let $h^{-1}(x) = \{b \in \mathcal{B} : x \in h(b)\}$. $h^{-1}(x)$ is an ultrafilter of $\mathcal{B}$.

Symbols $\wedge, \leq$ are the obvious abbreviations.

THEOREM 1 *Every boolean algebra is representable [Stone1936].*

PROOF:

Given the **BA** $\mathcal{B}$, let X be the set of all ultrafilters over $\mathcal{B}$. Represent $\mathcal{B}$ by

$$h(b) = \{x \in X : b \in x\}$$

$\Box$

Definitions

- An atom is a minimal non-zero element of $\mathcal{B}$. $\mathcal{B}$ is atomic if every non-zero element contains an atom. The set of all atoms of $\mathcal{B}$ is denoted $Ats(\mathcal{B})$.
- An infinite join $\bigvee S$ of a subset S of a **BA** $\mathcal{B}$ is the supremum of the set if it exists and is undefined otherwise. Similarly we can define $\bigwedge S$ to be the infimum of S if it exists.
- A *complete representation* of a **BA** $\mathcal{B}$ is a representation also satisfying

  $$h(\bigvee S) = \bigcup h[S]$$

  wherever $\bigvee S$ is defined. [By $h[S]$ we mean $\{h(s) : s \in S\}$.] Check that a complete representation also preserves infinite 'conjunctions'.
- An *atomic representation* $h : \mathcal{B} \to \mathcal{P}(X)$ is a representation such that for each $x \in X$, the ultrafilter $h^{-1}(x)$ is a principal ultrafilter, or alternatively x is in the range of some atom of $\mathcal{B}$.

THEOREM 2 *A representation h of a boolean algebra $\mathcal{B}$ is atomic if and only if it is complete.*

PROOF:

First suppose h is an atomic representation. Note that for any $b \in \mathcal{B}$ we have

$$h(b) = \bigcup\{h(\beta) : \text{atoms } \beta \leq b\}$$

To show that h is a complete representation, let S be a subset of $\mathcal{B}$ and suppose the supremum $\bigvee S$ exists. Then

$$\begin{array}{rcll} x \in h(\bigvee S) & \Leftrightarrow & x \in h(\beta) & \text{some atom } \beta \leq \bigvee S \\ & \Leftrightarrow & x \in h(\beta) & \text{some atom } \beta \text{ some } s \text{ with } \beta \leq s \in S \\ & \Leftrightarrow & x \in h(s) & \text{some } s \in S \\ & \Leftrightarrow & x \in \bigcup h[S] & \end{array}$$

Conversely, let h be a complete representation and let $x \in X$. In order to show that h is an atomic representation we must show that the ultrafilter $h^{-1}(x)$ is principal. 0 is a lower bound of $h^{-1}(x)$. If 0 is the greatest lower bound $(0 = \bigwedge h^{-1}(x))$ then since h is complete

$$\emptyset = h(0) = h(\bigwedge h^{-1}x) = \bigcap h[h^{-1}x]$$

but $x \in \bigcap h[h^{-1}x]$ which gives a contradiction. Therefore there is a non-zero lower bound b of $h^{-1}(x)$ in which case b must be an atom (follows from the fact that $h^{-1}(x)$ is an ultrafilter) and then b is the greatest lower bound so $h^{-1}(x)$ is principal, as required. Thus h is an atomic representation. □

Note If a Boolean algebra $\mathcal{B}$ admits an atomic representation, then it is atomic. Moreover, a representation $h : \mathcal{B} \to \mathcal{P}(X)$ is atomic if and only if $\langle h(\beta) : \beta \in Ats(\mathcal{B})\rangle$ is a partition of X. So, every non-atomic representation $h : \mathcal{B} \to \mathcal{P}(X)$ of an atomic Boolean algebra can be turned in an atomic representation as follows: Let $Y = \bigcup Ats(\mathcal{B})$ and let $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ be the map defined by $f(Z) = Z \cap Y$. The composition $f \circ h$ is an atomic representation of $\mathcal{B}$.

COROLLARY 3 *A boolean algebra has a complete representation if and only if it is an atomic boolean algebra.*

PROOF:

If $\mathcal{B}$ is atomic then let $X = Ats(\mathcal{B})$. The representation h defined by

$$h(b) = \{\beta \in Ats(\mathcal{B}) : \beta \leq b\}$$

is an atomic representation and hence a complete representation.

Conversely, suppose $\mathcal{B}$ has a complete representation h. By the previous theorem h must be an atomic representation. For any $b \in \mathcal{B}$ pick any point $x \in h(b)$. Since h is atomic, there is an atom β with $x \in h(\beta)$, hence $\beta \leq b$ so $\mathcal{B}$ is atomic. □

3 Relation Algebra

3.1 Basics

A *relation algebra* $\mathcal{A} = (A, 0, 1, \vee, -, Id, ^{\smile}, ;)$ satisfies

1. The reduct $(A, 0, 1, \vee, -)$ is a boolean algebra.
2. ; is an associative binary operator on $\mathcal{A}$
3. $(a^{\smile})^{\smile} = a$
4. $Id; a = a; Id = a$

5. $a; (b \vee c) = a; b \vee a; c$
6. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
7. $(a - b)^{\smile} = a^{\smile} - b^{\smile}$
8. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
9. $(a; b) \wedge c^{\smile} = 0 \Leftrightarrow (b; c) \wedge a^{\smile} = 0$.

A relation algebra is *simple* if every non-zero element r satisfies

$$1; r; 1 = 1$$

Every relation algebra can be decomposed as a direct sum of simple relation algebras [Jónsson and Tarski1948]. An *integral* relation algebra satisfies $a; b = 0 \Rightarrow (a = 0) \vee (b = 0)$. A relation algebra is integral if and only if the identity is an atom.

A *representation* $h : \mathcal{A} \to \mathcal{P}(X \times X)$ of a relation algebra $\mathcal{A}$ is a map such that

- h acts as a boolean representation from the boolean reduct of $\mathcal{A}$ to $\mathcal{P}(h(1))$.
- $(x, y) \in h(Id)$ if and only if $x = y$.
- $(x, y) \in h(r)$ if and only if $(y, x) \in h(r^{\smile})$.
- $(x, z) \in h(r; s)$ if and only if $(\exists y \in X)\ (x, y) \in h(r) \wedge (y, z) \in h(s)$.

where r, s are arbitrary elements of $\mathcal{A}$ and $x, y, z \in X$.

A relation algebra is representable if and only if all its simple components are.

Many of the definitions and results of the previous section carry over to the boolean reduct of a relation algebra. In particular we can define a *complete representation* to be a representation that preserves arbitrary unions wherever they are defined, and an *atomic representation* is a representation such that for any pair $(x, y) \in h(1)$ there is a (unique) atom $\beta \in \mathcal{A}$ with $(x, y) \in h(\beta)$. The proof of the following theorem is unchanged from theorem 2.

THEOREM 4 *Let h be a representation of a relation algebra. h is a complete representation if and only if it is an atomic representation.*

But corollary 3 does not carry over to relation algebras: there are atomic (in fact finite) relation algebras with no representation at all (e.g. [McKenzie1970] page 286), the so-called *point algebra* with three atoms $<, >, =$ has only infinite representations [Villain and Kautz1986] and there are atomic, representable relation algebra possessing no complete representations ([Maddux1978] pages 154 - 173, or see theorem 10 herein). So, in contrast to boolean algebra, it does not seem likely that we can build complete representations out of the atoms of a relation algebra.

3.2 Networks and Games

Let $\mathcal{A}$ be an atomic relation algebra. Define an atomic $\mathcal{A}$-network (N) to be a function $N : \alpha \times \alpha \to Ats(\mathcal{B})$ for some set of nodes α such that for all nodes $l, m, n \in \alpha$

- $N(m, n) \leq Id$ if $m = n$.
- $N(m, n) = N(n, m)^{\smile}$ and
- $N(l, n) \leq N(l, m); N(m, n)$.

$Anets(\mathcal{A})$ (or simply $Anets$ if the context is clear) denotes the class of atomic $\mathcal{A}$-networks.

Let N be an atomic network. It can be seen that Id forms a congruence relation on the nodes of N. For each node $n \in \alpha$ let $[n]$ denote the equivalence class of n or $\{m \in \alpha : N(m,n) = Id\}$ and let α/Id denote the set of all such equivalence classes. We can form a quotient network N/Id with nodes α/Id. An edge of N/Id is labelled with Id if *and only if* it is a reflexive edge $([n],[n])$. Define a map $N^* : \mathcal{A} \to \mathcal{P}(\alpha/Id \times \alpha/Id)$ by

$$N^*(a) = \{([m],[n]) \in \alpha/Id \times \alpha/Id : N(m,n) \leq a\}$$

LEMMA 5 *Let $\mathcal{A}$ be a simple relation algebra. N^* is a complete representation of $\mathcal{A}$ if and only if for all $m,n \in \alpha$ and for all $r,s \in Ats(\mathcal{A})$ with $r;s \geq N(m,n)$ there is a node $l \in \alpha$ such that $N(m,l) = r$ and $N(l,n) = s$.*

PROOF:

Since N is an atomic network, N^* is an 'atomic map', meaning that every pair of points in α/Id is in the range of some atom. So, by theorem 4, if N^* is a representation of $\mathcal{A}$ then it is automatically a complete one. We have already seen that N^* preserves the identity, clearly N^* is a boolean representation and it is straight-forward to check that it preserves converse.

N^* preserves composition if and only if for all $r,s \in Ats(\mathcal{A})$ and $[m],[n] \in \alpha/Id$

$$([m],[n]) \in N^*(r;s) \Leftrightarrow (\exists [l] \in \alpha/Id)\ [([m],[l]) \in N^*(r) \wedge ([l],[n]) \in N^*(s)]$$

which is equivalent to $(\forall m,n \in \alpha)(\forall r,s \in Ats(\mathcal{A}))$

$$N(m,n) \leq r;s \Leftrightarrow (\exists l \in \alpha)\ [N(m,l) = r\ \wedge\ N(l,n) = s] \qquad (1)$$

The implication from right to left in formula 1 holds automatically because of the final part of the definition of an atomic network, so the formula is equivalent to the condition in the lemma.

□

Frequently, we do not have a special name for the set of nodes of the network N but abuse the notation by letting N stand for the network and the nodes of the network too. Thus $n \in N$ means that n is a node of the network N.

Definition We define a game, $G_\omega(N,\mathcal{A})$, with two players, $\forall$ and $\exists$ who construct a countable sequence of finite, atomic networks

$$N = N_0 \subseteq N_1 \subseteq \ldots$$

For each $i \geq 0$ the atomic network N_{i+1} extends N_i by at most a single node. In the ith turn $(i = 1,2,\ldots)$ the play so far is $N = N_0 \subseteq \ldots \subseteq N_{i-1}$. $\forall$ picks any two nodes m,n from N_{i-1} and any two atoms $r,s \in \mathcal{A}$ such that $r;s \geq N_{i-1}(m,n)$. $\exists$ responds (if possible) by extending N_{i-1} to N_i with at most one extra node such that there is a node $l \in N_i$ with $N_i(m,l) = r$ and $N_i(l,n) = s$. (Possibly such a node already exists in N_{i-1} in which case $\exists$ does not need to extend the network). If she can do this each round she wins the play.

The game $G_\omega(\emptyset,\mathcal{A})$ is a special case with a "zero'th round" in which $\forall$ picks any atom a of $\mathcal{A}$ and $\exists$ responds with any network N_0 with an edge $e \in N_0^2$ such that $N_0(e) = a$. Without loss, $\exists$ can choose $|N_0| \leq 2$. The remaining rounds of a play of the game are as before.

The games $G_n(N,\mathcal{A}), G_n(\emptyset,\mathcal{A})$ are similar to $G_\omega(N,\mathcal{A}), G_n(\emptyset,\mathcal{A})$ but there are only n rounds. If $\exists$ is able to survive to the end of the nth round she has won the play. The final network in a play of $G_n(\emptyset,\mathcal{A})$ is N_n and has at most $n+2$ nodes.

For notational reasons, let us assume that each node is a natural number.

Let $\sigma : Anets^{<\omega} \times \omega \times \omega \times Ats(\mathcal{A}) \times Ats(\mathcal{A}) \to Anets$. In the ith round let $\forall$ pick the nodes m,n and atoms r,s. $\exists$ can respond (possibly illegally in which case she loses the play immediately) with the atomic network $\sigma(N_0, N_1, \ldots, N_{i-1}, m, n, r, s)$. σ is a *winning strategy* for $\exists$ if in any play of $G_\omega(N, \mathcal{A})$ where $\exists$ always uses σ she is guaranteed to win the play. [It can be shown that the ith-level Lyndon conditions [Lyndon1950] are equivalent to saying that $\exists$ has a winning strategy for $G_i(\emptyset, \mathcal{A})$. To use the terminology of [Maddux1978], this is equivalent to saying that $\mathcal{A}$ has an i-dimensional basis.]

Definition Let N be an atomic network and $h : \mathcal{A} \to \mathcal{P}(X \times X)$ a representation of $\mathcal{A}$. Say that N *embeds* in h if there is a map $\Xi : N \to X$ such that for all nodes $m, n \in N$,

$$(\Xi(m), \Xi(n)) \in h(N(m, n))$$

LEMMA 6 *1. Let $\mathcal{A}$ be a simple, atomic relation algebra with at most countably many atoms. $\exists$ has a winning strategy in the game $G_\omega(N, \mathcal{A})$ if and only if N embeds in a complete representation of $\mathcal{A}$.*

2. Let $\mathcal{A}$ be an atomic (not necessarily simple) relation algebra with at most countably many atoms. $\exists$ has a winning strategy in the game $G_\omega(\emptyset, \mathcal{A})$ if and only if $\mathcal{A}$ has a complete representation.

PROOF:

1. Suppose Ξ is an embedding from N to some complete representation h. In the first round let $\forall$ pick the nodes $m, n \in N$ and atoms r, s. The points $\Xi(m), \Xi(n)$ are in the domain of the representation and $(\Xi(m), \Xi(n)) \in h(N(m, n)) \subseteq h(r; s)$ since according to the rules of the game $r; s \geq N(m, n)$. Therefore there is a point x in the representation with $(\Xi(m), x) \in h(r)$ and $(x, \Xi(n)) \in h(s)$. Since h is a complete representation it is an atomic representation and therefore every two points are related by an atom. So the points $\{x\} \cup \{\Xi(l) : l \in N\}$ define an atomic network N_1 extending N_0 as required. If $\exists$ responds with N_1, it is still the case that N_1 embeds in h and so she can continue in this way forever.

   Conversely, suppose $\exists$ has a winning strategy for $G_\omega(N, \mathcal{A})$. Consider a play of the game in which $\forall$ eventually picks every possible pair of nodes m, n that appear in the play and every legitimate pair of atoms r, s. He can do this, because there are only countably many nodes that appear in the play and countably many atoms in $\mathcal{A}$. If $\exists$ uses her winning strategy the limit will be an atomic network N_ω satisfying the conditions of lemma 5 and therefore N^*_ω will be a complete representation.

2. It is required to show that each simple component of $\mathcal{A}$ has a complete representation (as a complete representation of $\mathcal{A}$ can then be obtained as the disjoint union of the complete representations of the components). For each simple component $\mathcal{A}_\lambda$ of $\mathcal{A}$, let a_λ be some atom of $\mathcal{A}_\lambda$. If $\forall$ picks a_λ in his first turn and then continues as in the previous part, the limit will be a complete representation of $\mathcal{A}_\lambda$. Since this can be done for each simple component the result follows.

$\square$

It is worth noting that it is not necessary to assume in the previous lemma that $\mathcal{A}$ is a relation algebra. The result still holds under the weaker conditions that $\mathcal{A}$ is an atomic boolean algebra enriched with constant Id, a unary operator $\smile$ and a binary ; such that $\smile$, ; are *normal operators* i.e. for all $a, b, c \in \mathcal{A}$, $0^\smile = 0, (a \vee b)^\smile = a^\smile \vee b^\smile, a; 0 = 0; a = 0$ and $a; (b \vee c) = a; b \vee a; c, (a \vee b); c = a; c \vee b; c$. Then a winning strategy for $\exists$ in $G_0(\mathcal{A}, \emptyset)$ is equivalent to saying that $\mathcal{A}$ obeys axioms 1, 3, 4 in the axiomatisation of relation algebra on page 3. A winning strategy in $G_1(\mathcal{A}, \emptyset)$ is equivalent to those axioms plus axioms 5, 6, 7, 8 and 9 and a winning strategy in $G_2(\mathcal{A}, \emptyset)$ also entails that the associative axiom (2) holds too.

3.3 The Class of Completely Representable Relation Algebras is Not Elementary

In this section we find two elementary equivalent relation algebras $\mathcal{A}, \mathcal{A}'$ and show that $\mathcal{A}'$ has a complete representation but $\mathcal{A}$ does not.

Let $\mathcal{B}$ be any non-principal ultraproduct of relation algebras $\mathcal{A}_i$ $(i < \omega)$ $(\mathcal{B} = \Pi_D \mathcal{A}_i)$. Let N be a finite, atomic $\mathcal{B}$-network. Now, for each $i < \omega$, pick an $\mathcal{A}_i$-graph N_i (an $\mathcal{A}_i$-graph is just a completely labelled graph with each edge labelled by an atom of $\mathcal{A}_i$) on the same set of nodes such that

$$N = [\langle N_i : i \in \omega \rangle]_D$$

Since N is an atomic network, for any three nodes $l, m, n \in N$ we have $N(l,m); N(m,n) \geq N(l,n)$. This implies that $\{i \in \omega : N_i(l,m); N_i(m,n) \geq N_i(l,n)\} \in D$ (definition of ultraproduct) and hence, regardless of the choice of N_i, $\{i : N_i$ is an atomic $\mathcal{A}_i$-network $\} \in D$ (as N is finite and this set is a finite intersection of members of D).

THEOREM 7 *If there is a natural number k such that $\{i \geq k : \exists$ has a winning strategy for $G_{i-k}(N_i, \mathcal{A}_i)\} \in D$ then $\exists$ has a winning strategy for $G_\omega(N, \mathcal{B})$.*

PROOF:

[One way of proving this is as follows. First show that a winning strategy for $\exists$ in $G_{i-k}(N_i, \mathcal{A}_i)$ is equivalent to a first-order formula $\phi(i,k)$ and hence, for each $i < \omega$ the formula $\exists k \phi(i,k)$ holds in $\mathcal{B}$ (first-order formulas are preserved under ultraproducts). Next, it can be shown that ω-saturation is sufficient to imply that $\exists$ has a winning strategy for $G_\omega(N, \mathcal{B})$. Finally, use [Chang and Keisler1973] theorem 6.1.1 which says that $\mathcal{B}$ is ω_1-saturated.]

Instead we prove the result directly. In $G_\omega(N, \mathcal{B})$ let $\forall$ play (e, a, b), so e is an edge of N and a, b are atoms of $\mathcal{B}$ such that $a; b \geq N(e)$. Let $a = [\langle a_i : i \in \omega \rangle]_D$, $b = [\langle b_i : i \in \omega \rangle]_D$ where $a_i, b_i \in \mathcal{A}_i$. Check that $\{i : (e, a_i, b_i)$ is a valid $\forall$-move in $G_{i-k}(N_i, \mathcal{A}_i)\} \in D$. It follows that
$\{i : (e, a_i, b_i)$ is a valid $\forall$-move in $G_{i-k}(N_i, \mathcal{A}_i)$ and $\exists$ has a winning strategy$\} \in D$. For each such i let $\exists$ respond to the move (e, a_i, b_i) with her winning strategy in $G_{i-k}(N_i, \mathcal{A}_i)$ and call the resulting atomic $\mathcal{A}_i$-network N_i^+. For other i, let N_i^+ be arbitrary. Let $\exists$ respond to $\forall$s move in the game $G_\omega(N, \mathcal{B})$ with the atomic network

$$N = [< N_i^+ : i \in \omega >]_D$$

Check that N is an atomic $\mathcal{B}$-network (this is true because the components N_i are atomic $\mathcal{A}_i$-networks for 'many' i) and a legal response.

If $\exists$ has a winning strategy for $G_{i-k}(N_i, \mathcal{A}_i)$ then she also has a winning strategy for $G_{i-(k+1)}(N_i^+, \mathcal{A}_i)$. So, replacing k by $k+1$,

$$\{i \geq k+1 : \exists \text{ has a winning strategy for } G_{i-(k+1)}(N_i^+, \mathcal{A}_i)\} \supseteq$$

$$\{i \geq k+1 : \exists \text{ has a winning strategy for } G_{i-k}(N_i, \mathcal{A}_i)\}$$

The latter set differs from

$$\{i \geq k : \exists \text{ has a winning strategy for } G_{i-k}(N_i, \mathcal{A}_i)\}$$

by at most a single point (k) and therefore belongs to the non-principal ultrafilter D. Thus the conditions of the theorem are maintained for the next round of the game and in this way $\exists$ can survive forever. $\Box$

COROLLARY 8 *If $\exists$ has a winning strategy for $G_i(\emptyset, \mathcal{A})$ (each i) and $\mathcal{B} = \Pi_D \mathcal{A}$ is any non-principal ultrapower of $\mathcal{A}$ then $\exists$ has a winning strategy for $G_\omega(\emptyset, \mathcal{B})$.*

PROOF:

Let $k = 0$ in the last theorem.
$\square$

Note Observe that the limit M of the game $G_\omega(\emptyset, \mathcal{B})$ is a countable, atomic $\mathcal{B}$ network. This means, for every three points $a, b, c \in M$, that $\{i < \omega : M(a,b)_i ; M(b,c)_i \geq M(a,c)_i\}$ belongs to the ultrafilter D. But because the network is infinite, it does not follow for any dimension i that any representative M_i forms an atomic $\mathcal{A}_i$-network. Indeed, in the example constructed in theorem 10 this will hold for no values of i.

We have nearly constructed a relation algebra, elementary equivalent to $\mathcal{A}$, with a complete representation. The only remaining problem is that the ultrapower $\mathcal{B}$ may be uncountable.

THEOREM 9 *If $\mathcal{A}$ is an atomic relation algebra and $\exists$ has a winning strategy for $G_i(\emptyset, \mathcal{A})$ (each i) then $\mathcal{A}$ is elementary equivalent to a completely representable relation algebra.*

PROOF:

Let $\mathcal{B}$ be any non-principal ultrapower of $\mathcal{A}$. We know (corollary 8) that $\exists$ has a winning strategy in $G_\omega(\emptyset, \mathcal{B})$. But $\mathcal{B}$ may be uncountable so we cannot conclude that $\mathcal{B}$ has a complete representation. The job of this proof is to find a countable, elementary subalgebra $\mathcal{A}'$ of $\mathcal{B}$ such that $\exists$ has a winning strategy for $G_\omega(\emptyset, \mathcal{A}')$. Recall, from theorem 6, that this implies that $\mathcal{A}'$ has a complete representation.

To construct $\mathcal{A}'$ we build a countable chain of countable, elementary subalgebras.

$$\mathcal{B}_0 \preceq \mathcal{B}_1 \preceq \ldots \qquad\qquad (\preceq \mathcal{B})$$

and define $\mathcal{A}'$ to be the union of the chain. The chain is defined inductively. Let $\mathcal{B}_0$ be the minimal elementary subalgebra of $\mathcal{B}$. Suppose we have defined the countable relation algebra $\mathcal{B}_j$ (some $j < \omega$). We know that $\exists$ has a winning strategy, say σ, in the game $G_\omega(\emptyset, \mathcal{B})$. σ is a function which takes an initial play of the game $\emptyset \subseteq N_1 \ldots N_k$, a move by $\forall$ (say (e, a, b) where e is an edge of N_k and $a, b \in \mathcal{B}$) and gives some extension N_{k+1} of N_k. But σ can be defined alternatively as a function taking a sequence of $\forall$-moves $\langle (e_0, a_0, b_0), \ldots (e_k, a_k, b_k) \rangle$ and giving the atomic network that $\exists$ should respond with, given that $\forall$s first k moves are $\langle (e_0, a_0, b_0), \ldots (e_k, a_k, b_k) \rangle$. This is because $\exists$ previous moves are *determined* by $\forall$s moves and σ. Defining σ in this way consider $\sigma|_{\mathcal{B}_j}$, the restriction of σ to $\mathcal{B}_j$, that is, restrict $\forall$s moves so that the atoms a_k, b_k can only be selected from $\mathcal{B}_j$ (all k). The range of $\sigma|_{\mathcal{B}_j}$ will be a countable set of atomic $\mathcal{B}$-networks and there will be a countable set S of atoms that occur in some atomic network from the range of $\sigma|_{\mathcal{B}_j}$. Let $\mathcal{B}_{j+1}$ be a countable, elementary subalgebra containing S in $\mathcal{B}$ (follows from downward Löwenheim-Skolem-Tarski theorem, or see [Chang and Keisler1973] theorem 3.1.6).

By construction, if $\forall$ plays only atoms from $\mathcal{B}_j$, then $\exists$ can win the play while only choosing atomic $\mathcal{B}_{j+1}$-networks.

Now let $\mathcal{A}'$ be the union of the chain $\mathcal{A}' = \bigcup_j \mathcal{B}_j$. Then $\mathcal{A}'$ is an elementary extension of each $\mathcal{B}_j$ (elementary chain theorem, [Chang and Keisler1973] theorem 3.1.13) hence elementary equivalent to $\mathcal{B}$ and $\mathcal{A}$. Furthermore, $\exists$ has a winning strategy for the game $G_\omega(\emptyset, \mathcal{A}')$. Since $\mathcal{A}'$ is countable we can use theorem 6 and conclude that $\mathcal{A}'$ has a complete representation. $\square$

THEOREM 10 *There is a relation algebra $\mathcal{A}$ such that $\exists$ has a winning strategy for $G_n(\emptyset, \mathcal{A})$ for each $n < \omega$, but $\mathcal{A}$ has no complete representation.*

PROOF:

Let us define the atomic relation algebra $\mathcal{A}$ by listing its atoms, stating their converses and then defining composition on the atoms. Any relation algebra based on this atom structure will then do, for example we can take the full power set of this set of atoms and define converse and composition by an infinite distribution rule. Or, if we wish to keep the cardinality of $\mathcal{A}$ countable, just take the subalgebra of the full power set algebra *generated* by the atoms.

$\mathcal{A}$ will be an integral, symmetric relation algebra, so the identity is an atom (no colour) and all elements are self-converse. The non-identity atoms are of five different colours: red, green, yellow, black and white[1], and here they are:

$$\begin{aligned} \text{Red} &= \{r_i^n : 0 \leq i < n < \omega\} \\ \text{Green} &= \{g_i : i < \omega\} \\ \text{Yellow} &= \{y\} \\ \text{Black} &= \{b, b_{ij} : i, j < \omega\} \\ \text{White} &= \{w, w_{ij} : i, j < \omega\} \end{aligned}$$

The composition of an atomic relation algebra is defined by listing all triples of atoms (x, y, z) such that $x; y \geq z^{\smile}$. If (x, y, z) is such a triple then so are (y, z, x) and (z, x, y) and, since a symmetric relation algebra is commutative, the triples (x, z, y), (z, y, z) and (y, z, x) must also be consistent. For $\mathcal{A}$ we define its atoms structure to consist of all triples *except for* permutations of the following.

(Id, r, s)	unless $r = s$(any atoms r, s)
(g_i, g_j, g_k)	any $i, j, k < \omega$
(g_i, g_j, B)	$B = b$ or b_{ij} for any $i, j < \omega$
(y, y, y)	
(y, y, W)	$W = w$ or $W = w_{ij}$ for any $i, j < \omega$
(r_i^l, r_j^m, r_k^n)	unless $l = m = n$ and $\lvert\{i, j, k\}\rvert = 3$
(w_{ij}, g_k, y)	unless $k = i$ or j
(b_{ij}, g_k, y)	unless $k = i$ or j

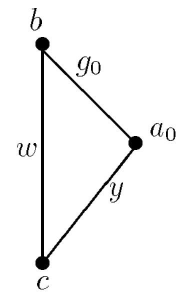
Note Let N be an atomic network with each edge $(k, l) \in N$ $(k \neq l)$ labelled by a red atom. Then since on each triangle all three edges are red the upper index must be the same throughout the network. Let the common upper index be n. Now fix a single node $k \in N$. Each pair of edges (k, l) and (k, m) (with $l \neq m$) must be labelled by *distinct* atoms r_i^n and r_j^n $(i \neq j)$. This is because the edges on an all-red triangle must be distinct. Since there are only n different red atoms with upper index n, it follows that the size of the network N cannot exceed $n + 1$.

Claim 1. $\exists$ has no winning strategy for $G_\omega(\emptyset, \mathcal{A})$ By theorem 6 this implies that $\mathcal{A}$ has no complete representation.

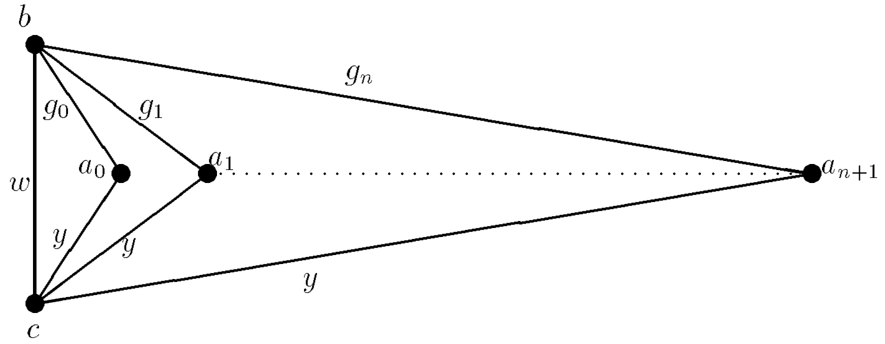
Proof of claim 1 Let $\forall$ start with the following sequence of moves.

1. Let $\forall$s first move be the atom w and let $\exists$ respond with the network N_0 with two nodes b, c with $N_0(b, c) = w$. In his second turn $\forall$ picks the edge (b, c) and the atoms g_0 and y. $\exists$ responds by adding one extra node a_0 and labelling the edges as in the diagram.

[1] There is a heated debate taking place in the school playground of my seven year old daughter as to whether black and white should be regarded as colours or not. The majority view is that white and black can be treated as a colours for certain purposes and that is the view we take in this article. For those who are unhappy about this position, please substitute other 'colours', say purple and orange. RH



2. Next $\forall$ picks the edge (b,c) and chooses the atoms (g_1, y). $\exists$ must add a new node a_1 and label the missing edge (a_0, a_1). The label must be an atom x such that $x \leq g_0; g_1 \wedge y; y$. Thus x can only be a red atom, say $x = r^n_k$ for some $k < n < \omega$. Now $\exists$ can choose n as large as she likes, but it will be finite. In the following n moves $\forall$ will be able to defeat her.
3. The next n moves by $\forall$ each use the same edge (b,c) and the pair of atoms (g_i, y) (for $i = 2, 3, \ldots n+1$). Each time $\exists$ must respond by adding a new node a_i and labelling the missing edges (a_i, a_j) $(j < i)$ and in each case they can only be labelled by red atoms. If $\exists$ were to win there would have to be a subnetwork with nodes $a_0, \ldots, a_{n+1}$ such that each pair were related by a red atom. Thus the upper index of all edges in this subnetwork must be n and by the foregoing remark, an all-red atomic network where the upper index is n cannot have size greater than $n+1$. However, if $\exists$ were still winning the atomic network on the nodes $a_0, \ldots, a_{n+1}$ would be an all-red atomic network of size $n+2$ where the upper index was n. This is impossible, so $\forall$ must have won.



Claim 2. $\exists$ has a winning strategy for each game $G_n(\emptyset, \mathcal{A})$ It follows from this, by theorem 9, that $\mathcal{A}$ is elementary equivalent to a completely representable relation algebra. Since the representable relation algebras form a variety we can conclude that $\mathcal{A}$ is a representable relation algebra.

Proof of claim 2 For each turn in a play of $G_n(\emptyset, \mathcal{A})$, $\exists$ plays using the following strategy. At the start of the $m+1$'th round, let the play so far be

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_m$$

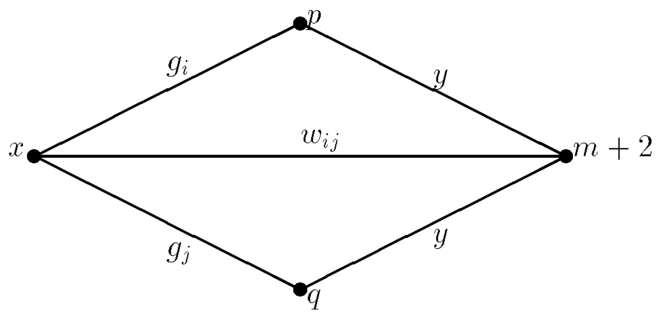
where $|N_0| \leq 2$ and $|N_k| \leq k+2$ for each $k \leq m$. Without loss, let us assume that the nodes of N_k are a subset of $\{0, 1, \ldots, k+1\}$. Now let $\forall$ pick the edge $(p,q) \in N^2_m$ and a pair of atoms $c, d \in \mathcal{A}$ with $c; d \geq N_m(p,q)$. If there is already a node $z \in N_m$ such that $N_m(p,z) = c$ and $N_m(z,q) = d$ then $\exists$ does nothing but sets

$$N_{m+1} = N_m. \tag{2}$$

Otherwise $\exists$ adds a new node '$m+2$' and has to define an atomic labels $N_{m+1}(x, m+2)$, $N_{m+1}(m+2, x)$ on each missing edge $(x, m+2), (m+2, x)$ for $x \in N_m \setminus \{p, q\}$ in such a way that the resulting graph forms an atomic network, N_{m+1}. The label $N_{m+1}(m+2, x)$ will always be the same as $N_{m+1}(x, m+2)$ so we only define $N_{m+1}(x, m+2)$. Her strategy selects atoms for $N_{m+1}(x, m+2)$ and the colour of that atom is always white if it is consistent, otherwise black if possible, or finally red. First the choice of colour.

1. If $(x, m+2)$ does not complete any triangle $(y, y, \_)$ then choose the colour white. (Note that other edges chosen by $\exists$ cannot create an inconsistency here as she never chooses yellow.)
2. If $(x, m+2)$ does complete a $(y, y, \_)$ triangle but does not complete any $(g_i, g_j, \_)$ (any i, j) then choose the colour black (again, $\exists$ does not choose green).
3. If $(x, m+2)$ completes both a $(y, y, \_)$ and a $(g_i, g_j, \_)$ triangle then choose red.

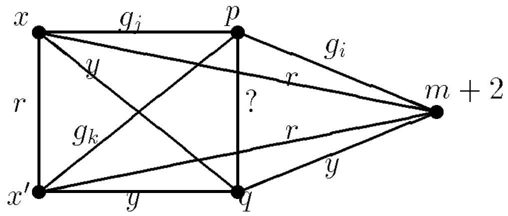
Now the indexes.



4. For cases 1 and 2, $\exists$ gives subscripts in some cases (see diagram above). $\exists$ adds the subscript ij to either w or b if either (i) $N_m(x, p) = g_i, N_m(x, q) = g_j$ and $\forall$ chose y and y for $(p, m+2)$ and $(q, m+2)$ (ii) the same as (i) with p and q reversed (iii) the same as (i) or (ii) with x and $m+2$ reversed.
5. If $\exists$ is forced to choose a red colour (case 3) then the atom she chooses is $r^{(n+2)^2}_{(n+2).x+m+1}$. The idea here is that in a game of length n there will not be more than $n+2$ distinct nodes and $(n+2)^2$ edges. By choosing $(n+2)^2$ as the superscript there will be 'enough room' for $\exists$ to choose distinct red colours with the same superscript, for all the red edges she chooses.

It must be shown that in each case $\exists$ constructs an atomic network, i.e. that every new triangle constructed must be consistent with the definition of composition in $\mathcal{A}$. In each case there is one new triangle chosen by $\forall$ and this is assumed consistent. It remains to check the other new triangles added to N_m. These have one old edge, already in N_m, plus two new edges. For cases 1 and 2, $\exists$ chooses white or black colours which do not complete (y, y, W) or (g_i, g_j, B) triangles. These are consistent. Now consider case 3. A red edge is always consistent unless it completes an all red triangle. So suppose there are two nodes $x, x' \in N_m$ where $N_m(x, x')$ is already a red atom and $\exists$ sets $N_{m+1}(x, m+2) = r^{(n+2)^2}_{(n+2).x\ +m+1}$ and $N_{m+1}(x', m+2) = r^{(n+2)^2}_{(n+2).x'\ +m+1}$ (see diagram on page 11). Let $N_m(x', p) = g_k$ and $N_m(x', q) = y$. Without loss, suppose $x' > x$. Now if the label on (x, x') was chosen by $\exists$ in a previous round then her strategy would have selected the atom $r^{(n+2)^2}_{(n+2).x+x'-1}$ in which case the triangle $(x, x', m+2)$ does not violate the definition of composition as $(n+2).x+x'-1 < (n+2).x\ +m+1 < (n+2).x'\ +m+1$ and the definition of the atom structure permits all-red triangles with equal upper indices provided the lower indices are all distinct. But what if the label on (x, x') were chosen by $\forall$? To deal with this case, first note the following proposition.

PROPOSITION 11 *If N is an atomic network constructed in any play of $G_n(\emptyset, \mathcal{A})$ then for any quadrangle $Q \subseteq N$, at least one of the six edges of Q was selected by $\exists$.*



The proposition is not hard to check. Now returning to the case in hand, consider the quadrangle with nodes x, x', p, q. We are assuming that the edge (x, x') was chosen by $\forall$ so one of the others was chosen by $\exists$. The edges $(x, p), (x, q), (x', p)$ and (x', q) are labelled by yellow or green atoms, but $\exists$'s strategy never selects those colours so none of these could have been selected by her. That leaves the edge (p, q) but here, her strategy would have led her to pick the atom w_{jk} or b_{jk} (by case 4) which forces $i = j$ or $i = k$ and this makes the new node $m + 2$ into a duplicate of either x or x'. In either case $\exists$ can respond using equation 2 i.e. by doing nothing.

We have shown that $\exists$'s strategy always provides her with a legal move and hence it is a winning strategy for $G_n(\emptyset, \mathcal{A})$. This proves claim 2 and completes the proof of the theorem. $\Box$

Note This example answers an open question of Maddux by constructing an integral, representable relation algebra with no complete representation. The example has the further property of being symmetric.

COROLLARY 12 *The class of all completely representable relation algebras is not elementary.*

PROOF:

By theorem 9, the relation algebra $\mathcal{A}$ just constructed has no complete representation but it is elementary equivalent to a completely representable relation algebra $\mathcal{A}'$ and so the class cannot be elementary. $\Box$

4 Cylindric Algebra

An equivalent result for finite dimensional cylindric algebras can be obtained, but is not entirely straight-forward. This will be included in a new paper jointly by Ian Hodkinson and Robin Hirsch. For infinite dimensional cylindric algebras the problem is decidedly delicate, many of the theorems and lemmas of this paper do not go through for the infinite dimensional case. It is conjectured, very tentatively at this stage, that the completely representable infinite dimensional cylindric algebras *do* form an elementary class.

References

[Andréka *et al.*1991] H Andréka, D Monk, and I Németi. *Algebraic Logic*. North-Holland, 1991.

[Chang and Keisler1973] C.C. Chang and H.J. Keisler. *Model Theory*. North-Holland, Amsterdam, 2nd edition, 1973.

[Henkin *et al.*1985] L Henkin, J Monk, and A Tarski. *Cylindric Algebras Part II*. North-Holland, 1985.

[Hirsch and HodkinsonSubmitted] R Hirsch and I Hodkinson. Step by step - building representations in algebraic logic. *Journal of Symbolic Logic*, Submitted.

[Jónsson and Tarski1948] B Jónsson and A Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, 54, 1948.

[Lyndon1950] R Lyndon. The representation of relational algebras. *Annals of Mathematics*, 51(3):707–729, 1950.

[Lyndon1956] R Lyndon. The representation of relation algebras, ii. *Annals of Mathematics*, 63(2):294–307, 1956.

[Maddux1978] R Maddux. *Topics in Relation Algebra*. PhD thesis, University of California, Berkeley, 1978.

[Maddux1982] R Maddux. Some varieties containing relation algebras. *Transactions of the AMS*, 272(2):501–526, 1982.

[McKenzie1970] R McKenzie. Representation of integral relation algebras. *Michigan Mathematics Journal*, 17:279–287, 1970.

[Monk1993] D Monk. Lectures on cylindric set algebras. In *Algebraic Methods in Logic and in Computer Science*, volume 28 of *Institute of Mathematics, Polish Academy of Sciences*. Banach Center publications, 1993.

[Stone1936] M Stone. The theory of representations for boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[Venema1992] Y Venema. *Many-Dimensional Modal Logic*. PhD thesis, University of Amsterdam, 1992.

[Villain and Kautz1986] M Villain and H Kautz. Constraint propagation algorithms for temporal reasoning. *Knowledge Representation*, pages 377–382, 1986.