Hard inverse problems in a probabilistic formulation

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Abstract

A hard (or exponential-time) computational problem is a problem where the time needed to obtain a solution increases at least exponentially with the size of the input (number of input parameters). Similarly, we can define a hard inverse problem as a problem whose solution time increases more than exponentially with the dimension of the model (parameter) space.

Being an exponential-time problem is (as suggested by the terminology) a property that is independent of the algorithm used for the analysis. In fact, we follow the convention used in the theory of algorithmic complexity, and say that a problem is 'exponential-time' (or hard) when a best conceivable algorithm used on the 'worst case' version of the problem is exponential-time.

Full probabilistic (Bayesian) analysis of an inverse problem, using Monte Carlo sampling or any other conceivable method, requires that we can collect sufficient information about the posterior probability density to ensure that an approximate reconstruction of the density is possible. If it is known a priori that we are sampling, e.g., a Gaussian posterior, an efficient sampling scheme can be designed. In fact, this sampling problem is not hard (it is a 'polynomial-time' problem). However, if for instance the structure of the posterior is only known to be smooth in some sense, sampling can be a huge computational task, and sometimes hard.

The difficulty of solving highly nonlinear inverse problems is illustrated by specific examples. For certain problems, a `brick wall effect' is observed, namely that the problem is practically solvable up to a certain number of unknown model parameters, but requires excessive computer resources if only a few more model parameters were added. In some cases, it is likely that the brick wall effect can be attributed to an exponential increase in the number of secondary maxima for the posterior probability density, when the number of model parameters increases. Some inverse problems in seismology are likely to suffer from this difficulty.

Intuitively, one would expect that problems with none or few local maxima are comparatively easy to solve. That this statement is not necessarily true follows from a result concerning determination of the volume of an (unknown) n-gon shaped region. This problem, which is similar to the problem of characterizing a probability density which is constant over an n-gon shaped region, has been shown to be a hard problem (Khachiyan, 1989).

The existence of hard inverse problems of relevance for physical applications can be demonstrated. The consequence of this is that certain physical model reconstruction problems with many unknown parameters may be practically unsolvable by any conceivable method based on digital computing.